# Modelling continuous positive and negative reasons in decision aiding

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**Abstract**

The use of positive and negative reasons in inference and decision aiding is a recurrent issue of investigation. A language enabling to explicitly take into account such reasons is Belnap's logic and the four valued logics derived from it. In this paper, we explore the interpretation of a continuous extension of a four-valued logic as a necessity degree (in possibility theory). It turns out that, in order to take full advantage of the four values, we have to consider "sub-normalised" necessity measures. Under such a hypothesis four-valued logics become the natural logical frame for such an approach.

## Introduction

Classic logic is not always suitable to formalise real life problem situations since it is unable to handle incomplete and/or inconsistent information. In decision aiding such situations are regular and indeed classic logic has often been criticised as a language used for decision support models formulation (Dubois & Prade 1988; 2001; Roy 1989; Tsoukiàs & Vincke 1995; Perny & Roubens 1998). On the other hand, both in decision theory and in logic a recurrent idea has been to separate positive and negative reasons supporting a decision and/or a logical inference. For some early contributions the reader can see (Raju 1954; Dubarle 1989; Rescher 1969; Belnap 1976; 1977). Under such a perspective we study the possibility to extend a four valued logic (Tsoukiàs 2002) in situations where it is possible to make continuous valuations on the presence of truth.

The four values (t, f, k, u) introduced by Belnap have a clear epistemic nature. Given a proposition $\alpha$, four situations may appear: (or truth values, the symbol $\Delta$ representing the "presence of truth"):
- true ($t$): there is evidence that $\alpha$ is true ($\Delta\alpha$) and there is no evidence that $\alpha$ is false ($\neg\Delta\neg\alpha$)
- false ($f$): there is no evidence that $\alpha$ is true ($\neg\Delta\alpha$) and there is evidence that $\alpha$ is false ($\Delta\neg\alpha$)
- contradictory ($k$): there is evidence that $\alpha$ is true ($\Delta\alpha$) and there is evidence that $\alpha$ is false ($\Delta\neg\alpha$)
- unknown ($u$): there is no evidence that $\alpha$ is true ($\neg\Delta\alpha$) and there is no evidence that $\alpha$ is false ($\neg\Delta\neg\alpha$)

However, the sources of uncertainty are not limited to pure incomplete and/or contradictory situations. The evidence "for" or "against" a certain sentence might not be necessarily of a crisp nature. In this case, we can introduce "positive reasons" and "negative reasons" supporting or not a certain sentence (Tsoukiàs, Perny, & Vincke 2002). Considering a continuous valuation of such reasons, we can introduce a continuous extension of any four-valued logic. This continuous extension may help us to deal with uncertainty due to doubts about the validity of the knowledge; imprecision due to the vagueness of the natural language terms; incompleteness due to the absence of information; apparent inconsistency due to contradictory statements. Such situations are all the more relevant in decision aiding and preference modelling.

More precisely in this paper, we consider two variants of Belnap's logic: DDT logic (Tsoukiàs 2002) which extends Belnap's logic to a first order language and its continuous extension suggested in (Perny & Tsoukiàs 1998).

The aim of this paper is to verify whether it is possible to associate to the DDT logic an uncertainty distribution, possibly of the possibility/necessity type and if so, under which conditions. Section 2 introduces the basic concepts of the four-valued logic and its continuous extension through the concept of positive and negative membership. In Section 3 we try to establish a first relation between four-valued logic and possibility theory. Some related problems are discussed. In Section 4 we suggest the use of "sub-normalised" necessity distributions and we show why four-valued logic can be considered a language for associating such type of uncertainty distributions.

## Four-valued logic and its continuous extension

### Syntax

Belnap's original proposition (Belnap 1976; 1977) aimed to capture situations where hesitation in establishing the truth of a sentence could be associated either to ignorance (poor information) or to contradiction (excess of information). He suggested the use of four truth values forming a bi-lattice (see figure 1). It has been shown that such a bi-lattice is the smallest nontrivial interlaced bi-lattice (Ginsberg 1988; Fitting 1991).

