Preference representation with 3-points intervals

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Abstract. In this article we are interested in the representation of qualitative preferences with the help of 3-points intervals (a vector of three increasingly ordered points). Preferences are crucial when an agent has to autonomously make a choice over several possible actions. We provide first of all an axiomatization in order to characterize our representation and then we construct a general framework for the comparison of 3-points intervals. Our study shows that from the fifteen possible different ways to compare 3-points intervals, seven different preference structures can be defined, allowing the representation of sophisticated preferences. We show the usefulness of our results in two classical problematics: the comparison of alternatives and the numerical representation of preference structures. Concerning the former one, we propose procedures to construct non classical preference relations (intransitive preferences for example) over objects being described by three ordered points. Concerning the latter one, assuming that preferences on the pairwise comparisons of objects are known, we show how to associate a 3-points interval to every object, and how to define some comparison rules on these intervals in order to have a compact representation of preferences described with these pairwise comparisons.

1 Introduction

The notion of preference, initially introduced by economists ([3, 6]) and researchers on Decision Making (DM) ([13, 11, 18, 17, 15]), has recently received an increasing attention in AI where artificial agents play the role of automated decision makers ([19, 7]).

In DM, preferences are used for two different problematics ([21]): the *comparison problem* and the *numerical representation problem*. These two problems arise naturally in AI since comparing objects and establishing preference (or any other order relations) is a key issue in knowledge representation and elicitation.

The comparison problem deals with the construction of preference relations over each pair of alternatives. In such a case evaluations of alternatives are known and may have different nature: numbers, colors, symbols, figures, intervals, fuzzy numbers, etc. The construction of relations may not be an easy task even with quantitative evaluations. For instance, consider a maximization problem with three alternatives (a, b and c) evaluated by numbers ($g(a) = 25$, $g(b) = 11$ and $g(c) = 9$). Depending on the context and/or the decision maker, we may have different relations. One solution may be to say that there are only strict preferences (a is strictly preferred to b and c (aPb and aPc) and b is strictly preferred to c (bPc)) while in a different context the alternatives b and c may be considered as indifferent (bIc) since the difference between their evaluation is not significant. It is clear that the relations obtained in the two different contexts do not have the same properties and they do not lead to the same model.

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The numerical representation problem goes in the opposite way. The preference on each pair of alternatives being known, the problem is to check if there exists (and under which conditions) one or more real valued functions which, when applied to the set of alternatives, will return the preferences of the decision maker. As an example, consider three alternatives a, b and c for which the decision maker claims that he is indifferent between c and b and he strictly prefers c to a and b to a. There are several different numerical representations which could account for such preferences. One option may be to associate intervals $[0, 1]$ to a, $[2, 4]$ to b and $[3, 6]$ to c under the rules "x is preferred to y iff the interval of x is completely to the right of the interval of y (no intersection) ".

We consider both types of problems with a special attention to an interval representation. Comparing intervals is a problem relevant to several disciplines. We need intervals in order to take into account intransitivity of indifference due to the presence of one or more thresholds, to compare time intervals ([2]), or to represent imprecision or uncertainty (the price of x lies between A and B, the quality of y lies between "medium" and "good" ...). In this article, we make use of a special type of intervals that we call "3-*points intervals*" (intervals with an intermediate point). Such intervals contain only ordinal information (the distances between points are not important) which allows us to represent qualitative evaluations. Qualitative approaches become more and more attractive in AI since the only existing knowledge may be qualitative or it may be easier to get qualitative information from experts or qualitative rules may be easier and faster (see [4] and [5]).

The main contribution of this article is to propose a general framework for the comparison of 3-points intervals. The general advantage of these intervals is their capacity of representation, especially for sophisticated preferences. Our results are useful for both of the problematics. Concerning the comparison problem, our work shows how to compare two intervals having only ordinal information in order to fit some desired properties such as transitivity of preference, intransitivity of indifference etc. Concerning the numerical representation problem, there are two main advantages. First of all the use of 3-points intervals allows to represent complex preferences. For instance, the use of simple numbers remains inefficient in the majority of cases (only total orders and weak orders have a representation with numbers), such a reason has led to the use of intervals for different preference structures ([12, 8, 20, 16]). There are many results concerning the classical intervals (2-points intervals), however such intervals may appear insufficient face to more complex preferences (for example when the preference is intransitive). For that reason we are interested in 3-points intervals for which there is a limited number of research ([10]). Another advantage is related to the cardinality of the set of alternatives. When there are too many alternatives (let n be the number of alternatives), it can be preferable to stock only the 3-points interval representation of each alternative ($3 * n$ information) instead of stocking all the pairwise comparisons of alternatives ($\frac{n(n-1)}{2}$ information). From this point of view we can say that 3-points interval representation proposes a compact representation for complex preferences.

