SCIENTIFIC TOOLS FOR DECISION MAKING

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Lecture 2-1

Linear Programming

Introduction & Modelling Examples
Introduction

- Most classical and used tool in OR
- Powerful framework for modelling and solving several concrete problems
- 80% of companies use LP
  - small companies: microprocessing, . . .
  - big companies: automobile, oil structures, . . .
  - Management field → Excel Solver
Example 1

**Problem:** A factory manufactures two finished products $P_1$ and $P_2$ using three raw materials $M_1, M_2$ and $M_3$

<table>
<thead>
<tr>
<th>Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
</tr>
<tr>
<td>$P_2$</td>
</tr>
<tr>
<td>$P_1$</td>
</tr>
<tr>
<td>Resources</td>
</tr>
<tr>
<td>--------------</td>
</tr>
<tr>
<td>$M_1$</td>
</tr>
<tr>
<td>$M_2$</td>
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<tr>
<td>$M_3$</td>
</tr>
<tr>
<td>Selling price (€) per unit</td>
</tr>
<tr>
<td>--------------</td>
</tr>
<tr>
<td>$P_1$</td>
</tr>
<tr>
<td>$P_2$</td>
</tr>
</tbody>
</table>

**Manager objective:** Organize the production in order to achieve the highest total revenue
Example 1 - Linear programming modelling

Decision variables definition

What the decision is about? → How many $P_1$ and $P_2$ to produce?

Variables

- $x_1$: number of $P_1$ units to produce
- $x_2$: number of $P_2$ units to produce
Example 1 - Linear programming modelling

Constraints specification

Availability of raw materials

- \( M_1 \):
  - 1 unit of \( P_1 \) \( \rightarrow \) 1 unit of \( M_1 \)
  - \( x_1 \) units of \( P_1 \) \( \rightarrow \) \( x_1 \) units of \( M_1 \)
  - 1 unit of \( P_2 \) \( \rightarrow \) 6 units of \( M_1 \)
  - \( x_2 \) units of \( P_2 \) \( \rightarrow \) 6\( x_2 \) units of \( M_1 \)

\[ x_1 \text{ units of } P_1 \& x_2 \text{ units of } P_2 \Rightarrow x_1 + 6x_2 \text{ units of } M_1 \]

Constraint: Qty of \( M_1 \) used \( \leq \) Availability of \( M_1 \)

\[ x_1 + 6x_2 \leq 30 \]
Example 1 - Linear programming modelling

Constraints specification

Availability of raw materials

- $M_1$: 
  \[ x_1 + 6x_2 \leq 30 \]
- $M_2$: 
  \[ 2x_1 + 2x_2 \leq 15 \]
- $M_3$: 
  \[ 4x_1 + x_2 \leq 24 \]

Non-negativity constraints

\[ x_1 \geq 0, x_2 \geq 0 \]
Example 1 - Linear programming modelling

Objective function specification

\[ x_1 \text{ unit of } P_1 \rightarrow 2x_1 \ \€ \]
\[ x_2 \text{ units of } P_2 \rightarrow 3x_2 \ \€ \]

Objective:

\[ \text{Maximize } 2x_1 + 3x_2 \]
Example 1 - Linear programming modelling

\[
\begin{align*}
\text{max} & \quad 2x_1 + 3x_2 \\
\text{s.t.} & \quad x_1 + 6x_2 \leq 30 \\
& \quad 2x_1 + 2x_2 \leq 15 \\
& \quad 4x_1 + x_2 \leq 24 \\
& \quad x_1 \geq 0, x_2 \geq 0
\end{align*}
\]
Example 2

Problem: An oil refinery produces 2 types of fuels $P_1$ and $P_2$ by blending 3 types of oils $C_1$, $C_2$ and $C_3$. 

Limited resources

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3000</td>
<td>2000</td>
<td>1000</td>
</tr>
</tbody>
</table>

Requirements

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$&lt; 30%$</td>
<td>$&gt; 40%$</td>
<td>$&gt; 50%$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$&lt; 50%$</td>
<td>$&gt; 10%$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Prices per barrel (€)

<table>
<thead>
<tr>
<th>Purchase</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>12</td>
<td>24</td>
<td>20</td>
</tr>
<tr>
<td>$C_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Selling

<table>
<thead>
<tr>
<th></th>
<th>$P_1$</th>
<th>$P_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>22</td>
<td>18</td>
</tr>
</tbody>
</table>

Aim: look for the daily production structure that maximizes the operating margin of the refinery.
**Example 2 - Linear programming modelling**

