Ordered sets with interval representation and (m, n)-Ferrers relation

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ABSTRACT: Semiorders may form the simplest class of ordered sets with a not necessarily transitive indifference relation. Their generalization has given birth to many other classes of ordered sets, each of them characterized by an interval representation, by the properties of its relations or by forbidden configurations. In this paper, we are interested in preference structures having an interval representation. For this purpose, we propose a general framework which makes use of n-point intervals and allows a systematic analysis of such structures. The case of 3-point intervals shows us that our framework generalizes the classification of Fishburn by defining new structures. Especially we define three classes of ordered sets having a non-transitive indifference relation. A simple generalization of these structures provides three ordered sets that we call "d-weak orders", "d-interval orders" and "triangle orders". We prove that these structures have an interval representation. We also establish some links between the relational and the forbidden mode by generalizing the definition of a Ferrers relation.

keywords: preference modelling, intransitivity, interval representation, $m+n$ posets.

1 Introduction

The representation of the decision maker's preferences over the set of alternatives is one of the important steps of decision aiding. In general, such a representation is done by the help of binary relations. The nature and the number of these relations depend on the context of the decision problem, the information that we are able to handle and the expectations of the decision maker. In this paper we are interested in representations having two binary relations: P and I. P (respectively I) represents a strict preference (respectively an indifference) between two alternatives. We suppose that the set of alternatives, denoted by A, is nonempty and finite. We define P as an irreflexive and transitive relation. I is the symmetric complement of P. Hence, by definition, I is symmetric but not necessarily transitive.

Following the well-known work of Luce ([Luc56]), a new line of research appeared in preference modelling with the introduction of a non-transitive indifference relation. A number of studies have shown that the transitivity of indifference can be empirically falsifiable in some context. Undoubtedly, the most famous example on this subject is the one given by Luce on a cup of sweetened coffee ([Luc56]). Before him, some authors already suggested this phenomenon ([Arm39, GR36, Fec60, Hal55] and [Poi05]). Some historical comments on the subject are presented by Fishburn and Monjardet ([FM92]).

Relaxing the transitivity of indifference results in different structures. Semiorders can be considered as the simplest ones. Many other structures generalizing them have been studied recently. Fishburn ([Fis97]) distinguishes ten nonequivalent ordered sets having a non-transitive indifference. These are semiorders, interval orders, split semiorders, split interval orders, tolerance orders, bitolerance orders, unit tolerance orders, bisemiorders, semitransitive orders and subsemitransitive orders.

Three different modes can be used to characterize such structures: the *numerical mode*, the *relational mode* and the *forbidden mode*[1].

The *numerical mode* characterizes an ordered set by the existence of a map from A into real intervals (with or without interior points) that are ordered in a way that preserves P [2]. For example, an interval order is defined by the existence of $f, g : A \to \mathbb{R}$ such that for all x in A, $f(x) \geq g(x)$ and for all x, y in A, xPy if and only if $g(x) > f(y)$. We call such a representation an "interval representation" since $g(x)$ (resp. $f(x)$) can be seen as the left (resp. right) endpoint of an interval associated to x.

The *relational mode* characterizes an ordered set by necessary and sufficient conditions on the properties of its binary relations. For example, an ordered set is a weak order if and only if I is transitive.

The *forbidden mode* characterizes an ordered set by the absence among all induced orders of P of every member of the family of minimal forbidden orders. For example, a semiorder is defined by the interdiction of two posets, $1+3$ and $2+2$, where the notation $m+n$ denotes a poset on $m+n$ points with two disjoint chains on m and n points and xIy holds whenever x and y are in different chain[3].

The ordered sets of the Fishburn classification may be characterized in one of these three modes. Most of these sets have an interval representation and for a few of them the characterization in all the three modes are known.

We begin this paper by presenting the classification of Fishburn and by showing the interrelations between its ten classes. Then, we introduce our general framework and analyse the case of 3-point intervals. Our study allows us to define new structures. Section 4 presents three structures which are defined as a generalization of the new ordered sets of 3-point intervals case. We call them "d-weak orders", "d-interval orders" and "triangular orders". These structures, defined by a given dimension, have a non-transitive indifference relation. We prove that they have an interval representation.

In the second part of this paper we are interested in the relations between the forbidden mode and the relational mode. We are motivated by the absence of a known characterization of semitransitive and subsemitransitive orders in the relational mode. The known characterizations of these structures call on a class of forbidden orders ($m+n$ posets). We introduce the notion of "(m, n)-Ferrers relation" in order to provide a relational characterization for these two structures.

[1]The numerical mode and the forbidden mode are defined in [Fis97].
[2]Real numbers can be seen as degenerate intervals.
[3]More details can be found in Section 5.

2 Ordered sets with interval representation

A small difference of evaluation between two alternatives can remain inadequate to affirm a strict preference between them. This leads to the introduction of a positive threshold q in such a way that the alternative a is said to be preferred to the alternative b if and only if the evaluation of a is greater than the evaluation of b plus the threshold q. The most classical structure respecting such an idea is a semiorder. Before presenting the formal definition of a semiorder we remind that P and I are two binary relations defined on the nonempty finite set A where P is irreflexive and transitive and I is the symmetric complement of P. We denote the relation $P \cup I$ by R (P and I are respectively the asymmetric and the symmetric part of R). In the rest of the paper we will present only the representation according to the relation P. The representation of I can be directly obtained by the one of P since $\forall x, y,\ xIy \Longleftrightarrow \neg xPy \wedge \neg yPx$.

Definition 1 (Semiorder) *[PV97] A relation R on a finite set A is a semiorder if and only if there exists a real-valued function g, defined on A, and a nonnegative constant q such that*

$$\forall x, y \in A,\ xPy \Longleftrightarrow g(x) > g(y) + q.$$

Such a representation can be transformed into an interval representation. It is sufficient to note that associating a value $g(x)$ and a strictly positive value q to each element x of A is equivalent to associating two values: $f_1(x) = g(x)$ (representing the left extremity of an interval) and $f_2(x) = g(x)+q$ (representing the right extremity of the interval) to each element x. Obviously: $f_2(x) \geq f_1(x)$ always holds.

Figure 1 illustrates the representation of a semiorder with intervals.

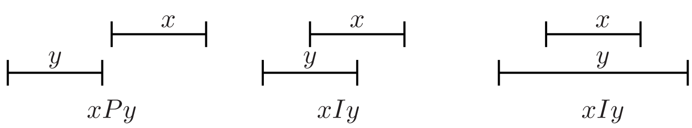


Figure 1: Semiorder

A generalization of semiorders can be done by relaxing the uniform threshold feature and/or by adding some interior points. For instance, interval orders do not use necessarily constant thresholds; the interval representations of split semiorders and split interval orders make use of one interior point; and the ones of bitolerance orders, tolerance orders, unit tolerance orders and bisemiorders have two interior points. Fishburn, [Fis97], presented nine nonequivalent classes for a generalization of semiorders. Notice that none of them, except interval orders, has a characterization in the three different modes. We present in the following seven of them being defined in the numerical mode ([Fis97])[4].

