A framework for decision aiding (part 1)

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A framework for decision aiding

My aims

- Recall of the general framework
- Step-by-step analysis of four cases
- Presentation of some specific tools
- *Discussion* (?)

Summary

- (1) The general framework (2) Five main features
- (3) Some basic tools (4) Going to the origin
- (5) Conclusions (part 1) (6) Tools p.2: risk analysis
- (7) Tools p.3: pairw. compar. (8) Tools p.4: group decision
- (9) Service design (10) Conclusions (part 2)
The general framework
A path *between the client and the analyst*

- **Ground info** → client’s statements (in his natural language)
- **Learning protocols** → procedures to identify ordering relations
- **Primitives** → information (strictly) necessary to decide
- **Modelling tools** → analytic tools to define-produce the model
- **Input** → adaptation of the info to a specific algorithm
Four examples

ExA → Palio di Siena

(an Italian horse race)
http://en.wikipedia.org/wiki/Palio_di_siena

ExB → St. John Hospital

(a nurse recruitment)
http://www.stjohnprovidence.org/default.aspx

ExC → Sausages

(a food classification)
http://en.wikipedia.org/wiki/Sausage

ExD → Le salaire de la peur

(a safe path problem)
http://www.youtube.com/watch?v=U3ssM6K1IdA
Problem
The client is a horse race gambler; the race is Palio di Siena (10 “contrade”)
In order to assess the “value” of each possible bet he takes into account 3 different information:
• the quality of the horse,
• the quality of the jockey,
• the weather conditions.
The client wants to rank all possible bets.

Learning protocols
Procedures through which the analyst will try to gather the client indications.

In our case:
• which horses does he prefer ?
• with which jockey ?
• under which weather conditions ?
• etc.
Ground information for ExB

**Problem**
A hospital is considering the recruitment of nurses for three of their departments: General medicine (MG), Oncology (ON), Pediatrics (PE). The selections are managed by the chief surgeon P and the vice-surgeon V. Candidates fill an application form and go through an interview. The result is a report where the two managers consider three information:

- the age,
- the specialisation (if any),
- the motivations of the candidate.

The managers want to assign the candidates in the best way.

**Learning protocols**
*Procedures through which the analyst will try to gather the client indications.*

In our case:

- what is a “good nurse” for a given department?
- what specialisations with respect to the dept. requirements?
- how the age influences the fitting of a candidate to a given dept.?
- etc.
Problem
The Nutrition Agency (NA) has to classify a lot of different kind of sausages. The result is a clustering of all the sausages to K categories of "similarity" (the names will be defined after the clustering). Each sausage is defined by a mix of m attributes. There are some indications "against" to be considered (if the difference between two kinds of sausages is too high, the sausages have to go to different clusters, etc.).

Learning protocols
Procedures through which the analyst will try to gather the client indications.

In our case:
- what are the attributes to be considered?
- what is the meaning of differences? (rate of fat? others?)
- how to deal with indications "against"?
- how to deal with continuous and discrete data? (see the example)
- etc.
Ground information for ExD

**Problem**
Your company have to deliver potentially dangerous goods to a regional system of clients. You are worried about the safety of such deliveries; costs represent a secondary issue. Deliveries are done using dedicated trucks. Road network is known and you also have the annual statistics of accidents for each segment of the network. In your decision you have to consider safety and costs.
The issue is how to define a set of “sufficiently safe” paths or how to choose the most safe path within the network (for a given delivery).

**Learning protocols**
The l. p. are procedures through which the analyst will try to gather information.

In our case:
- given a specific delivery (destination), which are the alternatives? (it is the whole set of feasible paths connecting the origin to the destination, but note that this set can be extremely large)
- given two paths x and y, when can you consider the relation “x at least as good as y”? (if the overall likelihood of having an accident through path x is not superior of the corresponding through path y → you prefer less risky paths)
- what about costs?

**Modelling.** We need to define in a formal way the set A of alternatives. In order to do so we represent the road network as a graph (N, A). To each couple of nodes i and j related by an arc we associate a binary variable xij. For each variable we have info:
- dij : the length of the arc,
- vij : the daily traffic,
- aij : the annual number of accidents,
- etc.
The information categories *(primitives)*

**Values** (related to attributes).

An alternative $x$ can be described by a set of attributes; each attribute is characterized by a scale (scales can be nominal, ordinal, ratio or interval ones). This description ($x$ is 10cm long; $y$ is yellow, etc.) must be reinforced by further information represented by preferential statements (she prefers long tables to short ones; I do not like yellow shoes), possibly of more complex content (he prefers a train travel to Paris than a flight to Amsterdam; my preference of apples against oranges is stronger than my preference of peaches against apricots). We distinguish two types:

- **comparative sentences**, where the alternatives are compared among them (under more attributes) in order to express a preference;
- **absolute sentences**, where an alternative is directly assessed with respect to some value structure (under one or more attributes).