**Decision variables definition**

What the decision is about? → Inventory of quantities to know for:

- **Objective:**
  - Daily quantities of $P_1$ and $P_2$ produced
  - Daily quantities of $C_1$, $C_2$ and $C_3$ used

- **Constraints:**
  - Resources limitation of oils
  - Ratio: daily quantities of $C_i$ intervening in $P_j$
GENERAL PRESENTATIONS AND EXAMPLES

Example 2 - Linear programming modelling

Decision variables definition

- $x_{ij}$: daily quantities of oil $C_i$ intervening in fuel $P_j$, for $i = 1, 2, 3$ and $j = 1, 2$ (in barrel)
- $u_i$: daily quantity of oil $C_i$ used, for $i = 1, 2, 3$
- $v_j$: daily quantity of fuel $P_j$ produced, for $j = 1, 2$

Remark: $u_i, v_j$ can be written according to $x_{ij}$:

- $u_i = x_{i1} + x_{i2} = \sum_{j=1}^{2} x_{ij}$ \hspace{1cm} (i = 1, 2, 3)
- $v_j = x_{1j} + x_{2j} + x_{3j} = \sum_{i=1}^{3} x_{ij}$ \hspace{1cm} (j = 1, 2)

→ called auxiliary variables
Example 2 - Linear programming modelling

Constraints specification

Non-negativity constraints

\[ x_{ij} \geq 0, \ i = 1, 2, 3; \ j = 1, 2 \]

Availability of oils

\[
\begin{align*}
- \quad u_1 & \leq 3000 & x_{11} + x_{12} & \leq 3000 \\
- \quad u_2 & \leq 2000 & x_{21} + x_{22} & \leq 2000 \\
- \quad u_3 & \leq 1000 & x_{31} + x_{32} & \leq 1000
\end{align*}
\]

\[ \sum_{j=1}^{2} x_{ij} \leq d_i, \ i = 1, 2, 3 \ (d_i: \text{availability of } C_i) \]
Example 2 - Linear programming modelling

Constraints specification

Ratios satisfaction

- $P_1$:
  - Ratio of oil C$_1$ in fuel $P_1$ produced
    \[
    \frac{x_{11}}{v_1} \Rightarrow \frac{x_{11}}{x_{11} + x_{21} + x_{31}}
    \]
    \[
    \Rightarrow x_{11} \leq 0.3 \times (x_{11} + x_{21} + x_{31}) \Leftrightarrow 7x_{11} - 3x_{21} - 3x_{31} \leq 0
    \]
  - Ratio of oil C$_2$ in fuel $P_1$ produced
    \[
    x_{21} \geq 0.4 \times (x_{11} + x_{21} + x_{31}) \Leftrightarrow 2x_{11} - 3x_{21} + 2x_{31} \leq 0
    \]
  - Ratio of oil C$_3$ in fuel $P_1$ produced
    \[
    x_{31} \geq 0.5 \times (x_{11} + x_{21} + x_{31}) \Leftrightarrow x_{11} + x_{21} - x_{31} \leq 0
    \]
Example 2 - Linear programming modelling

Constraints specification

Ratios satisfaction

- $P_2$:
  - Ratio of oil $C_1$ in fuel $P_2$ produced
    \[ x_{12} \leq 0.5 \times (x_{12} + x_{22} + x_{32}) \iff x_{12} - x_{22} - x_{32} \leq 0 \]
  - Ratio of oil $C_2$ in fuel $P_2$ produced
    \[ x_{22} \geq 0.1 \times (x_{12} + x_{22} + x_{32}) \iff x_{12} - 9x_{22} + x_{32} \leq 0 \]
Example 1 - Linear programming modelling

Objective function specification

Maximize (Profit - Cost)

\[ \Rightarrow \max (22v_1 + 18v_2) - (12u_1 + 24u_2 + 20u_3) \]

\[ \Leftrightarrow \max 22(x_{11} + x_{21} + x_{31}) + 18(x_{12} + x_{22} + x_{32}) - 12(x_{11} + x_{12}) - 24(x_{21} + x_{22}) - 20(x_{31} + x_{32}) \]

\[ \Leftrightarrow \max 10x_{11} - 2x_{21} + 2x_{31} + 6x_{12} - 6x_{22} - 2x_{32} \]
Example 1 - Linear programming modelling

