Scientific tools for decision making

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Lecture 2-3

Graphical sensitivity analysis and economical interpretation

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Sensitivity analysis on Excel
Graphical introduction to sensitivity analysis
Example 1

\[
\begin{align*}
\text{max} & \quad 2x_1 + 3x_2 \\
\text{s.t.} & \quad x_1 + 6x_2 \leq 30 \quad (C1) \\
& \quad 2x_1 + 2x_2 \leq 15 \quad (C2) \\
& \quad 4x_1 + x_2 \leq 24 \quad (C3) \\
& \quad x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

Each solution: \((x_1, x_2) \rightarrow\) a point \(x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\) of \(\mathbb{R}^2\)

Set of feasible solutions: \(E \subseteq \mathbb{R}^2\) sub set of points
Optimal sol. $B = \begin{pmatrix} 3 \\ 9/2 \end{pmatrix} \rightarrow$ Producing 3 units of $P_1$ & 4.5 units of $P_2 \rightarrow 19.5\€$
Example 1

An objective function $c_1x_1 + c_2x_2$

Optimal solution is $B$ while

$$-1 \leq -\frac{c_1}{c_2} \leq -\frac{1}{6} \quad \rightarrow \quad \frac{1}{6} \leq \frac{c_1}{c_2} \leq 1$$

- If $P_2$ price is certain ($3 \in \mathbb{E}$) but not $P_1$ price
  Same production plan? Yes if $\frac{1}{6} \leq \frac{c_1}{3} \leq 1 \quad \Leftrightarrow \quad \frac{1}{2} \leq c_1 \leq 3$

- If $P_1$ price is certain ($2 \in \mathbb{E}$) but not $P_2$ price
  Same production plan? Yes if $\frac{1}{6} \leq \frac{2}{c_2} \leq 1 \quad \Leftrightarrow \quad 2 \leq c_2 \leq 12$

Remark

Same production plan but not the same objective function value
**Example 1**

**Problem:** Up to how much I can restrict the availabilities of $M_3$ without changing the optimal solution.
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

Problem: Up to how much I can restrict the availabilities of $M_3$ without changing the optimal solution

Constraint: $4x_1 + x_2 \leq d_3$
**Example 1**

**Problem:** Up to how much I can restrict the availabilities of $M_3$ without changing the optimal solution.

Constraint: $4x_1 + x_2 \leq d_3$  
$d_3 \downarrow$: limit value when goes beyond $B$  
$4b_1 + b_2 = d_3 \Rightarrow d_3 = 16.5$  
$\Rightarrow d_3 \in [16.5, +\infty[$ same opt. sol
**MODIFICATION OF THE RIGHT HAND SIDE VALUE**

**Example 1**

**Problem:** Possibility to have more $M_2$. Is it interesting? Up to how much?
**MODIFICATION OF THE RIGHT HAND SIDE VALUE**

**Example 1**

**Problem:** Possibility to have more $M_2$. Is it interesting? Up to how much?

1 unit more $\rightarrow 2x_1 + 2x_2 \leq 16 \Rightarrow$ New solution $\left(\begin{array}{c} 3.6 \\ 4.4 \end{array}\right)$ of value 20.4
**Modification of the Right Hand Side Value**

**Example 1**

**Problem:** Possibility to have more $M_2$. Is it interesting? Up to how much?

1 unit more $\rightarrow 2x_1 + 2x_2 \leq 16 \Rightarrow$ New solution $\begin{pmatrix} 3.6 \\ 4.4 \end{pmatrix}$ of value 20.4

Interesting if we buy $M_2$ less than $20.4 - 19.5 = 0.9€$
Example 1

Problem: Possibility to have more $M_2$. Is it interesting? Up to how much?
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

Problem: Possibility to have more $M_2$. Is it interesting? Up to how much?

Possible only if $(D2)$ does not go beyond $E$ with $(D1) \cap (D3) = \{E\}$

$E \in (D2)$ with $E = \begin{pmatrix} 96/23 \\ 114/23 \end{pmatrix}$: \[ 2e_1 + 2e_2 = d_2 \Rightarrow d_2 = 18.26 \]
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

Problem: Possibility to have more $M_2$. Is it interesting? Up to how much?

Possible only if $(D2)$ does not go beyond $E$ with $(D1) \cap (D3) = \{E\}$

$E \in (D2)$ with $E = \left(\frac{96}{23}, \frac{114}{23}\right)$: $2e_1 + 2e_2 = d_2 \Rightarrow d_2 = 18.26$

$\Rightarrow$ Valid for $18.26 - 15 = 3.26$ units more
**Example 1**

**Problem:** One unit less of $M_2$ makes lose of 0.9€. How far is the decrease?
Example 1

Problem: One unit less of \( M_2 \) makes lose of 0.9€. How far is the decrease?