We organize the paper in the following way: in section 2 we introduce basic notations, we propose an axiomatization for the characterization of the 3-points interval representation. A general type of representation satisfying such axioms are also presented in this section. In section 3 we propose an exhaustive analysis of all the preference structures having a 3-points interval representation and in section 4 we conclude with some future research directions.

2 Basic notions and 3-points interval representation

In this paper we study complete preference structures with two binary relations: the strict preference relation P which is an asymmetric relation and the indifference relation I which is the symmetric complement of P. We introduce first of all some notions that we will use in the axiomatization.

We call a "3-*points interval*" an interval $x = [f_1(x), f_3(x)]$ with an intermediate point $f_2(x)$ (*i.e.* $f_1(x) < f_2(x) < f_3(x)$).

Then, we introduce a new notion that we call the "*relative position*" and that we denote by φ. The notation $\varphi(x, y)$ represents the position of the interval x with respect to the interval y ($\varphi(x, y) \neq \varphi(y, x)$).

Definition 1 (Relative position) *The relative position $\varphi(x, y)$ is the 3-tuple $(\varphi_1(x, y), \varphi_2(x, y), \varphi_3(x, y))$ where $\varphi_i(x, y)$ represents the number of j such that $f_i(x) < f_j(y)$.*

Intuitively, φ represents to what extend the position of two intervals is close to the case of two disjoint intervals, case which guarantees a strict preference. The following example illustrates the previous definition.

Example 1 *Let x and y be two 3-points intervals represented in figure 1, then $\varphi(x, y) = (1, 0, 0)$. $\varphi_1(x, y) = 1$ since there is only $f_3(y)$ being greater than $f_1(x)$ and $\varphi_2(x, y) = \varphi_3(x, y) = 0$ since $f_2(x)$ and $f_3(x)$ are greater than all the points of y.*

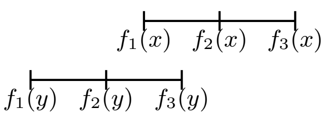

Figure 1. Relative position $\varphi(x, y) = (1, 0, 0)$

Let us remark that there are 20 ($\frac{(2*3)!}{(3!)^2}$) different relative positions when two 3-points intervals are compared.

The strict preference between two intervals depends on their relative positions and naturally there are some relative positions which are more suitable for the representation of a strict preference than others. For example the case where two intervals are disjoint is more suitable for a strict preference than a case where one interval is included to another. For such a purpose we introduce a new binary relation, called "*stronger than*", on the set of relative positions.

Definition 2 ("Stronger than" relation) *Let φ and φ' be two relative positions, then we say that φ is "stronger than" φ' and note $\varphi \rhd \varphi'$ if $\forall i \in \{1, \dots, n\},\ \varphi_i \leq \varphi'_i$.*

We present an example showing how we define a "stronger than" relation.

Example 2 *Let $\varphi(x, y)$ and $\varphi(x, t)$ be two relative positions of the figure 2. We have $\varphi(x, y) = (1, 1, 0)$, $\varphi(x, t) = (2, 1, 0)$. We get "$\varphi(x, y)$ is stronger than $\varphi(x, t)$" since $1 \leq 2,\ 1 \leq 1$ and $0 \leq 0$.*

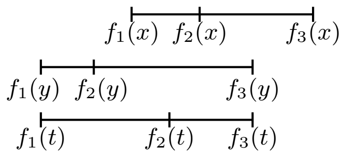

Figure 2. Example: $(1, 1, 0) \rhd (2, 1, 0)$

The "stronger than" relation satisfies some classical properties:

Proposition 1 *$\rhd$ is a partial order (reflexive, antisymmetric and transitive) defining a lattice on the set of possible relative positions.*

Proof. $\rhd$ is a partial order since it is induced from the relation "$<$" which is reflexive, antisymmetric and transitive. ■

Let us remark that the relation $\rhd$ is not complete: for example we have $(2, 0, 0) \not\rhd (1, 1, 0)$ and $(1, 1, 0) \not\rhd (2, 0, 0)$. We present in figure 3 the graph of the relation $\rhd$.