DDT logic (for details see Tsoukiàs,2002) extended Belnap's logic in a first order language endowed with a weak negation ($\not\sim$). DDT is a boolean algebra. This logic allows a distinction between the strong negation ($\neg$) and the

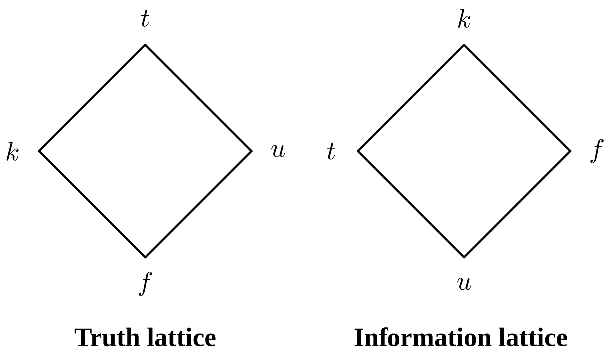


Figure 1: The two lattices suggested by Belnap

| $\alpha$ | $\sim\alpha$ | $\neg\alpha$ | $\not\sim\alpha$ |
|---|---|---|---|
| t | f | f | k |
| k | u | k | t |
| u | k | u | f |
| f | t | t | u |

Table 1: The truth tables of negations and complement

complementation ($\sim$); see table 1. It is easy to check that $\sim\alpha \equiv \neg \not\sim \neg \not\sim \alpha$.

Besides ordinary four valued sentences, in DDT it is possible to formulate two valued sentences such as:

- $\Delta\alpha$ (there is presence of truth in $\alpha$);
- $\Delta\neg\alpha$ (there is presence of truth in $\neg\alpha$);
- $\mathbf{T}\alpha$ ($\alpha$ is true);
- $\mathbf{K}\alpha$ ($\alpha$ is contradictory);
- $\mathbf{U}\alpha$ ($\alpha$ is unknown);
- $\mathbf{F}\alpha$ ($\alpha$ is false);

through the following formulas:

- $\Delta\alpha \equiv (\alpha \wedge \neg\sim\alpha) \vee (\not\sim\alpha \wedge \not\sim\neg\alpha) \equiv \mathbf{T}(\alpha) \vee \mathbf{K}(\alpha)$
- $\mathbf{T}\alpha \equiv \alpha \wedge \neg\sim\alpha \equiv \Delta\alpha \wedge \neg\Delta\neg\alpha$.

**Example 1.** Why the above is a relevant language in decision aiding problems? Let us take the example of a Parliament which is preparing to vote for a new proposal ($\alpha$) concerning an ethical issue. Members of the Parliament (MPs) can vote "for" or "against" this proposal or can "not vote". Suppose that the Parliament has the following rule for adopting laws concerning ethics: a "strong" majority has to vote "for" (more than 50%) and less than 1/3 can vote "against" (in order to defend minorities; the reader can see the Nice Treaty establishing the decision rules of the enlarged European Union for more complicated similar examples). This kind of voting can be captured by the four valued logic as in the following (the parliament having 100 Mps): Let us note $V(\alpha)$: number of MPs voting for $\alpha$, $V(\neg\alpha)$: number of MPs voting against $\alpha$

$\Delta\alpha = 1$ iff $\frac{V(\alpha)}{100)} \geq 0.51$, $\Delta\neg\alpha = 1$ iff $\frac{V(\neg\alpha)}{100} \geq 0.33$

Four different cases are presented in table 2. In the first two cases there is no hesitation since in the first one the proposition is clearly accepted , while in the second is clearly rejected. In the third case, the majority of MPs are for

| Case | $V(\alpha)$ | $V(\neg\alpha)$ | $\Delta\alpha$ | $\Delta\neg\alpha$ | Value |
|---|---|---|---|---|---|
| 1 | 75 | 20 | 1 | 0 | True |
| 2 | 48 | 40 | 0 | 1 | False |
| 3 | 60 | 40 | 1 | 1 | Contradictory |
| 4 | 41 | 25 | 0 | 0 | unknown |

Table 2: The truth table of example 1

the acceptance of the proposal but at same time the number of MPs against $\alpha$ is remarkable too; the proposition will not be accepted, but is clear that we are facing a conflict, a contradictory case. Finally, in the fourth case, the votes for and against $\alpha$ are insufficient to make a decision which is expressed here with the unknown value. From a decision aiding point of view is clear that the recommendation of an analyst towards a decision maker facing any of the above situations will be different. In the third case is necessary to work towards the opposants (perhaps negotiating in order to meet some of their claims), while in the fourth case is necessary to convince the "non voters" (perhaps strengthening the contents of the law). The reader should note that both situations 3 and 4 would lead the decision maker to an hesitation (a state represented by most of the formalisms used in order to take into account uncertainty). However, it is only the explicit separation between situations of (relative) ignorance and (relative) contradiction that allows to provide the decision maker useful operational recommendations. More examples and applications of this approach in decision aiding and preference modelling can be seen in (Tsoukiàs & Vincke 1997; Tsoukiàs, Perny, & Vincke 2002).