[4]To our knowledge, semitransitive orders and subsemiorders which belong also to the Fishburn classification do not have a characterization in the numerical or relational mode. We will analyze their case in the last section of this paper.

A relation R on a finite set A is

- an *interval order*[5] if and only if there exist two real-valued functions f_1 and f_2, defined on A such that
$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow f_1(x) > f_2(y), \\ \forall x \in A,\ f_2(x) \geq f_1(x)\ ; \end{cases}$$

- a *split interval order* if and only if there exist three real-valued functions f_1, f_2 and f_3 defined on A such that
$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_1(x) > f_2(y), \\ f_2(x) > f_3(y), \end{cases} \\ \forall x \in A,\ f_3(x) \geq f_2(x) \geq f_1(x)\ ; \end{cases}$$

- a *split semiorder* if and only if it is a split interval order with $f_2(x) - f_1(x) = q$ and $f_3(x) - f_1(x) = p$ for all x in A, where q, p are nonnegative constants ;

- a *bitolerance order* if and only if there exist four real-valued functions f_i, $i \in \{1, 2, 3, 4\}$, defined on A such that
$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_1(x) > f_2(y), \\ f_3(x) > f_4(y), \end{cases} \\ \\ \forall x \in A,\ f_4(x) \geq f_2(x) \geq f_1(x) \quad \text{and} \quad f_4(x) \geq f_3(x) \geq f_1(x)\ ; \end{cases}$$

- a *unit bitolerance order* if and only if it is a bitolerance order with $f_4(x) - f_1(x) = 1$, for all x in A ;

- a *tolerance order* if and only if it is a bitolerance order with $f_2(x) + f_3(x) = f_1(x) + f_4(x)$, for all x in A ;

- a *bisemiorder* if and only if it is a bitolerance order and there exists a positive real q such that $(f_2(x) - f_1(x)) = (f_3(x) - f_2(x)) = (f_4(x) - f_3(x)) = q$, for all x in A.

Figure 2 illustrates the interrelationships (inclusions and equivalences) between the ten classes of preference structures having non-transitive indifference. Some of the preference structures such as trapezoid orders, bi-interval orders, etc., will be presented in Section 4. Linear orders and weak orders are added in order to have a complete view of class inclusions. We present in the following a theorem recapitulating all these interrelationships.

[5] The name "interval order" first appeared in print in Fishburn [Fis70a, Fis70b]. However in 1914, Wiener had defined a *relation of complete sequence* as an irreflexive relation P satisfying $P.I.P \subset P$

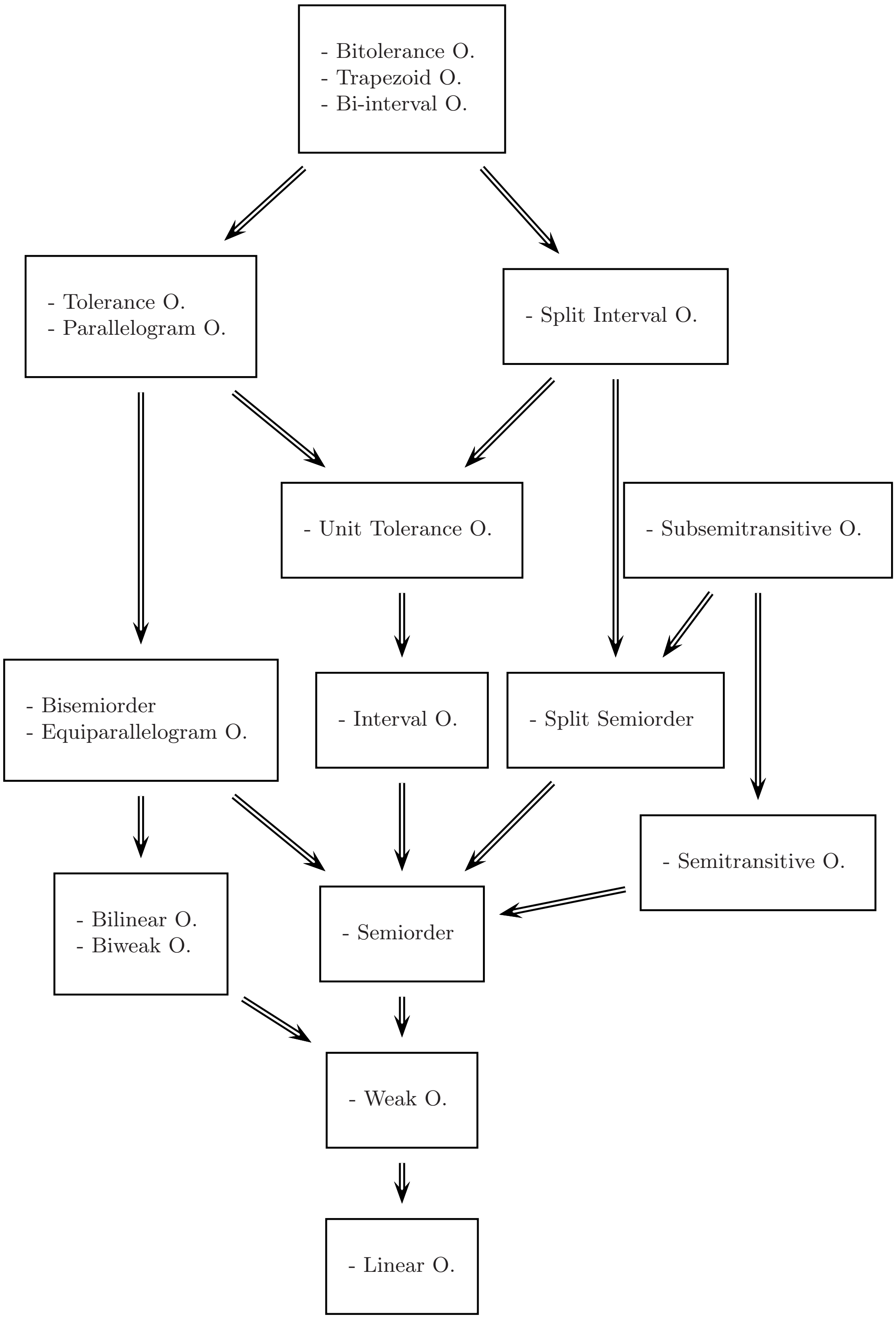


Figure 2: Inclusions between structures having non-transitive indifference

Theorem 1 *[Fis97] The classes of preference structures defined by each row of figure 2 are identical within each box. The box classes are partially ordered by inclusion from bottom to top according to the arrows.*

After this brief review of literature, we present now a general framework that we propose for a systematic analysis of preference structures having an interval representation. An interested reader can find more details on this subject in [ÖT06] and [Özt05].

3 A general framework for interval comparison

As the numerical representations of structures belonging to the Fishburn classification show, intervals are appropriate tools for the representation of intransitivity. Although there exists a large literature on the preference modelling and intransitivity, a study for unifying the representation of such structures is missing. We contribute to fill this gap by giving a general framework with a special interval representation. Our objective is to provide a common language for the study of preference structures having an interval representation by defining a mapping to a class of preference structures through a function from the set of objects to the set of intervals.