**Opinions** (related to stakeholders).

Decisions can be affected by the judgments and opinions of many stakeholders. In this case preference statements have to be associated to opinions. It is reasonable however, to distinguish among:

- **comparative opinions** (stakeholder $i$ prefers $x$ to $y$), where preferences are expressed among elements of the alternative set;
- **absolute opinions** (stakeholder $i$ considers $x$ as worthy), where preferences are expressed under form of value assessments.

**Likelihoods** (related to scenarios).

Preferences often depend from uncertain conditions. If we focus on decision situations the primitives we need to consider will be preference statements of the type “under scenario $j$, he prefers $x$ to $y$” or of the type “under scenario $j$, $x$ is unworthy”.

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**From ground info to primitives**

**Ground info**

The problem has 2 variables.
The DM gives a set of couples “good” for him and other couples “not good”.

For instance →

<table>
<thead>
<tr>
<th>(0, 0)</th>
<th>x1 neg</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 0)</td>
<td>x2 neg</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>x1 &gt; 2</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>x2 &gt; 2</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>etc.</td>
<td>etc.</td>
</tr>
</tbody>
</table>

**Primitives**

So you can define a region $X$ of feasible solution like this ...

**Algorithm ?**

You can use an (integer) LP, but region $X$ is not convex !
From primitives to input

Region & algorithm

The region \( X \) is not convex, but you have \( X = P \cup Q \) (you have an OR)

Input

(Int)LP works in convex region. You can transform the problem:

\[
\begin{align*}
0 & \leq x_1 \leq 2 \\
0 & \leq x_2 \leq 2 \\
x_1 - x_2 & \leq 0 + M_P y_P \\
x_1 + x_2 & \leq 2 + M_Q y_Q \\
y_P + y_Q & \leq 1 \\
y_P, y_Q & = \{0, 1\}
\end{align*}
\]

This is a classical Int.LP, so you can solve it in the usual way!
Three elements & eight problems

1. Math. Programming (MP)
2. Risk analysis (RA)
3. Multiple criteria (MC)
4. Group choice (GC)
5. 6. 7. 8. → Game theory, etc.

Information (scenarios)
Criteria (attributes)
Dec. makers (stakeholders)

- complete
- partial → state identif. & risk an.
- one
- more → trade-off
- one
- more → cooper. vs conflicts
An **ideal** decision problem *(point 1)*

- **Someone who decides**
  - with respect to one clear **objective**
  - with a set of well defined **constraints**
  - with all the suitable **information**

  in presence of a \( \begin{array}{c}\text{finite} \\ \text{infinite} \end{array} \) \( \text{set of alternatives} \)

- **Examples**
  - \( (1) \rightarrow \text{an ideal discrete case} \)
  - \( (2) \rightarrow \text{an ideal continuous case} \)
Ideal example 1

Combinatorial optimization

Your chorus is defining the storyboard of a concert and you must choose between a set of mottetti (a “mottetto” is a choral musical composition). Each mottetto \( m_1, m_2, \ldots, m_n \) has a time of execution \( t_j \) and a level of success \( s_j \) \( (j = 1, \ldots, n) \).

The total time of the exhibition is \( T \) min.

What can you do?

If you want, consider this specific instance:

\[
\begin{align*}
n & = 4; & t & = (10, 22, 37, 9); & s & = (60, 55, 100, 15); & T & = 45
\end{align*}
\]

(i) What are the variables?
(ii) How many solutions?
(iii) What is the optimal choice?
Ideal example 2

Linear programming (LP)

You must define the week production of a (small) firm that has only 2 products, A and B. One item of A needs 4 units of the resource R1 and 2 unit of the resource R2. One item of B needs 1 unit of the resource R1 and 3 units of the resource R2. You have (weekly) 200 units of R1 and 480 units of R2, and you know that the maximum possible sale for B is 110 items. The net revenue for item A is 500 €, for item B is 300 €.

What can you do?