## Semantics

The logic introduced deals with uncertainty. A set $A$ may be defined, but the membership of an object $a$ to the set may not be certain either because the information is not sufficient or because the information is contradictory.

In order to distinguish these two principal sources of uncertainty, the knowledge about the *"membership"* of $a$ to $A$ and the *"non-membership"* of $a$ to $A$ are evaluated independently since they are not necessarily complementary. From this point of view, from a given knowledge, we have two possible entailments, one positive, about membership and one negative, about non-membership. Therefore, any predicate is defined by two sets, its positive and its negative extension in the universe of discourse. Since the negative extension does not necessarily correspond to the complement of the positive extension of the predicate we can expect that these two extensions possibly overlap (due to the independent evaluation) and that there exist parts of the universe of discourse that do not belong to either of the two extensions. The four truth values capture such situations. More formally:

Consider a first order language $\mathcal{L}$. A *similarity type* $\rho$ is a finite set of predicate constants $R$, where each $R$ has arity $n_R$. Every alphabet uniquely determines a class of *formulas*. Relative to a given similarity type $\rho$, $R(x_1, \ldots, x_m)$ is an atomic formula iff $x_1, \ldots, x_m$ are individual variables, $R \in \rho$, and $n_R = m$. In this paper, formulas are denoted by the letters $\alpha$, $\beta$, $\gamma$, $\cdots$, possibly subscripted.

A *structure* or *model* $M$ for similarity type $\rho$ consists of a non-empty domain $|M|$ and, for each predicate symbol $R \in \rho$, an ordered pair $R^M = \langle R^{M^+}, R^{M^-} \rangle$ of sets (not necessarily a partition) of $n_R$-tuples from $|M|$. In fact, an individual can be in the two sets or in neither of them. A *variable assignment* is a mapping from the set of variables to objects in the domain of the model. Capital letters from the beginning of the alphabet are used to represent variable assignments.

The truth definition for DDT is defined via two semantic relations, $\models_t$ (true entailment) and $\models_f$ (false entailment), by simultaneous recursion as in the following definition (due to the structure introduced, the case of "not true entailment" $\not\models_t$ does not coincide with the false entailment and the case of "not false entailment" $\not\models_f$ does not coincide with the true entailment). Each formula is univocally defined through its model which is however, a couple of sets, the "positive" and "negative" extensions of the formula. When possible we are going to simplify notation using the ordered couple $\langle R^+, R^- \rangle$ in order to represent the models of $R$.

**Definition 1** *Let $M$ be a model structure and $A$ a variable assignment.*
- $M \models_t R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in R^{M^+}$.
- $M \models_f R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in R^{M^-}$.
- $M \not\models_t R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in |M| \setminus R^{M^+}$.
- $M \not\models_f R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in |M| \setminus R^{M^-}$.
- $M \models_t \neg \alpha[A]$ *iff* $M \models_f \alpha[A]$.
- $M \models_f \neg \alpha[A]$ *iff* $M \models_t \alpha[A]$.
- $M \not\models_t \neg \alpha[A]$ *iff* $M \not\models_f \alpha[A]$.
- $M \not\models_f \neg \alpha[A]$ *iff* $M \not\models_t \alpha[A]$.
- $M \models_t \not\sim \alpha[A]$ *iff* $M \models_t \alpha[A]$.
- $M \models_f \not\sim \alpha[A]$ *iff* $M \not\models_f \alpha[A]$.
- $M \not\models_t \not\sim \alpha[A]$ *iff* $M \not\models_t \alpha[A]$.
- $M \not\models_f \not\sim \alpha[A]$ *iff* $M \models_f \alpha[A]$.