As in the whole paper, we study preference structures having two binary relations, P (an asymmetric relation) and I (the symmetric complement of P). However, in this part we do not require the transitivity of P.

Consider a finite set A, each object x of A is represented by an interval. We associate to each interval a finite number n of ordered points. We call such an element an *n-point interval*. If not otherwise mentioned, we will use indifferently the notation x for the object x and its associated interval. An "*n-point interval*" is an interval $x = [f_1(x), f_n(x)]$ with $n-2$ interior points $f_i(x)$ (i in $\{2, \ldots, n-1\}$) such that for all x in A and i in $\{1, \ldots, n-1\}$, $f_i(x) < f_{i+1}(x)$. Then, we introduce the notion of "*relative position*" that we denote by φ. The notation $\varphi(x, y) (\neq \varphi(y, x))$ represents the position of the interval x with respect to the interval y.

Definition 2 (Relative position) *Relative position $\varphi(x, y)$ is an n-tuple $(\varphi_1(x, y), \ldots, \varphi_n(x, y))$ where $\varphi_i(x, y)$ represents the number of j such that $f_i(x) \leq f_j(y)$.*

The preference relation between two objects depends on their relative position. We pay special attention to the case of two disjoint intervals since in the rest of the paper we will consider this case as a reference point. Intuitively, n-tuple of a relative position shows to what extent the position of two intervals is close to the disjoint case: $\varphi_i(x, y)$ represents the number of the points of interval y that point $f_i(x)$ must beat in order to have interval x completely on the right of interval y.

Example 1 *Let x and y be two 3-point intervals represented in Figure 3, then $\varphi(x, y) = (1, 0, 0)$ since there is only $f_3(y)$ being greater than $f_1(x)$, and $f_2(x)$ and $f_3(x)$ are greater than all the points of y.*

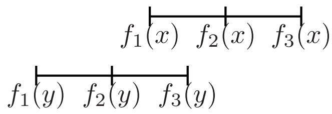


Figure 3: Relative position $\varphi(x, y) = (1, 0, 0)$

Naturally there are some relative positions which suit better to a preference relation than others. In order to represent this difference we introduce a new binary relation that we call "*stronger than*", defined on the set of relative positions.

Definition 3 ("*Stronger than*" relation) *Let φ and φ' be two relative positions, then we say that φ is "stronger than" φ' and note $\varphi \rhd \varphi'$ if $\forall i \in \{1, \dots, n\},\ \varphi_i \leq \varphi'_i$.*

Example 2 *Let $\varphi(x, y)$ and $\varphi(x, t)$ be two relative positions of Figure 4. We have $\varphi(x, y) = (1, 1, 0)$ and $\varphi(x, t) = (2, 1, 0)$. "$\varphi(x, y)$ is* stronger than *$\varphi(x, t)$" since $1 \leq 2,\ 1 \leq 1$ and $0 \leq 0$.*

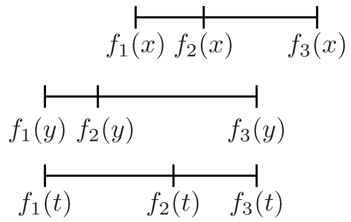


Figure 4: Example: $(1, 1, 0) \rhd (2, 1, 0)$

The *stronger than* relation is a partial order (reflexive, antisymmetric and transitive). We present in Figure 5 the graph of the relation $\rhd$ associated to 3-point intervals.

One of the objective of our framework is to determine all the preference structures having an n-point interval representation under some conditions. For this purpose, we will use the notion of relative position (intuitively we will define a preference relation by a set of relative positions and analyze its properties). Such a study needs some hypotheses. In what follows we present an axiomatization for this purpose:

Axiom 1 *The relation $P \cup I$ is complete, P is asymmetric and I is the complement of P (*i.e. *$I(x, y) \iff \neg P(x, y) \wedge \neg P(y, x)$).*

Axiom 2 *$P(x, y)$ and $I(x, y)$ depend only on the relative position of x and y.*

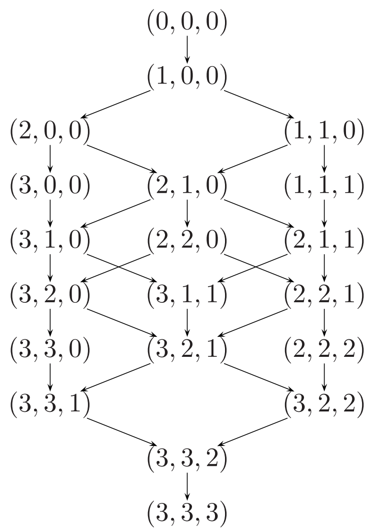


Figure 5: Graph of "stronger than" relation for $n = 3$

Axiom 3 *Given a relative position $\varphi(x, y)$, if $P(x, y)$ holds then for all t, z in A such that $\varphi'(t, z) \rhd \varphi(x, y)$, $P(t, z)$ holds.*

Axiom 4 *For all x, y in A, if $f_i(x) < f_i(y)$ for all i then $P(x, y)$ is not satisfied.*

Axiom 5 *Let Θ be the set of relative positions $\varphi(x, y)$ such that $P(x, y)$ holds. Then there is one and only one weakest relative position in the set Θ (a relative position is the weakest relative position of a set if all the other relative positions of the set* are stronger *than it).*

Axiom 1 shows that P and I are exhaustive and exclusive, Axiom 2 presents the comparison parameters and Axiom 3 guarantees the monotonicity. Every relative position is not a good candidate to represent a strict preference. Axiom 4 eliminates some undesired situations for the relation P. The role of the weakest relative position of a set of P is very important since we can determine all the other elements of the set by the weakest one (Axiom 3). Similarly, by forbidding the existence of more than one weakest relative position Axiom 5 guarantees a simple representation for the strict preference relation (as a consequence each preference relation can be represented by its unique weakest relative position).

In light of this axiomatization, we present a formal definition of preference structures induced by a relative position.

Definition 4 *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be an n-tuple where φ_i is in $\{0, 1, \dots, n\}$ for all i, and x and y two n-point intervals. Relations $P_{\leq\varphi}$ and $I_{\leq\varphi}$ associated to φ (i.e. φ represents the weakest relative position such that P holds) where $(n, n-1, n-2, \dots, 1) \not\rhd \varphi$ are defined as*

$$\begin{aligned} P_{\leq\varphi}(x, y) &\iff \varphi(x, y) \rhd \varphi, \\ I_{\leq\varphi}(x, y) &\iff \neg P_{\leq\varphi}(x, y) \wedge \neg P_{\leq\varphi}(y, x). \end{aligned}$$

The condition $(n, n-1, n-2, \ldots, 1) \not\geq \varphi$ guarantees the satisfaction of Axiom 4. It is easy to verify that the preference structure associated to an n-tuple φ characterized as in definition 4 verifies the Axioms 1, 2, 3, 4, 5.