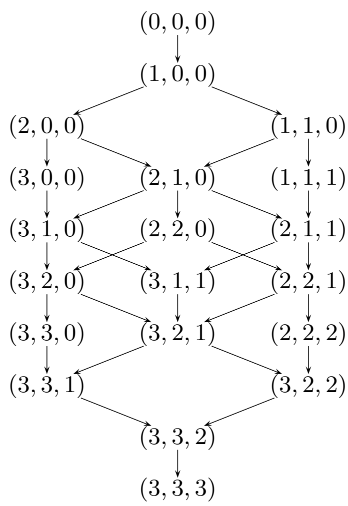

Figure 3. Graph of the stronger than relation

We are ready now to define the strict preference relation P and the indifference relation I. We will define P as a set of relative positions, satisfying some constraints, and construct I as the complement of P. For such a purpose we propose an axiomatization:

Axiome 1 *The relation $P \cup I$ is complete and I is the complement of P (i.e. $I(x, y) \Leftrightarrow \neg P(x, y) \wedge \neg P(y, x)$).*

Axiome 2 *The relations $P(x, y)$ and $I(x, y)$ depends only on the relative position of x and y.*

Axiome 3 *If a relative position φ is in the set of the strict preference P then all the relative positions which are stronger than φ are also in the set of P.*

Axiome 4 *If for all i, $f_i(x) < f_i(y)$ then $P(x, y)$ is not satisfied.*

Axiome 5 *The set of relative positions forming P has one and only one weakest relative position (relative position which is weak than every relative position of the set).*

Axiom 1 shows that P and I are exhaustive and exclusive, axiom 2 presents the comparison parameters and axiom 3 guaranties the monotonicity. Every relative position is not a good candidate to represent a strict preference. Axiom 4 eliminates some undesired situations in the definition of P. The role of the strongest relative position of a set of P is very important since we can determine all the other elements of the set by the help of the strongest one. Axiom 5 guarantees a unique representation for the strict preference relations by forbidding the existence of more than one strongest relative positions in their set.

It is easy to calculate the number of sets satisfying such axioms. Since every set has just one strongest relative position, every relative position may present one set, of course except the ones which do not satisfy the axiom 4. The number of relative positions with "for all i, $f_i(x) < f_i(y)$" is five ($\frac{1}{3+1}\binom{6}{3}$). Since there are, in total, twenty relative positions, the number of sets satisfying axioms 1-5 is fifteen.

We can present now the 3-points interval representation of a preference structure satisfying axioms 1-5. First of all, let's give a formal definition of the preference structure induced by the different possible relative positions of 3-points intervals.

Definition 3 *Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a 3-tuple in $\{0, 1, 2, 3\}$, and x and y two 3-points intervals. The preference relations $P_{\leq\varphi}, I_{\leq\varphi}$ associated to φ is defined as*

$$\begin{array}{lcl} P_{\leq\varphi}(x,y) & \iff & \varphi(x,y) \rhd \varphi \\ I_{\leq\varphi}(x,y) & \iff & \neg P_{\leq\varphi}(x,y) \wedge \neg P_{\leq\varphi}(y,x) \end{array}$$

Now, consider the preference relation $P_{\leq(2,0,0)}$. Then $P_{\leq(2,0,0)}(x,y)$ iff $f_1(y) < f_1(x)$, $f_3(y) < f_2(x)$ and $f_3(y) < f_3(x)$. We can remark that the third inequality is redundant. This motivates the definition of the component set of a triple φ.

Definition 4 *Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a 3-tuple in $\{0, 1, 2, 3\}$. The component set $Cp_{\leq\varphi}$ associated to φ is the set of couples $(3 - \varphi_i, i)$ such that there is no $i' < i$ with $\varphi_{i'} \leq \varphi_i$.*

For instance, $Cp_{\leq(2,0,0)} = \{(1, 1), (3, 2)\}$. Hence, $Cp_{\leq\varphi}$ represents the set of couples of points that are sufficient to be compared. Conditions on the elements of $Cp_{\leq\varphi}$ guarantees the minimality of the representation. The set $Cp_{\leq\varphi}$ contains all the information concerning the preference structure.

It is easy to verify that the preference structure associated to a triple φ verifies axioms 1,2,3 and 5. Following definition 4, one can show that axiom 4 is verified by $P_{\leq\varphi}$ iff $Cp_{\leq\varphi}$ contains at least one (i, j) with $i \geq j$.

