Positive and Negative Reasons in Interval Comparisons: Valued PQI Interval Orders

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Abstract

The paper presents some preliminary results concerning the comparison of intervals using a continuous evaluation of positive and negative reasons. More precisely we present a valued extension of the recently introduced concept of PQI interval order. The main idea is that, while comparing objects represented by intervals of values, there is a zone of hesitation between strict difference and strict similarity which could be modelled through valued relations. The paper presents suitable definitions of such valued relations fulfilling a number of interesting properties. The use of such a tool in data analysis and rough sets theory is discussed in the paper.

Keywords: interval orders, PQI interval orders, valued relations, valued similarity.

1 Introduction

Consider two objects whose length is between 10 and 12 for the first and between 11 and 13 for the second. Which is the shortest? Consider two objects whose price is between 100 and 130 for the first and between 90 and 120 for the second. Which do you prefer? Finally consider two objects whose quality is between fair and very good for the first and between good and very good for the second. Which is the better?

Comparing objects described under form of intervals dates back to the work of Luce, [5], where differences of utilities are perceived only when beyond a threshold (for a comprehensive discussion on the concepts of semi-order and interval order see [3], [12]). The basic idea is that when we compare objects under form of intervals they can be considered as different (preferred) iff their associated intervals have an empty intersection. Otherwise they are similar (indifferent). However, such an approach does not distinguish the specific case where one interval is “more to the right” (in the sense of the reals) of the other (both the left and right extreme of one interval is “more to the right” of the respective extremes of the other interval), but they have a non empty intersection. Such a situation can be viewed as an hesitation (denoted Q) between preference (dissimilarity, denoted P) and indifference (similarity, denoted I) and merits a specific attention.

Recently Tsoukiàs and Vincke [21] gave a complete characterisation of such a structure (denoted as PQI interval order: for further details the reader can also see [6], [7]). In this paper we extend such results considering the situation of hesitation under a continuous valuation of preference and indifference. The idea is that the intersection of the two intervals can be more or less large thus resulting in a more or less large hesitation represented by a value in the interval [0,1] for preference and indifference.

The paper is organised as follows. In the next section we introduce the basic notation and results on regular interval orders and semi-orders. In section 3 we introduce the concept of PQI interval order and semi-order. In section 4 we introduce a functional representation for preference (dissimilarity) and indifference (similarity) fulfilling a number of nice properties. Further research directions are included in the conclusions.

2 Interval Orders

In the following we will consider objects represented under form of intervals of values (for simplicity we consider continuous interval scales of values). Given a finite set A of objects we associate to each element of A two functions $l : A \mapsto \mathbb{R}$ and $r : A \mapsto \mathbb{R}$ (the left and right extreme of x respectively) such that $\forall x \; l(x) < r(x)$. Such a representation is equivalent to the one where to each element x of A is associated a function $g(x)$ and threshold function $t(x)$. We have $l(x) = g(x)$ and $r(x) = g(x) + t(x)$. In the rest of the paper we only use the $(l(x), r(x))$ representation.

Further on, consider a structure of two binary relations $P \subseteq A \times A$ and $I \subseteq A \times A$ (respectively named preference and indifference). From a data analysis point of view we can consider indifference as a similarity relation and the union of preference and its inverse as a dissimilarity relation. Hereafter, for sake of simplicity, we only use the terms of preference and indifference such that: P is asymmetric and irreflexive, I is symmetric and reflexive, $P \cup I$ is complete and $P \cap I = \emptyset$.

Given any two binary relations S, T on the set A we denote by $T.S$ the formula $\forall x, y \; \exists z : T(x, z) \wedge S(z, y)$ and by $T \subseteq S$ the formula $\forall x, y \; T(x, y) \Rightarrow S(x, y)$.

We are now able to give some basic definitions and theorems.

Definition 1 *A $\langle P, I \rangle$ preference structure on a set A is a PI interval order iff $\exists \; l, r : A \mapsto \mathbb{R}$ such that:*
$\forall \; x : \; r(x) \geq l(x)$
$\forall \; x, y : P(x, y) \; \Leftrightarrow \; l(x) > r(y)$
$\forall \; x, y : I(x, y) \; \Leftrightarrow \; l(x) \leq r(y)$ *and* $l(y) \leq r(x)$

Definition 2 *A $\langle P, I \rangle$ preference structure on a set A is a PI semi order iff $\exists \; l : A \mapsto \mathbb{R}$ and a positive constant k such that:*
$\forall \; x, y : P(x, y) \; \Leftrightarrow \; l(x) > l(y) + k$
$\forall \; x, y : I(x, y) \; \Leftrightarrow \; |l(x) - l(y)| \leq k$

Such structures have been extensively studied in the literature (see for example [3]). We recall here below the two fundamental results which characterise interval orders and semi orders.