Now, consider the strict preference relation, presented in Figure 6, having relative position (2,0,0) as the weakest relative position. Then $P_{\leq(2,0,0)}(x, y)$ if and only if $f_1(y) < f_1(x), f_3(y) < f_2(x)$ and $f_3(y) < f_3(x)$. Remark that the second inequality is redundant. In order to avoid such redundancies we introduce a new notion that we call *"the component set"* of an n-tuple φ and we denote it by $Cp_{\leq\varphi}$.

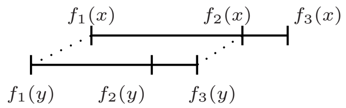


Figure 6: $P_{\leq(2,0,0)} : (0, 0, 0) \cup (1, 0, 0) \cup (2, 0, 0)$

For instance, $Cp_{\leq(2,0,0)} = \{(1, 1)(3, 2)\}$. Hence, $Cp_{\leq\varphi}$ represents the set of couples of points that are sufficient to be compared. Conditions on the elements of $Cp_{\leq\varphi}$ guarantees the minimality of the representation.

Definition 5 *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be an n-tuple in $\{0, 1, \ldots, n\}$ the component set $Cp_{\leq\varphi}$ associated to φ is the set of couples $(n - \varphi_i, i)$ such that there is no $i' < i$ with $\varphi_{i'} \leq \varphi_i$.*

Set $Cp_{\leq\varphi}$ contains all the information about a preference structure and offers a compact way for the characterization of different properties as it is shown in the following (proofs of the following propositions can be found in [Özt05]):

- $P_{\leq\varphi}$ is *transitive* if and only if $\forall (i, j) \in Cp_{\leq\varphi},\ i \geq j$,
- $I_{\leq\varphi}$ is *transitive* if and only if $\exists i,\ Cp_{\leq\varphi} = \{(i, i)\}$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *weak order* if and only if $\exists i,\ Cp_{\leq\varphi} = \{(i, i)\}$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *bi-weak order* if and only if $|Cp_{\leq\varphi}| = 2$ and $\forall (i, j) \in Cp_{\leq\varphi},\ i = j$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *Ferrers relation* if and only if $|Cp_{\leq\varphi}| = 1$.

Example 3 *We consider the preference relation presented in Figure 6. We saw that its component set is $Cp_{\leq(2,0,0)} = \{(1, 1)(3, 2)\}$. Hence, we can conclude that relation $P_{\leq(2,0,0)}$ is transitive (since $1 \geq 1$ and $3 \geq 2$) and relation $I_{\leq(2,0,0)}$ is not transitive (since $Cp_{\leq(2,0,0)}$ has more than one element).*

The propositions presented above impel us to conduct an exhaustive study of the 3-point intervals case.

There are 20 ($\frac{(2*3)!}{(3!)^2}$) different relative positions when two 3-point intervals are compared (it means there are 20 sets of relative positions having just one weakest relative position). The number of sets of relative positions satisfying our axioms is 15 (5 of the 20 sets do not satisfy Axiom 5). Our study shows that from these 15 sets, 7 preference structures can be characterized (some of them having more than one 3-point interval representation). From these 7 structures some of them belong to the classification given by Fishburn. These are weak orders, bi-weak orders, interval orders and split interval orders. However there exist three structures which do not have a known characterization. One of them has a non-transitive strict preference relation. Therefore, we call it an *"intransitive order"*. The other two structures can be characterized as an intersection of some known ordered sets. They are special cases of what we will call *d-weak order* and *triangle order* in the next section. The Section 4 is devoted to a detailed analysis of these structures. We present here their characterization by the help of component sets:

- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *d-weak order* if and only if $|Cp_{\leq\varphi}| = d$ and $\forall(i,j) \in Cp_{\leq\varphi},\ i = j$,

- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *triangle order* if and only if $Cp_{\leq\varphi} = \{(l,l),(i,j)\}$, and $i \geq j$.

Example 4 *Relation $P_{\leq(2,0,0)} \cup I_{\leq(2,0,0)}$ is a triangle order since $Cp_{\leq(2,0,0)} = \{(1,1)(3,2)\}$.*

Table 1 presents, by the help of the component sets, the 7 structures having a 3-point interval representation.

As it can be seen from Table 1, in our framework, 3-point intervals are not the minimal representation for some preference structures. For instance, the representation of weak orders makes use of just one point and the one of bi-weak orders and interval orders need two points. However, 3-point intervals propose a minimal representation for tree-weak orders, bi-interval orders, triangle orders and intransitive orders. Remark also that one preference structure can have more than one representation even if these are minimal (for example triangle orders).

We are ready now to present in more details d-weak orders and triangle orders. Their definition inspires also another characterization, the one of *d-interval order.*

4 Three ordered sets defined by dimension

The notion of dimension is important for the characterization of certain structures (especially bilinear orders, bisemiorders and bi-interval orders). The *dimension* of an ordered set R, denoted by $dim(R)$, is the minimal cardinality over the sets of linear extensions whose intersection gives the ordered set R. Clearly, $dim(R) = 1$ if and only if R is a linear order. Weak orders which are

Preference Structure	*3-point interval representation*
Weak Orders	$Cp_{\leq(3,3,0)} = \{(3,3)\}$ $Cp_{\leq(3,1,1)} = \{(2,2)\}$ $Cp_{\leq(2,2,2)} = \{(1,1)\}$
Bi-weak Orders	$Cp_{\leq(3,1,0)} = \{(2,2),(3,3)\}$ $Cp_{\leq(2,1,1)} = \{(1,1),(2,2)\}$ $Cp_{\leq(2,2,0)} = \{(1,1),(3,3)\}$
Three-Weak Orders	$Cp_{\leq(2,1,0)} = \{(1,1),(2,2),(3,3)\}$
Interval Orders	$Cp_{\leq(0,0,0)} = \{(3,1)\}$ $Cp_{\leq(3,0,0)} = \{(3,2)\}$ $Cp_{\leq(1,1,1)} = \{(2,1)\}$
Bi-Interval Orders	$Cp_{\leq(1,0,0)} = \{(3,2),(2,1)\}$
Triangle Orders	$Cp_{\leq(1,1,0)} = \{(2,1),(3,3)\}$ $Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$
Intransitive Orders	$Cp_{\leq(3,2,0)} = \{(3,3),(1,2)\}$ $Cp_{\leq(2,2,1)} = \{(1,1),(2,3)\}$

Table 1: Preference structures with 3-point interval representation

not linear orders have dimension 2 ($dim(weak\ order) \leq 2$) and semiorders which are not linear orders or weak orders have dimension 3 ($dim(semiorder) \leq 3$) ([Rab78, Tro92]). Bilinear orders are characterized with $dim(bilinear\ order) \leq 2$.

Similar studies can be done for the intersection of weak orders, semiorders and interval orders. The notion of dimension exists also for such intersections.