3 3-points interval comparisons

In this section we analyze in details the fifteen sets of P satisfying our axiomatization. Let us remind that each set represents a strict preference relation which has a 3-points interval representation and the component set $Cp_{\leq\varphi}$ has the whole information about this representation.

Our study shows that from the fifteen sets of P, seven different preference structures can be defined, some of them having more than one 3-points interval representation. For the sake of clarity, we will first of all present the classical definition of these seven structures, then give their equivalent characterization with 3-points intervals by the help of component sets and finally present in table 1 all the 3-points interval representations of these preference structures.

An exhaustive study of the 3-points intervals shows that weak orders and bi-weak orders have three different 3-points interval representations while three-weak orders have one, interval orders have three, split interval orders have one, triangle orders have two and intransitive orders have two. We present first of all the definition of each preference structure that we cited:

Let P be a binary relation on a finite set A and I be the symmetric complement of P, then

- $P \cup I$ is a *weak order* if and only if there exists a real-valued function f defined on A such that $\forall x, y \in A,\ xPy \iff f(x) > f(y)$
- $P \cup I$ is a *bi-weak order* if and only if there exist two different real-valued functions f_1 and f_2 defined on A such that $\forall x, y \in A,\ xPy \iff \begin{cases} f_1(x) > f_1(y) \\ f_2(x) > f_2(y) \end{cases}$
  It is easy to see that bi-weak orders are defined as the intersection of two weak orders.
- $P \cup I$ is a *3-weak order* if and only if it is defined as the intersection of three weak orders.
- $P \cup I$ is an *interval order* if and only if there exist two real-valued functions f_1 and f_2, defined on A such that $\begin{cases} \forall x, y \in A,\ xPy \iff f_1(x) > f_2(y) \\ \forall x \in A,\ f_2(x) > f_1(x) \end{cases}$
- $P \cup I$ is a *split interval order* if and only if there exist three real-valued functions f_1, f_2 and f_3 defined on A such that $\begin{cases} \forall x, y \in A, xPy \iff \begin{cases} f_1(x) > f_2(y), \\ f_2(x) > f_3(y), \end{cases} \\ \forall x \in A,\ f_3(x) > f_2(x) > f_1(x) \end{cases}$
- $P \cup I$ is a *triangle order* if and only if it is defined as the intersection of one weak order and one interval order.
- $P \cup I$ is an *intransitive order* if and only if P is intransitive.

From these seven structures weak orders are the most used ones. Their difference from linear orders (total orders) comes from the fact that weak orders may have equivalence classes (two different objects may be considered as indifferent) which is forbidden in the case of linear orders. Bi-weak orders are also known structures, especially for the researchers of DM. They are equivalent to bilinear orders (interested reader may find more details in [9]). Three-weak orders were born from the generalization of bi-weak orders (for more details see [14]). Interval orders have been introduced by Fishburn ([8]). The relaxation of the coherence condition of semiorders (semiorders have an interval representation where each interval has the same length) has led to interval orders which are especially used in the presence of discrimination thresholds in order to represent intransitive indifference. Split interval orders are especially studied by mathematicians ([10]) and allow the representation of very sophisticated preferences. The name of triangle orders comes from their classical representation: an object is preferred to another one if and only if the triangle representing the first object is completely to the right of the triangle representing the second one (no intersection)(more details can be found in [14]). Intransitive orders are marginal orders, however they are used in some special domains (such as the biology in the case of cellule comparison or the chemistry in the case of molecular connection [1]). Circles are used in order to represent such structures: an object is preferred to another one if and only if the circle representing the first object is completely to the right of the circle representing the second one (circles may have different diameters). Unfortunately, we can not give here more details about these seven structures, interested reader may find more information in the cited references.