Theorem 2.1 *A $\langle P, I \rangle$ preference structure on a set A is a PI interval order iff $P.I.P \subset P$.*

Proof. See [3].

Theorem 2.2 *A $\langle P, I \rangle$ preference structure on a set A is a PI semi order iff $P.I.P \subset P$ and $I.P.P \subset P$.*

Proof. See [3].

From the above results it appears clear the idea that two objects are considered indifferent if their associated intervals have a non empty intersection. A object is preferred to another if its associated interval is "completely to the right" (in the sense of the line of the reals) of the interval associated to the other object.

3 PQI interval orders

Recently [21] suggested that, while the conditions under which the relation P holds could be considered fixed, the conditions under which the relation I holds contain two different situations. One called "sure indifference" (where one interval is included to the other) and one called "weak preference" or "hesitation between indifference and preference" (where the intersection of the two intervals is non empty, but one interval is "more to the right of the other"). In this section we consider preference structures composed of three preference relations: P (which is asymmetric and irreflexive), Q (which is asymmetric and irreflexive) and I (which is symmetric and reflexive), $P \cup Q \cup I$ being complete and mutual intersections being empty and we have the following results.

Definition 3 *A $\langle P, Q, I \rangle$ preference structure on a finite set A is a PQI interval order, iff $\exists \; l, r : A \mapsto \mathbb{R}$ such that, $\forall x, y \in A, x \neq y$:*
- $r(x) \geq l(x)$;
- $P(x, y) \; \Leftrightarrow \; r(x) \geq l(x) > r(y) \geq l(y)$;
- $Q(x, y) \; \Leftrightarrow \; r(x) > r(y) > l(x) > l(y)$;
- $I(x, y) \; \Leftrightarrow \; r(x) \geq r(y) \geq l(y) \geq l(x)$ *or* $r(y) \geq r(x) \geq l(x) \geq l(y)$.

Theorem 3.1 *A $\langle P, Q, I\rangle$ preference structure on a finite set A is a PQI interval order, iff there exists a partial order L such that:*
i) $I = L \cup R \cup I_o$ where $I_o = \{(x, x),\ x \in A\}$ and $R = L^{-1}$;
ii) $(P \cup Q \cup L)P \subset P$;
iii) $P(P \cup Q \cup R) \subset P$;
iv) $(P \cup Q \cup L)Q \subset P \cup Q \cup L$;
v) $Q(P \cup Q \cup R) \subset P \cup Q \cup R$;

Proof. See [21]

Definition 4 *A PQI semi order is a PQI interval order that admits a representation where $\forall x\ r(x) - l(x)$ is constant.*

In other words, a PQI semi order is a $\langle P, Q, I\rangle$ preference structure for which there exists a real valued function $l : A \mapsto \mathbb{R}$ and a positive constant k such that $\forall\ x, y$:
- $P(x, y) \Leftrightarrow l(x) > l(y) + k$;
- $Q(x, y) \Leftrightarrow l(y) + k > l(x) > l(y)$;
- $I(x, y) \Leftrightarrow l(x) = l(y)$; (in fact I reduces to I_o).

Theorem 3.2 *A $\langle P, Q, I\rangle$ preference structure is a PQI semi order iff:*
i) I is transitive
ii) $PP \cup PQ \cup QP \subset P$;
iii) $QQ \subset P \cup Q$;

Proof See [21].

4 Valued PQI interval orders

The existence of a zone of hesitation between strict preference and indifference and the introduction of valued relations in order to take into account such an hesitation have been first considered by Roy ([13], [14]) in the case of the so-called pseudo-order and extensively studied in [10]. However, in this case they consider preference structures with two thresholds which is equivalent to a representation with intervals having an intermediate point used for comparing them. The hesitation occurs between the extremes of this second interval.

In this case we consider preference structures with only one threshold. The hesitation is due to the interval structure of the information associated to each object. The results presented in the previous section however, although they introduce the idea that comparing objects represented by intervals implies the existence of a zone of hesitation between preference and indifference, are unable to give a "measure" of such an hesitation.

Consider three objects whose cost is for the first (x) in the interval $[10, 18]$, for the second (y) in the interval $[11, 20]$ and for the third (z) in the interval $[17, 20]$. Using the previous approach we get $Q(x, y)$, $Q(x, z)$ and $I(y, z)$. However, it is intuitively clear that the hesitation which occurs when objects x and y are compared is not the same with the hesitation which occurs when objects x and z are compared. Moreover, although objects y and z are considered indifferent it is again intuitively clear that they are indifferent to some extent and not identical.