It is now possible to introduce an evaluation function $v(\alpha)$ mapping $\mathcal{L}$ in to the set of truth values $\{t, k, u, f\}$ as follows:
- $v(\alpha) = t$ iff $M \models_t \alpha[A]$ and $M \not\models_f \alpha[A]$
- $v(\alpha) = k$ iff $M \models_t \alpha[A]$ and $M \models_f \alpha[A]$
- $v(\alpha) = u$ iff $M \not\models_t \alpha[A]$ and $M \not\models_f \alpha[A]$
- $v(\alpha) = f$ iff $M \not\models_t \alpha[A]$ and $M \models_f \alpha[A]$

Clearly, such "truth values" have an epistemic nature. It is our knowledge and/or beliefs (in that precise moment) that allows a "positive" and/or a "negative" entailment. However, there is no use of epistemic modal operators in the logic.
From the above definitions we get for any two subsets of formula $\alpha$ and $\beta$:

**Proposition 1**
$\alpha \models_t \beta \ iff \ A^+ \subseteq B^+$
$\alpha \models_f \beta \ iff \ B^- \not\subseteq A^-$
$\alpha \not\models_t \beta \ iff \ A^+ \not\subseteq B^+$
$\alpha \not\models_f \beta \ iff \ B^- \subseteq A^-$

Finally we can introduce the concept of strong consequence:

**Definition 2 (Strong Consequence.)**
*A formula $\alpha$ is* true *in a model $M$ iff $M \models_t \alpha[A]$ and $M \not\models_f \alpha[A]$ for all variable assignments $A$ and we write $M \models\!\!\!\equiv \alpha[A]$. A formula $\alpha$ is* satisfiable *iff $\alpha$ is* true *in a model $M$ for some $M$. A set of formulas $\Gamma$ is said to has as* strong consequence *or to* strongly entail *a formula $\alpha$ (written $\Gamma \models\!\!\!\equiv \alpha$) when for all models $M$ and variable assignments $A$, if $M \models\!\!\!\equiv \beta_i[A]$, for all $\beta_i \in \Gamma$, then $M \models\!\!\!\equiv \alpha[A]$.*

Practically we get the following. Consider a universe of discourse and a predicate $S$ of arity $n$. Such an universe is partitioned into four subsets:
$S^t = S^+ \cap \sim S^-$
$S^k = S^+ \cap S^-$
$S^u = \sim S^+ \cap \sim S^-$
$S^f = \sim S^+ \cap S^-$
where $\sim S^+$ ($\sim S^-$) is the complement of $S^+$ ($S^-$) and $S^t$, $S^k$, $S^u$, $S^f$, represent the true, contradictory, unknown and false extensions of the predicate $S$ within the universe $A^n$. Hence $(\neg S)^+, (\neg S)^-, (\sim S)^+ and (\sim S)^-$ are defined as follows:
$(\neg S)^+ = S^-$
$(\neg S)^- = (S^+)$
$(\sim S)^+ = \sim (S^+)$
$(\sim S)^- = \sim (S^-)$
Obviously the following hold:
$S^t \cup S^k = S^+$
$S^t = (\neg S)^f = (\sim S)^f$
$S^f \cup S^k = S^-$
$S^k = (\neg S)^k = (\sim S)^u$
$S^t \cup S^u = \sim S^-$
$S^u = (\neg S)^u = (\sim S)^k$
$S^f \cup S^u = \sim S^+$
$S^f = (\neg S)^t = (\sim S)^t$
$S^t \cap S^k = S^t \cap S^u = S^t \cap S^f =$
$S^f \cap S^k = S^f \cap S^u = S^k \cap S^u = \emptyset$
$S^t \cup S^k \cup S^u \cup S^f = A^n$

## Continuous Extension

For the continuous extension of the previously introduced four valued logic, $S^+$ and $S^-$ can be considered as fuzzy sets and two membership functions can be introduced: $\mu_{S^+}$ and $\mu_{S^-}$. They can be considered for instance as degrees which represent to what extent we believe in $S(x)$ and non $S(x)$ respectively ($X$ representing an universe of discourse):

$$\mu_{S^+}: \quad X \to [0,1] \qquad\qquad \mu_{S^-}: \quad X \to [0,1]$$

We then have to define the fuzzy subsets $S^t$, $S^k$, $S^u$, $S^f$. Hence, we have to make explicit the intersection, the union and the complementarity to fuzzy subsets of $X$. To define these operators, we introduce a De Morgan triple $(N, T, V)$ where N is a strict negation on [0, 1], $T$ a continuous t-norm and $V$ is a continuous co-norm such that $V(x, y) = N(T(N(x), N(y)))$. If we denote $u = \mu_{S^t}(a)$, $v = \mu_{S^k}(a)$,