Let us remark that the classical representation of the majority of these structures do not make use of intervals (intervals can be seen as vectors of some ordered points). For instance weak orders use simple numbers while bi-weak orders (resp. three-weak orders) utilize two, not necessarily ordered numbers (resp. three points) (for instance we can have $f_1 < f_2$ or $f_2 < f_1$). Triangle orders are represented by triangles and intransitive orders by circles. Our study shows that all these seven structures have a 3-points interval representation. We will present now the general form of these representations by the help of component sets. We begin by some propositions concerning the transitivity properties since they are fundamental for some preference structures :

- $P_{\leq\varphi}$ *is transitive* if and only if $\forall(i,j) \in Cp_{\leq\varphi},\ i \geq j$,
- $I_{\leq\varphi}$ *is transitive* if and only if $\exists i,\ Cp_{\leq\varphi} = \{(i,i)\}$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *weak order* if and only if $\exists i,\ Cp_{\leq\varphi} = \{(i,i)\}$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *bi-weak order* if and only if $|Cp_{\leq\varphi}| = 2$ and $\forall(i,j) \in Cp_{\leq\varphi},\ i = j$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *3-weak order* if and only if $|Cp_{\leq\varphi}| = 3$ and $\forall(i,j) \in Cp_{\leq\varphi},\ i = j$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an *interval order* if and only if $Cp = \{(i,j)\}$ where $i \geq j$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *triangle order* if and only if $Cp_{\leq\varphi} = \{(l,l),(i,j)\},$ where $i \geq j$,
- $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *intransitive order* if and only if $\exists(i,j) \in Cp_{\leq\varphi},\ i < j$.

We present now the key-steps of the proofs of these general propositions.

- *The transitivity of* $P_{\leq\varphi}$:
  - If $\forall(i,j) \in Cp_{\leq\varphi},\ i \geq j$ then $P_{\leq\varphi}$ is transitive: obvious.
  - If $P_{\leq\varphi}$ is transitive then $\forall(i,j) \in Cp_{\leq\varphi},\ i \geq j$: we prove this result by showing that if $\exists(i,j) \in Cp_{\leq\varphi}\ i < j \implies \exists x,y,z,\ P_{\leq\varphi}(x,y) \wedge P_{\leq\varphi}(y,z)$ and$\neg P_{\leq\varphi}(x,z)$.
- *The transitivity of* $I_{\leq\varphi}$:
  - $Cp_{\leq\varphi} = \{(i,i)\}$ implies $I_{\leq\varphi}$ is transitive: obvious.
  - $I_{\leq\varphi}$ is transitive implies $Cp_{\leq\varphi} = \{(i,i)\}$: we prove this result by contradiction. Supposing that $I_{\leq\varphi}$ is transitive we analyze two different cases: $\exists(i,j) \in Cp_{\leq\varphi}, i \neq j$, and $\forall(i,j) \in Cp_{\leq\varphi}, i = j$ and $|Cp_{\leq\varphi}| > 1$. We show that these two cases are contradictory with the transitivity of $I_{\leq\varphi}$.
- *Weak order*:
  - If $Cp_{\leq\varphi} = \{(i,i)\}$ then $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a weak order: we prove that $I_{\leq\varphi}$ and $P_{\leq\varphi}$ are transitive and $P_{\leq\varphi} \cup I_{\leq\varphi}$ is reflexive and complete.
  - If $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a weak order then $Cp_{\leq\varphi} = \{(i,i)\}$: the key idea is the transitivity of $I_{\leq\varphi}$. If $P_{\leq\varphi} \cup I_{\leq\varphi}$ is a weak order then $I_{\leq\varphi}$ is transitive and if $I_{\leq\varphi}$ is transitive then $Cp_{\leq\varphi} = \{(i,i)\}$.
- *Bi-weak order*: the proof follows directly from the one of weak orders.
- *Three-weak order*: the proof follows directly from the one of weak orders.
- *Interval order*: for this proof we make use of the relational characterization of an interval order: $P \cup I$ is an interval order if $\begin{cases} P.I.P \subset P, \\ \text{P} \cup \text{I is reflexive and complete.} \end{cases}$
  where $P.I.P \subset P$ means $\forall x,y,z,t$, if $P(x,y) \wedge I(y,z) \wedge P(z,t)$ then $P(x,t)$.
  We prove first of all that $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi}$ iff $Cp = \{(i,j)\}$ where $i \geq j$:
  - If $Cp = \{(i,j)\}$ where $i \geq j$ then $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi}$: obvious.
  - If $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi}$ then $Cp = \{(i,j)\}$ where $i \geq j$: first of all if $Cp = \{(i,j)\}$ with $i < j$ then $P_{\leq\varphi}$ is not transitive. In this case it is easy to see that when $I_{\leq\varphi}$ is the identity $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi}$ is not satisfied. We prove then that if $|Cp_{\leq\varphi}| > 1$ then *not* $(P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi})$. We analyze two cases where $|Cp_{\leq\varphi}| > 1$: $\exists(i,j) \in Cp_{\leq\varphi},\ i < j$ and $\forall(i,j) \in Cp_{\leq\varphi},\ i \geq j$. The first one provides an intransitive $P_{\leq\varphi}$. The key point of the analysis of the second case is the definition of $I_{\leq\varphi}$ when $|Cp_{\leq\varphi}| > 1$: let $(i,j),(l,m)$ be elements of $Cp_{\leq\varphi}$ then $f_i(x) \geq f_j(y) \wedge f_l(y) \geq f_m(x)$ with $(i,j) \neq (l,m) \implies I_{\leq\varphi}(x,y)$. It is easy to see that this implication has one part where a point of x is greater than a point of y and another part which inverses such inequality. In this case one can always find four elements w,x,y,z such that $P_{\leq\varphi}(w,x), I_{\leq\varphi}(x,y), P_{\leq\varphi}(y,z)$ and $\neg P_{\leq\varphi}(w,z)$.