The basic idea introduced is that the extent to which the two intervals have a non empty intersection could be a "measure" of the hesitation between preference and indifference. Such an idea dates back to ([2]), but was applied there to conventional preference structures where a distribution of possibility can be associated to alternatives under the form of a fuzzy number. In this approach we consider flat distributions of uncertainty in the sense that any value of the interval has the same possibility to represent the "real" value. From this point of view it is meaningful to compare lengths of intervals in order to have a "measure" of the uncertainty. The approach however, can be easily generalised in the case of specific uncertainty distributions.

Intuitively the idea is that relations P and I are two valued relations represented by functions $p : A \times A \mapsto [0, 1]$ and $i : A \times A \mapsto [0, 1]$ such that (for a detailed discussion of such properties see [9]):

Definition 5 *A pair of valued preference relations p and i is a preference structure with valued hesitation iff p and i satisfy the following properties:*
$\forall x, y\ p(x, y) = 1 \Rightarrow i(x, y) = 0$
$\forall x, y\ i(x, y) = 1 \Rightarrow p(x, y) = 0$
$\forall x, y\ i(x, y) = i(y, x)$
$\forall x, y\ p(x, y) \geq 0 \Rightarrow p(y, x) = 0$

The part of $A \times A$ where both $1 > p(x, y) > 0$ and $1 > i(x, y) > 0$ corresponds to the crisp relation Q of a PQI preference structure. When $p(x, y) = 1$ we have the crisp P relation and when $i(x, y) = 1$ we have the crisp I relation. We denote functions p_I and i_I for the interval order case and p_S and i_S for the semi order case. We get:

$$p_I(x, y) = \min(1, \max(0, \min(\frac{r(x) - r(y)}{r(x) - l(x)}, \frac{l(x) - l(y)}{r(y) - l(y)})))$$

$$i_I(x, y) = \frac{\max(0, (\min(r(x) - l(y), r(x) - l(x), r(y) - l(y), r(y) - l(x))))}{\min(r(x) - l(x), r(y) - l(y))}$$

$$p_S(x, y) = \min(1, \max(0, \frac{l(x) - l(y)}{k}))$$

$$i_S(x, y) = \frac{\max(0, (\min(r(x) - l(y)), k, r(y) - l(x)))}{k}$$

k being the constant length of the PQI semi order interval.

It is easy to verify now the following propositions.

Proposition 4.1 *p_I and i_I satisfy the properties introduced in definition 5.*

Proposition 4.2 *p_S and i_S satisfy the properties introduced in definition 5.*

We can introduce the function $r(x, y)$ representing the valued relation $R = P \cup Q \cup I$ as follows:

$$r(x, y) = \max(0, \min(1, \max(\frac{r(x) - l(y)}{r(y) - l(y)}, \frac{r(x) - l(y)}{r(x) - l(x)})))$$

and verify that the following propositions hold (see [9]):

Proposition 4.3
$p_I(x, y) = 1 - r(y, x)$,
$i_I(x, y) = \min(r(x, y), r(y, x))$.

The above results show that we practically can consider three situations. The first, where $p(x, y) = 1$ (and $i(x, y) = 0$ and $p(y, x) = 0$), corresponds to the crisp strict preference relation. The second, where $i(x, y) = 1$ (and $p(x, y) = 0$ and $p(y, x) = 0$), corresponds to the crisp indifference relation. The third one, where $1 > p(x, y), i(x, y) > 0$ ($p(y, x) = 0$), represents the area of hesitation denoted by Q in the PQI interval order preference structure. We can summarise the results as follows.

1. PQI interval order

$r(x) > l(x) > r(y) > l(y)$	$\Leftrightarrow$	$p_I(x, y) = 1$	$i_I(x, y) = 0$	$p_I(y, x) = 0$
$r(x) > r(y) > l(x) > l(y)$	$\Leftrightarrow$	$0 < p_I(x, y) < 1$,	$0 < i_I(x, y) < 1$	$p_I(y, x) = 0$
$r(x) > r(y) > l(y) > l(x)$	$\Leftrightarrow$	$p_I(x, y) = 0$,	$i_I(x, y) = 1$	$p_I(y, x) = 0$
$r(y) > r(x) > l(x) > l(y)$	$\Leftrightarrow$	$p_I(x, y) = 0$,	$i_I(x, y) = 1$	$p_I(y, x) = 0$
$r(y) > r(x) > l(y) > l(x)$	$\Leftrightarrow$	$p_I(x, y) = 0$	$0 < i_I(x, y) < 1$	$0 < p_I(y, x) < 1$
$r(y) > l(y) > r(x) > l(x)$	$\Leftrightarrow$	$p_I(x, y) = 0$	$i_I(x, y) = 0$	$p_I(y, x) = 1$