The *weak order dimension* of an ordered set R is the minimum number of weak orders whose intersection gives the ordered set R. It is shown that bilinear orders are equivalent to biweak orders and the number of minimal forbidden posets for them is infinite ([BFR72, Fis97]). Naturally, they can be defined in the numerical mode as follows:

Definition 6 (Biweak order) *[Fis97] A relation R on a finite set A is a biweak order if and only if there exist two real-valued functions f and g, defined on A such that $\forall x, y \in A$,*

$$xPy \Longleftrightarrow \begin{cases} g(x) > g(y), \\ f(x) > f(y). \end{cases}$$

Such a representation is similar to an interval one but lacks clarity for the place of endpoints of intervals. For example, concerning object x, its interval representation can be $[f(x), g(x)]$ if $f(x) \leq g(x)$ or $[g(x), f(x)]$ if $g(x) \leq f(x)$. Such an ambiguity can be easily resolved thanks to an old theorem of Dushnik and Miller ([DM41]). Our presentation is based on the paper of Dushnik and Miller but the notation is coherent with our prior definitions.

Theorem 2 *[DM41] A relation R on a finite set A is a biweak order if and* *only if there exist two real-valued functions f_1 and f_2 on A such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} f_2(x) > f_2(y), \\ f_1(x) > f_1(y), \end{cases} \\ \forall x,\ f_2(x) \geq f_1(x). \end{cases}$$

We propose to generalize such an idea for the intersection of d weak orders. We call *d-weak order* an ordered set whose weak order dimension is d and we define them in the following way:

Definition 7 (d-weak order) *A relation R on a finite set A is a d-weak order if and only if there exist d real-valued functions g_i ($i \in \{1, \dots, d\}$), defined on A, such that ,*

$$\forall x, y \in A,\ xPy \Longleftrightarrow \forall i \in \{1, \dots, d\},\ g_i(x) > g_i(y). \tag{1}$$

This representation is not an interval one since there is no ordering condition on different g_i[6]. By generalizing the theorem of Dushnik and Miller, we propose in the following an interval representation for d-weak order.

Proposition 1 *A relation R on a finite set A is a d-weak order if and only if there exist d real-valued functions g_i ($i \in \{1, \dots, d\}$) defined on A such that $\forall x, y \in A$,*

$$\begin{cases} xPy \Longleftrightarrow \forall i \in \{1, \dots, d\},\ g_i(x) > g_i(y), \\ \forall x,\ \forall i \in \{1, \dots, d-1\},\ g_{i+1}(x) \geq g_i(x). \end{cases} \tag{2}$$

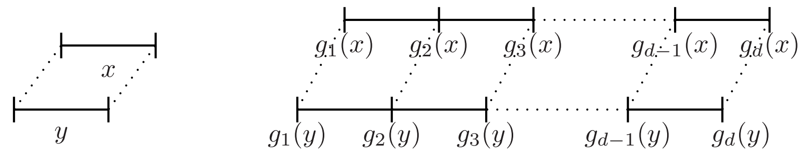
Proof.
- ($2 \Longrightarrow 1$): Obvious.
- ($1 \Longrightarrow 2$): Supposing that there exist d real-valued functions g_i ($i \in \{1, \dots, d\}$), defined on A, such that, $\forall x, y \in A,\ xPy \Longleftrightarrow \forall i \in \{1, \dots, d\},\ g_i(x) > g_i(y)$, we will show that one can always find d real-valued functions g'_i ($i \in \{1, \dots, d\}$) defined on A satisfying (2).

We define a constant M such that $M = \max_i \max_{x \in A} |g_i(x)|$ (A is a finite set) and we define $\forall x \in A,\ g'_i(x) = g_i(x) + i * (2M)$. It is easy to see that $g_i(x) > g_i(y) \Longleftrightarrow g'_i(x) > g'_i(y)$.

For the second inequality of the proposition, we have $g'_{i+1}(x) - g'_i(x) = g_{i+1}(x) - g_i(x) + 2|M|$ and $2|M| \geq g_{i+1}(x) - g_i(x)$ by definition. Hence we obtain $\forall x,\ \forall i \in \{1, \dots, d-1\},\ g'_{i+1}(x) \geq g'_i(x)$. ■

We showed that every d-weak order can be represented by intervals. Figure 7 illustrates such a representation.

[6] To our knowledge, no interval representation is proposed for this class of ordered set.



xPy Biweak order

xPy d-weak order

Figure 7: d-weak order

In the same way one can define interval order dimension. The *interval order dimension* ($I(R)$) of an ordered set R is the minimum number of interval orders whose intersection gives the ordered set R. Interval orders are characterized by $I = 1$. We call *d-interval order*, an ordered set whose interval order dimension is d.

Definition 8 (d-interval order) *A relation R on a finite set A is a d-interval order if and only if there exist d real-valued functions g_i ($i \in \{1, \ldots, d\}$) and d nonnegative functions q_i ($i \in \{1, \ldots, d\}$) defined on A, such that ,*

$$\forall x, y \in A,\ xPy \Longleftrightarrow \forall i \in \{1, \ldots, d\},\ g_i(x) > g_i(y) + q_i(y). \tag{3}$$

This representation is not an interval one and the interval representation of d-interval order is not widely studied in the literature. Therefore , we propose in the following an interval representation for these structures.

Proposition 2 *A relation R on a finite set A is d-interval order if and only if there exist $2d$ real-valued functions g_i ($i \in \{1, \ldots, 2d\}$) defined on A, such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \forall k \in \{1, \ldots, d\},\ g_{(2k-1)}(x) > g_{(2k)}(y), \\ \forall x,\ \forall i \in \{1, \ldots, 2d-1\},\ g_{i+1}(x) \geq g_i(x). \end{cases} \tag{4}$$

Proof.
-($4 \Longrightarrow 3$): Supposing that there exist $2d$ real-valued functions g_i ($i \in \{1, \ldots, 2d\}$) defined on A verifying (4), we show that the assertion (3) is satisfied. Let us define d real valued functions g'_i ($i \in \{1, \ldots, d\}$) on the set A in such a way that $\forall x \in A,\ g'_i(x) = g_{2i-1}(x)$ and d nonnegative functions q_i ($i \in \{1, \ldots, d\}$) on the set A such that $\forall x \in A,\ q_i(x) = g_{2i}(x) - g_{2i-1}(x)$. Consequently, for all x, y in A, $g_{(2k-1)}(x) > g_{(2k)}(y)$ implies $g'_i(x) > g'_i(y) + q_i(y)$.
-($3 \Longrightarrow 4$): Suppose that the assertion 3 is verified with d real-valued functions g_i ($i \in \{1, \ldots, d\}$) and d nonnegative functions q_i ($i \in \{1, \ldots, d\}$) on A. We define $2d$ real-valued functions g'_i ($i \in \{1, \ldots, 2d\}$) on A, such that $\forall x,\ g'_{(2k-1)}(x) = g_k(x) + (2k-1)M$ and $g'_{2k}(x) = g_k(x) + (2k-1)M + q_k(x)$ where $M = 2 * \max_i \max_x \max(g_i(x), q_i(x))$. Hence, we have $\forall x, y,\ g_k(x) > g_k(y) + q_k(y) \Longleftrightarrow g'_{(2k-1)}(x) > g'_{(2k)}(y)$.

