Is it hard to color a graph as a 4 years old child?

Édouard Bonnet¹, Florent Foucaud², Eunjung Kim³, and Florian Sikora³.

¹ Hungarian Academy of Sciences
² LIMOS – France
³ LAMSADE, Université Paris Dauphine, CNRS – France

JGA 2015
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\textsuperscript{1} Hungarian Academy of Sciences
\textsuperscript{2}LIMOS – France
\textsuperscript{3}LAMSADE, Université Paris Dauphine, CNRS – France

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(Wrote most of these slides)
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JGA 2015
Outline

Warm Up

Exact algorithms

Weak Grundy Coloring
Outline

Warm Up

Exact algorithms

Weak Grundy Coloring
Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.
Grundy Colorings – Definition(s)

The worst way of reasonably coloring a graph.

- Order the vertices $v_1, v_2 \ldots v_n$ to maximize the number of colors used by the greedy coloring: the Grundy Number (GN).
- That is, $v_i$ is colored with $c(v_i)$ the first color that is not in its neighborhood (first-fit).
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- **Connected version**: $\forall i, G[v_1 \cup \ldots \cup v_i]$ is connected.
Grundy Colorings – Definition(s)

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- Order the vertices $v_1, v_2 \ldots v_n$ to maximize the number of colors used by the greedy coloring: the Grundy Number (GN).
- That is, $v_i$ is colored with $c(v_i)$ the first color that is not in its neighborhood (first-fit).
- **Connected version:** $\forall i, G[v_1 \cup \ldots \cup v_i]$ is connected.
- **Weak version:** $v_i$ can be colored with any color in $\{1, \ldots, c(v_i)\}$.
Algorithmic motivations

- GN(G) upper bounds the number of colors used by any greedy heuristic for Min Coloring.
- GN(G) $\leq C \cdot \chi(G)$ on some classes of graphs gives a C-approximation for Min Coloring.
- See Sampaio’s PhD thesis for further motivations.
Can you achieve color 6?
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Can you achieve color 6?
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Can you achieve color 6?
Cycles

Grundy number =?
Cycles

Grundy number = ?
Cycles

Grundy number =?
Cycles

Grundy number $=$?
Cycles

Grundy number = 3
(even) Cycles

Connected Grundy number = ?
(even) Cycles

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(even) Cycles

Connected Grundy number = 2
$K_{n,m}$

Grundy number =?
$K_{n,m}$

Grundy number $= 2$
$K_{p,p}$ minus a perfect matching

Grundy number $=$?
$K_{p,p}$ minus a perfect matching

Grundy number $=?$
**K_{p,p}** minus a perfect matching

Grundy number = ?
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Connected Grundy number $=$?
$K_{p,p}$ minus a perfect matching

Connected Grundy number = ?
\( K_{p,p} \) minus a perfect matching

Connected Grundy number = ?
$K_{p,p}$ minus a perfect matching

Connected Grundy number $=$?
$K_{p,p}$ minus a perfect matching

Connected Grundy number $= 2$
(minimal) Witnesses

How many vertices (at least) did we need to achieve color $k$?
(easy)
(minimal) Witnesses

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(easy)

$k$ (Clique of size $k$).
(minimal) Witnesses

How many vertices (at most) did we need to achieve color $k$?
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4
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How many vertices (at most) did we need to achieve color \( k \)?

Binomial tree \( T_4 \).

\[
T_k | = \sum_{1 \leq i \leq k-1} T_i |, \quad T_1 | = 1.
\]
(minimal) Witnesses

How many vertices (at most) did we need to achieve color $k$?

Binomial tree $T_4$.

- $|T_k| = \sum_{1 \leq i \leq k-1} |T_i|$, $|T_1| = 1$.
- So $|T_k| = 2^{k-1}$
Warm Up

Exact algorithms

Weak Grundy Coloring

(minimal) Witnesses – Consequences

Algorithm:

- For every subset of $2^{k-1}$ vertices, check if there is a witness.

Theorem (Zaker '05)

The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

XP algorithm: $O(f(k)n^{g(k)})$: polynomial for fixed values of $k$. 
(minimal) Witnesses – Consequences

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Theorem (Zaker '05)
The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

XP algorithm: $O(f(k)n^{g(k)})$: polynomial for fixed values of $k$.

Can we do the same for the connected Grundy number?
Witness for the connected version

A connected Grundy number = 3 but unbounded witness.

Theorem: Connected Grundy Coloring is $\text{NP}$-complete even for $k = 7$.
Witness for the connected version

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Can't do the previous trick!

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**Witness for the connected version**

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**Theorem**
**Connected Grundy Coloring is NP-complete**
Witness for the connected version

Connected Grundy number = 3 but unbounded witness. Can’t do the previous trick!