2. PQI semi order

$$\begin{array}{llllll} l(x) > l(y) + k & \Leftrightarrow & p_S(x,y) = 1 & i_S(x,y) = 0 & p_S(y,x) = 0 \\ l(y) + k > l(x) > l(y) & \Leftrightarrow & 0 < p_S(x,y) < 1, & 0 < i_S(x,y) < 1 & p_S(y,x) = 0 \\ x = y & \Leftrightarrow & p_S(x,y) = 0, & i_S(x,y) = 1 & p_S(y,x) = 0 \\ l(x) + k > l(y) > l(x) & \Leftrightarrow & p_S(x,y) = 0 & 0 < i_S(x,y) < 1 & 0 < p_S(y,x) < 1 \\ l(y) > l(x) + k & \Leftrightarrow & p_S(x,y) = 0 & i_S(x,y) = 0 & p_S(y,x) = 1 \end{array}$$

5 Discussion

What do we get with such results? What can we do with such valued relation? We consider two cases.

The first, obvious, case concerns the domain of preference modelling. Having a functional representation of the type described in the above section enables to give an explicit representation of the uncertainty and hesitation which appears when we compare intervals and to overcome the difficulty associated to the use of crisp thresholds. In fact if a discrimination problem exists this will concern any type of comparison. Therefore even if we fix a discrimination threshold there always exist an interval around the threshold for which a discrimination problem has to be considered (and that recursively for any new threshold introduced). The valued representation solves this problem. In this particular case the solution does not require the introduction of two thresholds, but gives a valued version for preference and indifference in all cases intervals are compared.

The second case concerns more generally the problem of comparing objects not necessarily for preference modelling reasons. As already introduced we can always consider the concept of indifference equivalent to the one of similarity, the concept of preference becoming a directed dissimilarity. Establishing the similarity among objects is a crucial problem for several research fields such as statistics, data analysis (in archaeology, geology, medical diagnosis etc.), information theory, classification, case based reasoning, machine learning, temporal logic (see [1] etc.. A specific area of interest in the use of similarity relations is in rough sets theory ([8]).

In rough sets we consider objects described under a set of attributes and we establish a relation of indiscernibility (which is a crisp equivalence relation) in order to take into account our limited descriptive capability. In other terms real objects might be different, but due to our limited descriptive capability (represented by the set of attributes) we might be obliged to consider them as identical (indiscernible). Indiscernibility classes are then used in order to induce classification rules. However, equivalence relations can be very restrictive for several real cases where the more general concept of similarity is more suitable (see [15][16]). The use of a valued similarity has been considered in [4], [17], [18] [19], for several different cases. Thanks to such a relation it is possible to induce classification rules to which a credibility degree is associated. By this way it is possible to enhance the classification capability of a data set although a confidence degree inferior to 1 has to be accepted. The approach described in this paper enables to give a theoretical foundation for the case where objects have to be compared on attributes with continuous scales and where either a discrimination threshold has to be considered or the objects are represented by intervals.

A critical point in the above representation concerns the relation I. Indeed the solution suggested does not consider the separation of I in the relations L and R which take into account the inclusion of one interval into the other, inclusion which can be more or less large. This is the next step to undertake in order to fix a complete scheme of valued representation of hesitation in intervals comparison. Further on, we are expecting to be able to characterise more complex valued preference structures, allowing valued hesitation, adding further properties to the valued preference relations.

6 Conclusion

In this paper we present some preliminary results concerning the extension of PQI interval orders under continuous valuation. Particularly we give the functional representation for relations P and I such that the portion of interval which is common is considered as a "measure" of the hesitation associated to the interval comparison. Such functions fulfill a number of nice properties in the sense that they correspond to a fuzzy preference structure as defined in [9].

The use of such valued preference relations not only enhance the toolkit of preference modelling, but enables a more flexible representation in all cases where a similarity among objects is under question. The particular case of rough sets theory is discussed in the paper. Several research directions remain open such as:

- the problem of aggregating such valued relations in order to obtain a comprehensive relation (crisp or valued) when several attributes or criteria are considered;
- a further analysis of the formal properties fulfilled by such valued relations;
- the analysis of such preference structures under the positive/negative reasons framework as introduced in [11] and discussed in [20].

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