$x = \mu_{S^+}(a)$, $y = \mu_{S^-}(a)$, we have:

$$u = T(x, N(y)) \quad v = T(x, y) \quad x = V(u, v) \quad y = V(N(u), v)$$

As a consequence we should get:

$$\forall x, y \in [0,1], \quad x = V(T(x, N(y)), T(x, y))$$

Unfortunately, for such an equation generally there is no De Morgan triplet satisfying it. Thus, we have to investigate partial solutions. Following (Perny & Tsoukiàs 1998) we denote:

$$\mu_{S^+}(\alpha) = B(\alpha) \qquad \mu_{S^-}(\alpha) = B(\neg\alpha)$$

Thus the four truth values $t(\alpha)$, $k(\alpha)$, $u(\alpha)$, $f(\alpha)$ can be defined through $B(\alpha)$ and $B(\neg\alpha)$ as follows (using different T-norms):

$$\mu_{S^t}(\alpha) = t(\alpha) = T_1(B(\alpha), N(B(\neg\alpha))) \quad (1)$$
$$\mu_{S^k}(\alpha) = k(\alpha) = T_2(B(\alpha), (B(\neg\alpha)) \quad (2)$$
$$\mu_{S^u}(\alpha) = u(\alpha) = T_3(N(B(\alpha)), N(B(\neg\alpha))) \quad (3)$$
$$\mu_{S^f}(\alpha) = f(\alpha) = T_4(N(B(\alpha)), (B(\neg\alpha)) \quad (4)$$

In order to fulfill the definition of fuzzy partition ($t(\alpha) + k(\alpha) + u(\alpha) + f(\alpha) = 1$) we can use the following:

$$N = LN_\phi \quad T_2 = T_3 = LT_\phi \quad V = LV_\phi \quad T_1 = T_4 = \min$$

Where (LN $_\phi$, LT $_\phi$, LV $_\phi$) is *the Lukasiewicz triple* (Schweizer & Sclar 1983). We thus get

$$t(\alpha) = min(B(\alpha), 1 - B(\neg\alpha)) \quad (5)$$
$$k(\alpha) = max(B(\alpha) + B(\neg\alpha) - 1, 0) \quad (6)$$
$$u(\alpha) = max(1 - B(\alpha) - B(\neg\alpha), 0) \quad (7)$$
$$f(\alpha) = min(1 - B(\alpha), B(\neg\alpha)) \quad (8)$$

The reader can see further details in (Perny & Tsoukiàs 1998). We just mention that in order to generalise inference we associate to each formula a pair of values ($\langle\alpha, (B(\alpha), B(\neg\alpha))\rangle$) and we get for modus ponens:

$$\frac{\begin{array}{l}\langle\alpha, (B(\alpha), B(\neg\alpha))\rangle \\ \langle\alpha \to \beta, (B(\alpha \to \beta), B(\neg(\alpha \to \beta)))\rangle\end{array}}{\langle\beta, (B(\beta), B(\neg\beta))\rangle}$$

where $B(\beta) = \min(B(\alpha), B(\alpha \to \beta))$ and $B(\neg\beta) = \max(B(\neg\alpha), B(\neg(\alpha \to \beta)))$.

For another approach (without the fuzzy partition property) on the continuous extension of four-valued logics the reader can see (Fortemps & Słowiński 2002).

**Example 2.** We take again the example of the Parliament, but this time we are going to value the positive and negative reasons within the $[0, 1]$ interval. Positive reasons become strictly positive when at least 50% of the MPs vote "for" and become sure (equal to 1) when at least 80% vote "for". Negative reasons become strictly positive when at least 15% vote "against" and become sure (equal to 1) when at least 35% vote "against". The model is shown in figure 2 (for simplicity we considered the slopes of the membership functions linear).