Theorem
Connected Grundy Coloring is NP-complete even for \( k = 7 \).
Outline

Warm Up

Exact algorithms

Weak Grundy Coloring
Solving Grundy Coloring

Try all possible ordering of the vertices and check if at least $k$ colors are used by the greedy coloring: $\Theta(n!)$-time.
Solving Grundy Coloring

Try all possible ordering of the vertices and check if at least $k$ colors are used by the greedy coloring: $\Theta(n!)$-time.

We can have a $O(c^n)$ algorithm.
Solving Grundy Coloring

In a witness:

- Any color class is an independent dominating set in the graph induced by the next classes.
Solving Grundy Coloring

In a witness:

- Any color class is an independent dominating set in the graph induced by the next classes.
- \[ GN(S) = \max\{GN(S \setminus X), X \text{ ind. dom. set of } G[S]\} + 1. \]
Solving Grundy Coloring

- A minimal independent dominating set is a maximal independent set.
- One can enumerate all maximal independent sets in $O(1.45^n)$ time.
  - Filling a cell in the table takes $O(1.45^i)$ time for a subset of size $i$. 

Theorem

Under the ETH, Grundy Coloring cannot be solved in $O^{\ast}(c^{tw})$ for any constant $c$ (even $O^{\ast}(2^{o(tw \log tw)})$).

A cannot replace the $n$ by the treewidth (even feedback vertex set).
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Theorem

One can solve Grundy coloring in $\sum_{i=0}^{n} \binom{n}{i} 1.45^i = (1 + 1.45)^n$. 

**Theorem**

Under the ETH, Grundy Coloring cannot be solved in $O^{*}(c^{tw})$ for any constant $c$ (even $O^{*}(2^{o(tw) \log tw})$).
Solving Grundy Coloring

- A minimal independent dominating set is a maximal independent set.
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  - Filling a cell in the table takes $O(1.45^i)$ time for a subset of size $i$.

**Theorem**

One can solve Grundy coloring in $\sum_{i=0}^{n} \binom{n}{i} 1.45^i = (1 + 1.45)^n$.

Cannot replace the $n$ by the treewidth (even feedback vertex set).

**Theorem**

Under the ETH, Grundy Coloring cannot be solved in $O^*(c^{tw})$ for any constant $c$ (even $O^*(2^{o(tw \log tw)})$).
Outline

Warm Up

Exact algorithms

Weak Grundy Coloring
Are there graphs where weak Grundy exceeds Grundy number?
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Weak Grundy number = 3, (connected) Grundy number = 2.
Are there graphs where weak Grundy exceeds Grundy number?

Weak Grundy number $= 3$, (connected) Grundy number $= 2$. 
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Weak Grundy number = 3, (connected) Grundy number = 2.
Are there graphs where weak Grundy exceeds Grundy number?

Weak Grundy number = 3, (connected) Grundy number = 2.
Color Coding

- Add colors between 1 and k uniformly at random to the instance.
Color Coding

- Add colors between 1 and $k$ uniformly at random to the instance.
- The probability that a good solution was well colored is a function of $k$. 
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- Solving the instance is easier with this extra information.
Color Coding

- Add colors between 1 and k uniformly at random to the instance.
- The probability that a good solution was well colored is a function of k.
- Solving the instance is easier with this extra information.
- Do a function of k tries to have a small probability of failure.
Theorem
Weak Grundy Coloring is in FPT.

FPT algorithm: $O(f(k)n^c)$. 
Theorem

Weak Grundy Coloring is in FPT.

FPT algorithm: $O(f(k)n^c)$.

- Idea:
  - A witness is of size at most $2^{k-1}$.
  - This witness is well colored with probability is $\frac{1}{k^{2k-1}}$. 

**Theorem**

Weak Grundy Coloring is in FPT.

FPT algorithm: \(O(f(k)n^c)\).

- **Idea:**
  - A witness is of size at most \(2^{k-1}\).
  - This witness is well colored with probability is \(\frac{1}{k^{2k-1}}\).
  - After \(\log(\frac{1}{\varepsilon})k^{2k-1}\) tries, the probability of success is at least \(1 - \varepsilon, \forall \varepsilon > 0\).
Guess #1
Guess #1
Guess #1
Guess #2
Guess #2

Grundy Coloring

florian.sikora@dauphine.fr
Guess #2
\[ \ldots O(k^{2^k}) \] unsuccessful guesses later \ldots
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Open Problems

- Is Grundy Coloring solvable in $O(2^n)$?
- Is Grundy Coloring solvable in $O(f(tw)n^c)$?
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- Is Grundy Coloring solvable in $O(2^n)$?
- Is Grundy Coloring solvable in $O(f(tw)n^c)$?
- Is Grundy Coloring solvable in $O(f(k)n^c)$? Even for bipartite graph?
  - True for chordal, claw-free and bounded degree graphs or graph excluding a fixed graph as a minor.
  - ETH fails if Grundy Coloring is solvable in $O^*(2^{o(k)}2^{o(n+m)})$. 

florian.sikora@dauphine.fr
Merci !