\[
\begin{align*}
\text{max} & \quad 10x_{11} - 2x_{21} + 2x_{31} + 6x_{12} - 6x_{22} - 2x_{32} \\
\text{s.t.} & \quad x_{11} + x_{12} \leq 3000 \\
& \quad x_{21} + x_{22} \leq 2000 \\
& \quad x_{31} + x_{32} \leq 1000 \\
& \quad 7x_{11} - 3x_{21} - 3x_{31} \leq 0 \\
& \quad 2x_{11} - 3x_{21} + 2x_{31} \leq 0 \\
& \quad x_{11} + x_{21} - x_{31} \leq 0 \\
& \quad x_{12} - x_{22} - x_{32} \leq 0 \\
& \quad x_{12} - 9x_{22} + x_{32} \leq 0 \\
x_{ij} & \geq 0, \quad i = 1, 2, 3; j = 1, 2
\end{align*}
\]
GENERAL MODELLING SCHEME IN LP

4 steps:

1. Decision variables definition
2. Constraints specification
3. Objective function specification
4. LP writing

Remarks:

- Decision variables: what the decision is about?; what do we need to express the constraints and objective function?
- Never use non defined variables beforehand
- Auxiliary variables must be linked to principle variables
- Respect the homogeneity of terms in constraints and objective function
**General form of LP**

\[
\begin{align*}
\text{max} & \quad 3x_1 + 2x_2 + x_3 \\
\text{s.t.} & \quad x_1 + 3x_2 + 2x_3 \leq 6 \\
& \quad 2x_1 + 4x_2 + 2x_3 \leq 9 \\
& \quad 4x_1 + x_2 + 2x_3 \leq 10 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 3x_1 - x_2 \\
\text{s.t.} & \quad x_1 - 2x_2 + x_3 \geq -2 \\
& \quad 2x_1 + 3x_3 = 4 \\
& \quad x_1 + 3x_2 \leq 5 \\
& \quad x_1, x_3 \geq 0
\end{align*}
\]
GENERAL FORM OF LP

Remarks:

In general: $n$ real numbers $c_1, c_2, \ldots, c_n$

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n = \sum_{j=1}^{n} c_jx_j \rightarrow \text{linear function}$$
General form of LP

Remarks:

In general: \( n \) real numbers \( c_1, c_2, \ldots, c_n \)

\[
f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n = \sum_{j=1}^{n} c_jx_j \rightarrow \text{linear function}
\]

If \( f \) is a linear function and \( b \) is a real number:

\( f(x_1, x_2, \ldots, x_n) = b \rightarrow \text{linear equation} \)

\( f(x_1, x_2, \ldots, x_n) \leq b \rightarrow \text{linear inequality} \)

\( f(x_1, x_2, \ldots, x_n) \geq b \rightarrow \text{linear inequality} \)
**General form of LP**

**Matrix form**

\[ A.x = b \]

\[ A.x \leq b \]

\[ A.x \geq b \]

\[ A = [a_{ij}] : m \times n \text{ matrix} \]

\[ x = (x_1, \ldots, x_n) : n \text{ vector of decision variables} \]

\[ b = (b_1, \ldots, b_m)^t : m \text{ vector of constants (right hand side)} \]
**General form of LP**

**Developed algebraic form**

\[
\begin{align*}
&\leq \\
&\geq \\
&\sum_{i=1}^{n} a_{1i} x_i = b_1 \\
&\sum_{i=1}^{n} a_{2i} x_i = b_2 \\
&\vdots \\
&\sum_{i=1}^{n} a_{mi} x_i = b_m
\end{align*}
\]

**Compact algebraic form**

\[
\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m
\]
**General form of LP**

**Definition**

Linear programming problem is an optimisation problem where we want to determine values of decision variables $x_j\ (j = 1, \ldots, n)$ such that:

- Maximise (minimise) a linear function of the decision variables → **Objective function**
- decision variable values must satisfy the constraints. Each constraint is a linear equation or linear inequality
- Sign restriction is associated to every variable $x_j$ ($x_j \geq 0$ or $x_j \leq 0$ or $x_j$ w.n.r)
**General form of LP**

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{min} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m_1 \\
& \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = m_1 + 1, \ldots, m_1 + m_2 \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = m_1 + m_1 + 1, \ldots, m_1 + m_2 + m_3 \\
& x_j \geq 0 \quad j = 1, \ldots, n_1 \\
& x_j \leq 0 \quad j = n_1 + 1, \ldots, n_1 + n_2 \\
& x_j \text{ w.n.r} \quad j = n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3
\end{align*}
\]