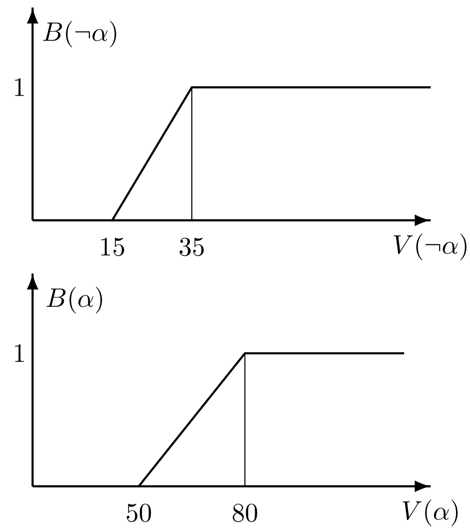


Figure 2: $B(\alpha)$ and $B(\neg\alpha)$ for example 2

| Case | $V(a)$ | $V(\neg a)$ | $B(a)$ | $B(\neg a)$ | $t(a)$ | $k(a)$ | $u(a)$ | $f(a)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 75 | 20 | 0.83 | 0.25 | 0.75 | 0.08 | 0 | 0.17 |
| 2 | 75 | 18 | 0.83 | 0.15 | 0.83 | 0 | 0.02 | 0.15 |
| 3 | 77 | 20 | 0.9 | 0.25 | 0.75 | 0.15 | 0 | 0.1 |
| 4 | 82 | 18 | 1 | 0.15 | 0.85 | 0.15 | 0 | 0 |
| 5 | 78 | 22 | 0.93 | 0.35 | 0.65 | 0.28 | 0 | 0.07 |
| 6 | 58 | 26 | 0.26 | 0.55 | 0.26 | 0 | 0.19 | 0.55 |
| 7 | 58 | 17 | 0.26 | 0.1 | 0.26 | 0 | 0.64 | 0.1 |
| 8 | 58 | 35 | 0.26 | 1 | 0 | 0.26 | 0 | 0.74 |
| 9 | 68 | 26 | 0.6 | 0.55 | 0.45 | 0.15 | 0 | 0.4 |
| 10 | 68 | 17 | 0.6 | 0.1 | 0.6 | 0 | 0.3 | 0.1 |

Table 3: The truth table for example 2

In table 3 we show the simulation of a number of votes on a set of issues. How can the decomposition in positive and negative reasons help a decision maker? First of all it is easy to observe that (with that precise decision rule) negative reasons grow faster than positive ones. Cases 1 to 3 show that convincing two non voters to vote "for" will not improve acceptability ($t(a)$), while convincing two opponents to not vote will do. Cases 4 and 5 show how acceptability and opposition will change due to opinion shifts from "for" to "against" when there are no "non voters". Cases 6 to 10 show the appearance of hesitation due to ignorance or conflict. The analysis of the positive and negative reasons helps in showing to a decision maker in what direction he should concentrate his efforts in order to pursue his policy.

## B(a) as a standard necessity

Although the four valued logic previously introduced allows to use a classical structure of "truth values" it is intuitive to consider such values as representations of uncertainty states, possibly qualitative. It is therefore natural to check any relations between this formalism and others, established since a long time, such as possibility theory (see (Dubois & Prade 1988)). Possibility measures should enable to take into account qualitative states of uncertainty and are expected to provide an ordinal representation of uncertainty as follows:

**Definition 3** ***Possibility Measure*** *Given a set of events $\Omega$, a possibility measure $\Pi$ is a function defined on the power set $2^{\Omega}$, ($\Pi : 2^{\Omega} \mapsto [0,1]$) such that:*
*1.* $\Pi(\emptyset) = 0, \Pi(\Omega) = 1$
*2.* $A \subseteq B \in 2^{\Omega} \rightarrow \Pi(A) \leq \Pi(B)$
*3.* $\forall A, B \in 2^{\Omega}$, $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$

The dual of the possibility measure, denoted necessity measure is defined as $N(a) = 1 - \Pi(\neg a)$.