We show in two parts that the inequality $\forall x, i,\ g'_{i+1}(x) \geq g'_i(x)$ holds:

- If i is odd, for all x:
  - $g'_{i+1}(x) = g_{\frac{(i+1)}{2}}(x) + q_{\frac{(i+1)}{2}}(x) + iM$,
  - $g'_i(x) = g_{\frac{(i+1)}{2}}(x) + iM$,

  since the function q_i is nonnegative for all i, we have $\forall x, i,\ g'_{i+1}(x) \geq g'_i(x)$.
- If i is even, for all x:
  - $g'_{i+1}(x) = g_{\frac{(i+2)}{2}}(x) + (i+1)M$,
  - $g'_i(x) = g_{\frac{i}{2}}(x) + q_{\frac{i}{2}}(x) + (i-1)M$,

  hence, we have $g'_{i+1}(x) - g'_i(x) = g_{\frac{(i+2)}{2}}(x) - g_{\frac{i}{2}}(x) - q_{\frac{i}{2}}(x) + 2M$. Following the definition of M this value is nonnegative, as a consequence we have $\forall x, i,\ g'_{i+1}(x) \geq g'_i(x)$.■

Figure 8 illustrates the interval representation satisfying the condition of our proposition for d-interval order.

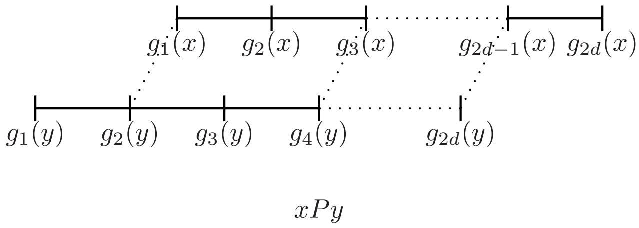


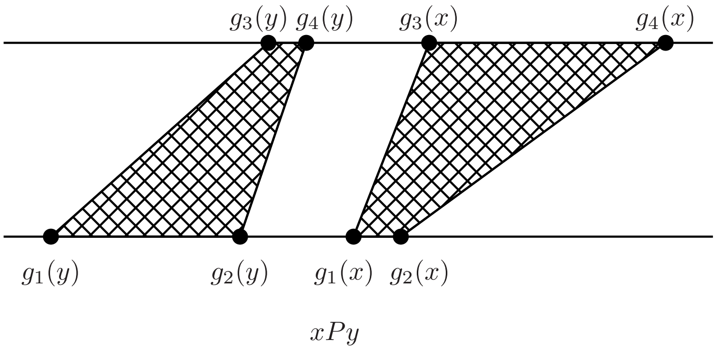
Figure 8: d-interval Order

A special case of d-interval orders are bi-interval orders which are also known as *trapezoid orders* since they have a representation by trapezoids (see figure 9): the preference relation P holds when there is no intersection between two trapezoids and all the remaining cases are expressed by the indifference I. It is proved that the class of bitolerance orders and trapezoidal orders are identical ([BT94]).

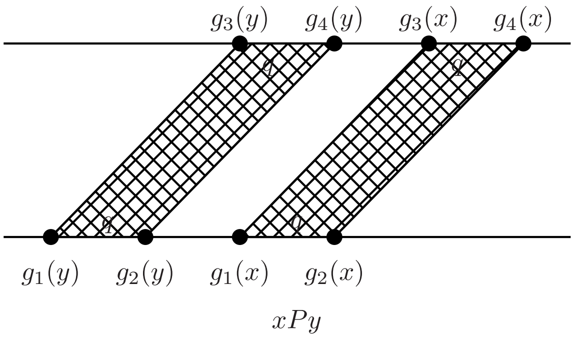
A subclass of trapezoid orders is the class of *equiparallelogram orders* where $g_2(x) - g_1(x)$ and $g_4(x) - g_3(x)$ is equal to a constant for all x in A. This subclass is identical to the class of bi-semiorders.

We present in Figure 9 the graphical representation of a trapezoid order and an equiparallelogram order.

Finally we propose an ordered set defined as the intersection of two different classes of orders: weak order and interval order. We call such a structure a *triangle order*.



Trapezoid Order



Equiparallelogram Order

Figure 9: 2-interval Orders

Definition 9 *A relation R on a finite set A is a triangle order if and only if there exist 2 real-valued functions g_i $(i \in \{1,2\})$ defined on A and one nonnegative function q on the set A such that*

$$\forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} g_1(x) > g_1(y), \\ g_2(x) > g_2(y) + q(y). \end{cases} \tag{5}$$

Like in the case of d-weak orders and d-interval orders, we propose an interval representation for triangle orders.

Proposition 3 *A relation R on a finite set A is a triangle order if and only if there exist 3 real-valued functions g_i $(i \in \{1,2,3\})$ defined on A, such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} g_1(x) > g_1(y), \\ g_2(x) > g_3(y), \end{cases} \\ \forall x,\ \forall i \in \{1,2\},\ g_{i+1}(x) \geq g_i(x). \end{cases} \tag{6}$$

Proof.
-($6 \Longrightarrow 5$): Suppose that there exist 3 real-valued functions g_i ($i \in \{1,2,3\}$) defined on A satisfying the assertion 6. One can always define 2 real-valued functions g'_i ($i \in \{1,2\}$) and one nonnegative function q on the set A such that $\forall x \in A$, $g'_1(x) = g_1(x)$, $g'_2(x) = g_2(x)$ and $q(x) = g_3(x) - g_2(x)$. These functions satisfy the assertion 5.
-($5 \Longrightarrow 6$): Suppose that there exist 2 real-valued functions g_i ($i \in \{1,2\}$) and one nonnegative function q on the set A satisfying the assertion 5. Let us define three real-valued functions g'_i ($i \in \{1,2,3\}$) defined on A, such that $\forall x$,

- $g'_i(x) = g_i(x) + i|M|, \ \forall i \in \{1,2\}$,
- $g'_3(x) = g_2(x) + 2|M| + q(x)$

where $M = 2 * \max_i \max_x(g_i(x))$. Hence, $\forall x, y$, ($g_1(x) > g_1(y)$ and $g_2(x) > g_2(y) + q(y)$) is equivalent to ($g'_1(x) > g'_1(y)$ and $g'_2(x) > g'_3(y)$).

The last inequality of 6 is also satisfied since

- $\forall x, \ g'_2(x) - g'_1(x) = g_2(x) - g_1(x) + |M|$ and by definition of M, $\forall x, \ g_2(x) - g_1(x) \leq |M|$;
-$\forall x, \ g'_3(x) - g'_2(x) = q(x)$ and q is a nonnegative function.■

Figure 10 illustrates the preference relation P of a triangle order. An object is preferred to another when its associated triangle is completely on the right of the triangle of the second. Remark that our proposition provides triangles oriented to the left. However, other representations where triangles are oriented to the right can provide identical ordered sets.