When meeting \( A = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \). Then \( 2a_1 + 2a_2 = d_2 \Rightarrow d_2 = 10 \)

⇒ Validity interval \([10, 18.26]\) where 1 unit variation of \( M_2 \) is valued 0.9€
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

**Problem:** Possibility to have more $M_1$. Is it interesting? Up to how much?
Example 1

Problem: Possibility to have more $M_1$. Is it interesting? Up to how much?

1 unit more $\rightarrow x_1 + 6x_2 \leq 31 \Rightarrow$ New solution $\left(\begin{array}{c}2.8 \\ 4.7\end{array}\right)$ of value 19.7

Marginal profit $= 19.7 - 19.5 = 0.2\text{€}$
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

Problem: Possibility to have more $M_1$. Is it interesting? Up to how much?
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

Problem: Possibility to have more $M_1$. Is it interesting? Up to how much?

Validity domain: Increase until $F = \begin{pmatrix} 0 \\ 7.5 \end{pmatrix}$: $f_1 + 6f_2 = d_1 \Rightarrow d_1 = 45$

$\Rightarrow$ allowable increase is 15 units
MODIFICATION OF THE RIGHT HAND SIDE VALUE

Example 1

Problem: Some units less of $M_1$. How far is the decrease?
Example 1

Problem: Some units less of $M_1$. How far is the decrease?

Validity domain: Decrease until $C$, with $(D2) \cap (D3) = \{C\}$

$C \in (D1)$ with $C = \left(\frac{5.5}{2}\right)$: $c_1 + 6c_2 = d_1 \Rightarrow d_1 = 17.5$

$\Rightarrow$ allowable decrease is 12.5 units
Sensitivity analysis on Excel
SENSITIVITY ANALYSIS ON EXEL SOLVER

Demonstration

Example 1

\[
\begin{aligned}
\text{max} & \quad 2x_1 + 3x_2 \\
\text{s.t.} & \\
& x_1 + 6x_2 \leq 30 \quad (C1) \\
& 2x_1 + 2x_2 \leq 15 \quad (C2) \\
& 4x_1 + x_2 \leq 24 \quad (C3) \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
\]
**Sensitivity analysis on Exel solver**

**Demonstration**

**Example 1**

<table>
<thead>
<tr>
<th>Variable Cells</th>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Reduced Cost</th>
<th>Objective Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>D5</td>
<td>x1</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1,5</td>
<td></td>
</tr>
<tr>
<td>E5</td>
<td>x2</td>
<td>4,5</td>
<td>0</td>
<td>3</td>
<td>9</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

| Constraints   | Cell  | Name     | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---------------|-------|----------|-------------|---------------------|--------------------|--------------------|
| F10           | 30    | 0,2      | 30          |                     | 15                 | 12,5               |
| F11           | 15    | 0,9      | 15          | 3,260869565         | 15                 | 5                  |
| F12           | 16,5  | 0        | 24          | 1E+30              | 7,5                |
Different results of a linear program
Usual case

- The LP has an unique optimal solution
  ⇒ Solution corresponding to a node of the polyhedra
  node: intersection of $n$ independent hyperplans.

Result

If a LP admits one or more optimal solutions, one of them is at least one node of the convex polyhedra of the feasible solutions.
Several optimal solutions

\[
\begin{align*}
\text{max} \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 2 \\
& x_1 \leq 1 \\
& x_1, x_2 \geq 0
\end{align*}
\]

- Set of optimal solutions: segment $AB$
  - Among optimal solutions: nodes $A$ and $B$
  - Set of optimal solutions is infinite
- Can happen when one hyperplane is parallel to the one associated to the objective function
No optimal solution

Case 1: Set of feasible solutions is empty

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 2 \\
& \quad x_1 \geq 3 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

No feasible solution $\Rightarrow$ No optimal solution
No optimal solution

Case 2: Set of feasible solutions is not bounded

\[
\begin{aligned}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_2 \geq 2 \\
& \quad x_1 \leq 1 \\
& \quad x_1, x_2 \geq 0
\end{aligned}
\]

→ Infinite optimum

Remark: Set of optimal solutions can be not bounded and admits an optimal solution
Example: \( \text{min} \ 2x_1 + x_2 \Rightarrow A \) is the optimal solution
RESULTS OF A LINEAR PROGRAM

Summary

- No optimal solution
- An unique optimal solution (one extreme point)
- An infinity of optimal solutions (at least two extreme points)
REDUNDANCY

\[
\begin{align*}
    & \text{max} & 2x_1 + x_2 \\
    \text{s.t.} & & x_1 + x_2 \leq 2 \\
                & & x_1 \leq 1 \\
                & & x_2 \leq 3 \\
                & & x_1, x_2 \geq 0
\end{align*}
\]

\(x_2 \leq 3\) is redundant
**Degeneracy**

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 2 \\
& \quad x_1 \leq 1 \\
& \quad x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

More than \( n \) constraints that define a node