**Definition 4** ***Necessity measure*** *Given a set of events $\Omega$, a necessity measure $N$ is a function defined on the power set $2^{\Omega}$, ($N : 2^{\Omega} \mapsto [0,1]$), such that:*
*1.* $N(\emptyset) = 0, N(\Omega) = 1,$
*2.* $A \subseteq B \in 2^{\Omega} \rightarrow N(A) \leq N(B)$
*3.* $\forall A, B \in 2^{\Omega}$, $N(A \cap B) = \min(N(A), N(B))$

As a result, we obtain the following properties:

$$max(\Pi(a), \Pi(\neg a)) = 1 \qquad (9)$$
$$\Pi(a) \geq N(a) \qquad (10)$$

$$If \quad N(a) \neq 0, \quad then \quad \Pi(a) = 1 \qquad (11)$$
$$If \quad \Pi(a) \neq 1, \quad then \quad N(a) = 0 \qquad (12)$$

By definition we can consider a possibility measure as the upper bound of the uncertainty associated to an event (or a sentence), the one carrying the less specific information. Dually the necessity measure will represent the lower bound: how sure we are about an event (or a sentence). Clearly three extreme situations are possible:
- $N(a) = 1, N(\neg a) = 0$, $a$ is the case;
- $N(a) = 0, N(\neg a) = 1$, $\neg a$ is the case;
- $N(a) = 0, N(\neg a) = 0$, nothing is sure and everything is possible.

A first attempt to interpret the continuous valuation of "presence of truth in $\alpha$" and "presence of truth in $\neg\alpha$" could be to consider them as necessity measures. Coming back to our notation, we consider $B(\alpha)$, as a standard necessity; as a consequence we have:
$B(\alpha) = N(\alpha) = 1 - \Pi(\neg\alpha)$
$B(\neg\alpha) = N(\neg\alpha) = 1 - \Pi(\alpha)$
Hence, we obtain the following definitions:

$$t(\alpha) = min(N(\alpha), \Pi(\alpha)) \qquad (13)$$
$$k(\alpha) = max(N(\alpha) - \Pi(\alpha), 0) \qquad (14)$$
$$u(\alpha) = max(\Pi(\alpha) - N(\alpha), 0) \qquad (15)$$
$$f(\alpha) = min(\Pi(\neg\alpha), N(\neg\alpha)) \qquad (16)$$

However, since $\Pi(\alpha) > N(\alpha)$ we can reformulate the equations 13-16:

$$t(\alpha) = N(\alpha) \qquad (17)$$
$$k(\alpha) = 0 \qquad (18)$$
$$u(\alpha) = \Pi(\alpha) - N(\alpha) \qquad (19)$$
$$f(\alpha) = N(\neg\alpha) = 1 - \Pi(\alpha) \qquad (20)$$

We first observe that interpreting $B(\alpha)$ as a standard necessity measure leads to $k(\alpha) = 0$. This is not surprising given the semantics of necessity. Let us study separately two situations, i.e $N(\alpha) = 0$ and $N(\alpha) > 0$:

When N($\alpha$)> 0: we get

$$t(\alpha) = N(\alpha) \qquad (21)$$
$$k(\alpha) = f(\alpha) = 0 \qquad (22)$$
$$u(\alpha) = \Pi(\neg\alpha) \qquad (23)$$

When N($\alpha$)= 0, we get:

$$t(\alpha) = k(\alpha) = 0 \qquad (24)$$
$$u(\alpha) = \Pi(\alpha) \qquad (25)$$
$$f(\alpha) = N(\neg\alpha) \qquad (26)$$

In other terms it appears that, while the necessity measure represents the "trueness" of a sentence (or, exclusively, of its negation), the possibility measure represents the "unknownness" of the same sentence. Although this is consistent with possibility theory it presents also some weak points:
- presence of truth and "trueness" are practically equivalent;
- there is no way to consider contradictory statements;
- there are several compositional problems (for instance $N(\alpha \vee \beta) = t(\alpha \vee \beta) = \max(t(\alpha), t(\beta)) = \max(N(\alpha), N(\beta))$, while this is not the case in possibility theory).