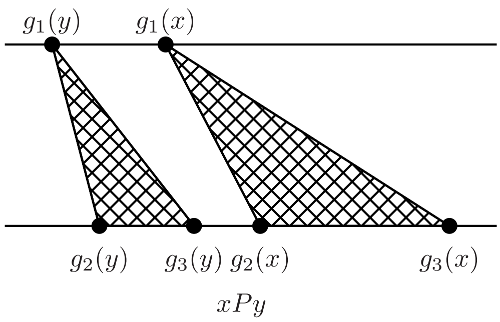


Figure 10: Triangle Order

In the two previous sections we paid attention to the numerical mode characterization of preference structures having a non-transitive indifference relation. Now we are interested in the two other modes, the relational and the forbidden ones.

5 $m + n$ posets and the relational mode

The *forbidden* mode defines an ordered set by the absence among all induced orders of P of every member of family of minimal forbidden orders (where P is asymmetric and transitive and I is its symmetric complement) and the *relational mode* defines an ordered set by necessary and sufficient conditions on the properties of its binary relations.

The orders encountered most often in the previous sections have as forbidden orders disjoint sums of two linear orders (or chains): For positive integers $m, n \in \{1, 2, \dots\}$, let $m+n$ denote a poset (irreflexive and transitive relation) on $m+n$ points. The poset $m+n$ has two disjoint chains, one on m points and the other on n points. We have not(xPy) and not(yPx) whenever x and y are in a different chain (it means that the points x and y are indifferent: xIy) and with (x_iPx_j) whenever x_i and x_j are in the same chain and the rank of x_i is greater than the rank of x_j. We present in the following the characterization in the forbidden mode of some of orders presented previously.

Theorem 3 *[SS58, Fis70a, Fis85] A binary relation P (asymmetric and transitive) on a finite set A is*

- *a linear order if and only if it has no* $1+1$ *;*
- *a weak order if and only if it has no* $1+2$ *;*
- *a semiorder if and only if it has no* $1+3$ *and no* $2+2$ *;*
- *an interval order if and only if it has no* $2+2$*.*

Figure 11 illustrates the graphical representations of the forbidden orders presented in Theorem 3.

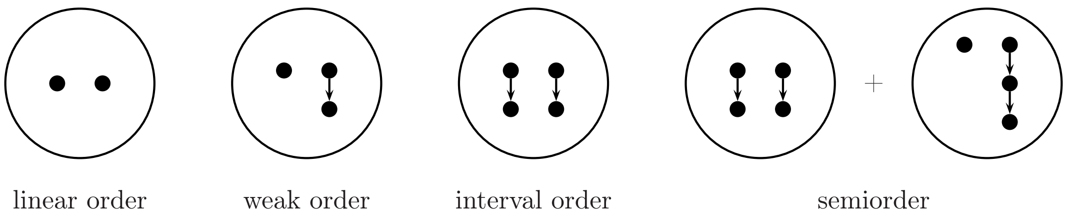


Figure 11: Forbidden orders for some preference structures

Semitransitive orders and *subsemiorders* ([Tre98]), which belong to the classification of Fishburn (see Figure 2), are also defined in the forbidden mode by the help of $m+n$ posets.

Definition 10 (Semitransitive orders and subsemiorders) *A binary relation P on a finite set A is*

- *a semitransitive order if and only if it has no* $1+3$ *;*
- *a subsemiorder if and only if it has no* $2+3$ *and no* $1+4$*.*

The definition of semitransitive orders is motivated by the separation of the two forbidden posets of semiorders ([Chi71]): interval orders have no 2+2 and semitransitive orders have no 1+3. The motivation of subsemiorders is different. The characterizations of classical orders such as linear orders, weak orders and semiorders make use of $m+n$ posets such that $m+n=2$, $m+n=3$ and $m+n=4$, respectively. Hence, in the continuity of this approach, Trenk defined subsemiorders as orders having no $m+n$ such that $m+n=5$ ([Tre98]).

Remark that all the structures characterized by the absence of $m+n \leq 4$ have an interval representation. But it seems that semitransitive orders and subsemiorders do not have a simple interval representation. There is a conjecture given by Fishburn ([Fis97]) in order to explain this phenomenon: *"Both semitransitive orders and subsemiorders contain bipartite partially ordered sets and there is no known nice interval representation for the $H=2$ class"* (the *height* of a poset R, $H(R)$, is the number of elements in a maximum chain; and a chain is *maximum* if it has the largest cardinality over all the chains of a poset). Their characterization in the numerical mode is difficult because of the previous conjecture and their characterization in the relational mode has not been studied in the literature. We will present at the end of this section their characterization in the relational mode thanks to some new results we obtained on the relation between the forbidden and the relational mode.

Although the relational and the forbidden modes are used in order to characterize preference structures, there is very little literature that defines the relationship between them. The rest of this section is dedicated to a study of such a relationship. Moreover, such a study allows us to characterize semitransitive orders and subsemiorders in the relational mode.

For this purpose, we need to introduce a property that we call (m,n)-*Ferrers relation* which is a simple generalization of the classical Ferrers relation defined by Riguet ([Rig50]) (a binary relation S on a set A is a Ferrers relation if and only if $\forall x,y,z,w \in A,\ (xSy \wedge zSw) \Longrightarrow (xSw \vee zSy)$).

Definition 11 ((m,n)-Ferrers relation) *A binary relation S on set A is (m,n)-Ferrers if $\forall x_1, x_{m+1}, y_1, y_{n+1} \in A$,*

$$S^m(x_1,x_{m+1}) \wedge S^n(y_1,y_{n+1}) \Longrightarrow S(x_1,y_{n+1}) \vee S(y_1,x_{m+1}).$$

where $S^m(x_1,x_{m+1})$ and $S^n(y_1,y_{n+1})$ represent the compositions of respectively m and n relations:

$$S^k(x_1,x_{k+1}) : \exists x_i, i \in \{2,\ldots,m\},\ (S(x_1,x_2) \wedge S(x_2,x_3) \wedge \cdots \wedge S(x_k,x_{k+1})).$$

Notice that the $(1,1)$-Ferrers relation is the classical Ferrers relation.

It is easy to show that there is an equivalence between the fact that P is (m,n)-Ferrers and it has no $(m+1)+(n+1)$:

Lemma 1 *Let P be an asymmetric and transitive relation. P is (m,n)-Ferrers if and only if it has no $(m+1)+(n+1)$.*

Proof.
-($\Longrightarrow$)
Let P be a (m,n)-Ferrers relation. Assume that we can find $m+n+2$ points $(x_1,x_2,\ldots,x_{m+1},y_1,y_2,\ldots,y_{n+1})$, such that the order induced by P of these points is a $(m+1)+(n+1)$.

By the definition of $(m+1)+(n+1)$ we have $P^m(x_1,x_{m+1})$ and $P^n(y_1,y_{n+1})$. Since P is (m,n)-Ferrers this implies $(P(x_1,y_{n+1})$ or $P(y_1,x_{m+1}))$. If we have

$P(x_1, y_{n+1})$ or $P(y_1, x_{m+1})$, it means that there exist at least two points belonging to two different chains and satisfying the relation P (contradiction with $(m+1)+(n+1)$).