  We can now proof the characterization of interval orders:
  - If $Cp = \{(i,j)\}$ where $i \geq j$ then $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an interval order: we prove that $P_{\leq\varphi} \cup I_{\leq\varphi}$ is reflexive and complete and $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi}$.
  - If $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an interval order then $Cp = \{(i,j)\}$ where $i \geq j$: if $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an interval order then $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subset P_{\leq\varphi}$which implies $|Cp_{\leq\varphi}| = 1$.
- *Triangle order*: the proof follows directly from the ones of weak orders and of interval orders.
- *Intransitive order*: the proof follows directly from the transitivity of $P_{\leq\varphi}$.

These propositions give us general representations of structures in the sense that these are also true for intervals having more than 3 points. We can conclude now this section by presenting all the 3-points interval representations for the seven preference structures in table 1.

Preference Structure	$\langle P_{\leq\varphi}, I_{\leq\varphi}\rangle$ interval representation
Weak Orders	$Cp_{\leq(3,3,0)} = \{(3,3)\}$ $Cp_{\leq(3,1,1)} = \{(2,2)\}$ $Cp_{\leq(2,2,2)} = \{(1,1)\}$
Bi-weak Orders	$Cp_{\leq(3,1,0)} = \{(2,2),(3,3)\}$ $Cp_{\leq(2,1,1)} = \{(1,1),(2,2)\}$ $Cp_{\leq(2,2,0)} = \{(1,1),(3,3)\}$
Three-Weak Orders	$Cp_{\leq(2,1,0)} = \{(1,1),(2,2),(3,3)\}$
Interval Orders	$Cp_{\leq(0,0,0)} = \{(3,1)\}$ $Cp_{\leq(3,0,0)} = \{(3,2)\}$ $Cp_{\leq(1,1,1)} = \{(2,1)\}$
Split Interval Orders	$Cp_{\leq(1,0,0)} = \{(3,2),(2,1)\}$
Triangle Orders	$Cp_{\leq(1,1,0)} = \{(2,1),(3,3)\}$ $Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$
Intransitive Orders	$Cp_{\leq(3,2,0)} = \{(3,3),(1,2)\}$ $Cp_{\leq(2,2,1)} = \{(1,1),(2,3)\}$

Table 1. Preference structures with 3-points interval representation

4 Conclusion

Our study provides an exhaustive view of the comparison of 3-points intervals. Concerning the comparison problem, we have defined and analyzed all the possible 3-points interval comparison procedures that satisfy our axiomatization. Our analysis allows us to know which procedure provides which preference structure and to detect the properties of the resulted preference relations. Concerning the numerical representation problem, we know now all the preference structures having a 3-points interval representation. When ordered points are used, the use of exactly three points is optimal for three-weak orders, triangle orders, split interval orders and intransitive orders, however weak orders, bi-weak orders and interval orders need less than three ordered points. The classical representation of triangle orders and intransitive orders make use of geometric figures (dimension two). As a result they have more complicated comparison rules (for example the comparison of circles is done by a quadratic function) and the representation of preferences needs more space. By proposing a 3-points interval representation we facilitate the comparison rules and the preference representation. We show also that preference structures may have more than one representation.

A possible interesting extension would be the generalization of our study in the case n-points intervals. Such a generalization would offer a general framework for the comparison of ordinal intervals and would allow a systematic study of all the preference structures having an interval representation (for instance for n=4 there are 56 sets of P, some of them are already known thanks to the propositions that we presented in this paper).

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