## B(a) as a sub-normalised necessity measure

An important feature of four-valued logics is the separation of negation from complementarity. Possibility theory does not make any difference between these two operators since it has been conceived as an uncertainty measure to be associated to classic logic. In this section, we suggest the idea of associating an uncertainty measure to a formalism such as DDT and study the consequences. In order to do that we recall the use in DDT of the "weak negation" $\not\sim$ (to be read as "perhaps"). We remind that such a weak negation is conceived so that the complement of a sentence $\sim \alpha$ can be established as $\neg \not\sim \neg \not\sim \alpha$. We further impose (consistently with the semantics of the DDT language) that an uncertainty distribution associated to a sentence of the language should fulfil the property:

$$B(\alpha) \;=\; B(\not\sim \alpha) \;=\; t(\alpha) + k(\alpha)$$

Denoting the dual measure of $B$ as $H$ ($H(\alpha) = 1 - B(\neg\alpha)$) and recalling that $B(\neg\alpha) = f(\alpha) + k(\alpha)$ as well as the principle of fuzzy partition we get that: $H(\alpha) = t(\alpha) + u(\alpha)$. Further on, due to the definitions of section 2 we have:
$t(\alpha) = f(\neg\alpha) = f(\neg \not\sim \neg \not\sim \alpha)$
$k(\alpha) = k(\neg\alpha) = u(\neg \not\sim \neg \not\sim \alpha)$
$u(\alpha) = u(\neg\alpha) = k(\neg \not\sim \neg \not\sim \alpha)$
$f(\alpha) = t(\neg\alpha) = t(\neg \not\sim \neg \not\sim \alpha)$

Therefore we have:
$H(\alpha) = t(\alpha) + u(\alpha) = f(\neg \not\sim \neg \not\sim \alpha) + k(\neg \not\sim \neg \not\sim \alpha) = t(\not\sim \neg \not\sim \alpha) + k(\not\sim \neg \not\sim \alpha) = B(\not\sim \neg \not\sim \alpha) = B(\neg \sim \alpha)$.

In other terms the dual measure of $B$ is equal to the measure of the negation of the complement. We can summarise the result in table 4.

| |
|---|
| $B(\alpha) = B(\not\approx \alpha) = H(\not\approx \neg \not\approx \alpha) = H(\not\approx \neg\alpha)$ |
| $B(\neg\alpha) = B(\not\approx \neg\alpha) = H(\neg \not\approx \neg \not\approx \alpha) = H(\not\approx \alpha)$ |
| $B(\neg \not\approx \neg \not\approx \alpha) = B(\neg \not\approx \neg\alpha) = H(\neg\alpha) = H(\neg \not\approx \alpha)$ |
| $B(\not\approx \neg \not\approx \alpha) = B(\neg \not\approx \alpha) = H(\alpha) = H(\neg \not\approx \neg\alpha)$ |

Table 4: Equivalence between $B$ and $H$

Table 4 shows that the introduction of the weak negation reduces the dual measures of the type necessity/possibility to a single one. Indeed we just need to know an uncertainty measure of a sentence and of its negation in order to know all about the uncertainty associated to this sentence. Further on, let us consider the first column of table 4. If we consider that only one uncertainty distribution is defined (say $B$) there is no reason to claim that $B(\not\approx \neg \not\approx \alpha) > B(\alpha)$ (the uncertainty associated to the negation of the complement of a sentence is not necessarily larger than the uncertainty associated to the sentence itself; they should be unrelated). However, since $B(\not\approx \neg \not\approx \alpha) = H(\alpha)$, if the above relation does not hold we are practically relaxing the normalisation principle of uncertainty measures used in possibility theory. What we see is that, while it is difficult to justify such distributions in a pure possibility theory frame, the use of the DDT logic allows to give a logical justification for their existence.

## Conclusion

In this paper we discuss two distinct tools used to deal with uncertainty: four valued logics and uncertainty distributions, both extensively used in decision aiding, the first in order to take into account positive and negative reasons in formulating a recommendation the second in order to take into account the poor or contradictory information present in the decision aiding process.

We first show how it is possible to extend a four valued logic using continuous valuations of positive and negative reasons. We then interpret such continuous valuations as standard necessity measures. On the one hand we obtain result consistent with possibility theory, but on the other hand we lose some of the expressive power of the four valued logic, mainly the possibility to distinguish contradictory statements from unknown ones. We then show that interpreting such valuations as sub-normalised necessity measures we are ably to fully exploit the expressivity of the four valued language, but at the price of losing the possibility to use two independent dual measures of uncertainty.

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