- ($\Longleftarrow$)
Let $x_1, x_{m+1}, y_1, y_{n+1}$ be four elements of A such that $P^m(x_1, x_{m+1})$ and $P^n(y_1, y_{n+1})$ hold. Consequently, there exist a chain with the points $x_1, \ldots, x_{m+1}$ and another chain with the points $y_1, \ldots, y_{n+1}$. Since $(m+1)+(n+1)$ is forbidden, we conclude that there exist two points x_i and y_j belonging to two different chains of P such that $x_i P y_j$ or $y_j P x_i$. We analyze now these two cases:

-if $x_i P y_j$ is true: thanks to the transitivity of P, we get $x_1 P y_{n+1}$ $(P(x_1, x_i) \wedge P(x_i, y_j) \wedge P(y_j, y_{n+1}) \Longrightarrow P(x_1, y_{n+1}))$,

-if $y_j P x_i$ is true: thanks to the transitivity of P we get $y_1 P x_{m+1}$ $(P(y_1, y_j) \wedge P(y_j, x_i) \wedge P(x_i, y_{m+1}) \Longrightarrow P(y_1, x_{m+1}))$.

As a result, $P(x_1, y_{n+1}) \vee P(y_1, x_{m+1})$ is true. Hence, we can conclude that $P^m(x_1, x_{m+1}) \wedge P^n(y_1, y_{n+1}) \Longrightarrow P(x_1, y_{n+1}) \vee P(y_1, x_{m+1})$ is also true.

As a consequence, P is (m, n)-Ferrers. ■

There is also an equivalence between the property of being a (m, n)-Ferrers relation and a specific relative product (composition) of the relation P.

Lemma 2 *Let P be an asymmetric and transitive relation. P is (m, n)-Ferrers if and only if $P^m.P^d.P^n \subset P$.*

The notation P^d corresponds to the dual of the binary relation P:

$$xP^d y \text{ iff } \neg(yPx)$$

We use the notation $P_1.P_2$ in order to represent a composition between these two relations. For instance,

$P^m.P^d.P^n \subset P \iff (\forall x, y, z, t,\ P^m(x, y) \wedge P^d(y, z) \wedge P^n(z, t) \Longrightarrow P(x, t))$.

Proof.

Let P be a binary relation. P satisfies $P^m.P^d.P^n \subset P$ if and only if

$$\forall x, y, z, t\ (P^m(x, y) \wedge P^d(y, z) \wedge P^n(z, t)) \Longrightarrow (P(x, t)). \tag{7}$$

Replacing $P^d(y, z)$ by $\neg(P(z, y))$ the assumption 7 becomes

$$\forall x, y, z, t,\ not(P^m(x, y) \wedge P^n(z, t)) \vee P(z, y) \vee P(x, t)$$

which is equivalent to

$$\forall x, y, z, t\ (P^m(x, y) \wedge P^n(z, t)) \Longrightarrow (P(z, y) \vee (P(x, t)).$$

As a consequence, P is (m, n)-Ferrers if and only if it satisfies $P^m.P^d.P^n \subset P$.■

Another relation that we can find is about a composition of P with its symmetric part I.

Lemma 3 *Let P be an asymmetric and transitive relation and I its symmetric complement, then*

$$P^m.P^d.P^n \subset P \Longleftrightarrow P^m.I.P^n \subset P$$

Proof.
Let us remark first of all that we have $P^d = P \cup I$ (because $P \cup I \cup P^{-1} = A \times A$). This result gives us the following equivalence where we decompose P^d in P and I:

$$P^m.P^d.P^n \subset P \Longleftrightarrow \begin{cases} P^m.P.P^n \subset P \\ P^m.I.P^n \subset P \end{cases} \tag{8}$$

- $P^m.P^d.P^n \subset P \Longrightarrow P^m.I.P^n \subset P$: obvious

-$P^m.I.P^n \subset P \Longrightarrow P^m.P^d.P^n \subset P$: by the transitivity of P we have $P^m.P.P^n \subset P$, so if $P^m.I.P^n \subset P$, we get $P^m.P^d.P^n \subset P$ because of the equivalence 8. ∎

These equivalences show different interpretations of a (m, n)-Ferrers relation and help us better understand the relation between relational and forbidden modes. We present in the following a proposition summarizing all the equivalences that we proved.

Proposition 4 *Let P be an asymmetric and transitive relation and I its symmetric complement. The following assertions are equivalent:*

i . *P has no $(m + 1) + (n + 1)$;*

ii . *P is (m, n)-Ferrers;*

iii . *$P^m.P^d.P^n \subset P$;*

iv . *$P^m.I.P^n \subset P$.*

The characterizations of semitransitive orders and subsemiorders in the relational mode may be obtained directly from Proposition 4.

Proposition 5 *Let P be an asymmetric and transitive relation and I its symmetric complement. The following assertions are equivalent:*

i . *P is a semitransitive order;*

ii . *P is (0+2)-Ferrers;*

iii . *$P^2.P^d \subset P$ (or equivalently $P^d.P^2 \subset P$);*

iv . *$P^2.I \subset P$ (or equivalently $I.P^2 \subset P$).*

This result shows that semitransitive orders are equivalent to what Monjardet called *reflexive S-relation* [Mon78]. A relation R on set A is a S-relation if and only if $\forall x, y, z, t \in A,\ xRy \wedge yRz \Longrightarrow xRt \vee tRz$. Monjardet showed that $R = P \cup I$ is reflexive and a S-relation if and only if R is complete and the affirmation (iii) of the Proposition 5 is satisfied.

We conclude this part with the relational representation of a subsemiorder.

Proposition 6 *Let P be an asymmetric and transitive relation and I its symmetric complement. The following assertions are equivalent:*

i . *P is a subsemiorder;*

ii . *P is (1+2)-Ferrers and (0+3)-Ferrers relation;*

iii . $\begin{cases} P.P^d.P^2 \subset P \ (\textit{or equivalently } P^2.P^d.P \subset P), \\ P^3.P^d \subset P \ (\textit{or equivalently } P^d.P^3 \subset P); \end{cases}$

iv . $\begin{cases} P.I.P^2 \subset P \ (\textit{or equivalently } P^2.I.P \subset P), \\ P^3.I \subset P \ (\textit{or equivalently } I.P^3 \subset P). \end{cases}$

6 Conclusion

In this paper, we were interested in structures having an interval representation. We based our study on the classification given by Fishburn ([Fis97]) who analyzed different structures having a non-transitive indifference. In order to enrich this classification and let the analysis of such structures systematic, we gave a general framework. Based on an axiomatization, this framework proposes a unified language for structures having an interval representation. It provides a very simple characterization of some classical properties. We also showed that the study of 3-point intervals is exhaustive within our framework and that it defines three new structures that we presented in detail.

In the last part of our paper we were interested in the relation between the forbidden and the relational mode. We introduced the property of (m, n)-Ferrers relation and we showed that, thanks to the relations between being a (m, n)-Ferrers relation and having no $(m + 1) + (n + 1)$, we can obtain the characterization of semitransitive orders and subsemiorders in the relational mode.

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