(i) What are the variables?
(ii) How many solutions?
(iii) What is the optimal choice? (solve with Excel …)
Ideal example 2: the model

LP properties ！

\[ z \text{ (max)} = 500 \, x_1 + 300 \, x_2 \]  
\[ \text{s.t.} \]  
\[ 4 \, x_1 + 1 \, x_2 \leq 200 \]  
\[ 2 \, x_1 + 3 \, x_2 \leq 480 \]  
\[ x_2 \leq 110 \]  
\[ x_1, \quad x_2 \geq 0 \]  

(objective function)  
(a set of constraints)  
(resource R1)  
(resource R2)  
(constr. \: x2 sale)  
(non neg. constr.)

The optimal choice: B  
\[ x_1=22.5, \quad x_2=110, \quad z=\ldots \]
A real decision problem

- **Complexity** (problem dimension, non linearity, …)
- **Uncertainties** (non-deterministic context, data mining)
- **Several stakeholders** (distributed decision power)
- **Different rationalities** (multiple criteria and preferences)
- **Various time horizons** (often)
- **Need of simulation models** (what … if …)

*the other points (2, 3, …) of the cube*
## Examples (position in the cube)

<table>
<thead>
<tr>
<th>Example</th>
<th>Task</th>
<th>Operation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>ExA - horse race</td>
<td>(ranking)</td>
<td>RA-MC</td>
<td>p. 6</td>
</tr>
<tr>
<td>ExB - nurses</td>
<td>(assign)</td>
<td>MC-GC</td>
<td>p. 5</td>
</tr>
<tr>
<td>ExC - sausages</td>
<td>(cluster)</td>
<td>MC</td>
<td>p. 3</td>
</tr>
<tr>
<td>ExD - safe paths</td>
<td>(rating/ranking)</td>
<td>MC</td>
<td>p. 3</td>
</tr>
</tbody>
</table>
Five main features
Features of a decision problem

1. Set of the alternatives
2. Problem statement
3. Independence (or not) of …
4. Differences of preferences
5. Pos./negative reasons

a quick survey
1. Set of alternatives

The set A of the alternatives:

i. A is a finite set of objects (countable enumeration)

ii. A is a subset of all possible combinations of attribute values

iii. A is the product of a set of discrete (binary) decision variables

iv. A is a vector space (all the admissible values of real variables)

Examples:

ExA (Palio) → finite set (10 contrade)
ExB (nurses) → finite set (n candidates)
ExC (sausages) → vector space (m dimensions)
ExD (paths) → product of a binary variable set (large !)
2. Problem statement  
(partition of the alternative set)

- Different partitions

- Classes:
  - ordered ➔ not ordered
  - not predefined ➔ predefined

- Two problems:
  - choice ➔ what you want (and the remaining …)
  - or
  - rejection ➔ what you don’t (and the remaining …)
The other features

3. Defining independence (or not) of …
There are two principal interpretations ($H$ is a set of criteria):
• “$x$ is at least as good as $y$, under $I'$, independently on what happens to $H \setminus I$ (preferential indep.)
• “$x$ is at least as good as $y$, under $I'$, provided a condition holds in some $J \subseteq H \setminus I$ (conditional indep.).
These two interpretations lead to completely different problem formulations and consequently to different methods and resolution algorithms: preferential independence allows to envisage a linear (additive) model representing preferences; conditional preferences lead to more complex preference structures (non linear aggregation functions).

4. Defining differences of preferences
Let’s consider the sentence “$x$ is strictly better than $y$ and these are both better than $z$”. We can represent this sentence giving numerical values to $x$, $y$, $z$ (for instance, $x = 3$, $y = 2$, $z = 1$). But we could choose the numerical representation $x = 100$, $y = 10$, $z = 1$ and it would be the same. In many cases we could either have richer information (we know for instance that $x$ is twice more heavy than $y$) or we would like to have information of the type “$x$ is much more better than $y$”. We need to reason in terms of “differences of preferences”. In other terms we need primitives of the type: “$xy$ is not less than $zw$” where $xy$ ($zw$) represents the difference of preference between $x$ and $y$ ($z$ and $w$). Primitives of this type can be used in order to express ordinal preferences, while the opposite is not true. We can claim that primitives should always be considered as sentences about differences of preferences, the ordinal case being a special one.

5. Defining pos./negative reasons
Consider a preference statement of the type: “I do not like $x$”, or “any candidate, but not $x$”. Such statements can be considered as explicit “negative preferential statements” to be considered independently from the “positive ones” (which are the usual ones). The idea here is that there are cases where decision makers need to express negative judgments and values which are not complementary to the positive ones (such as a veto on a specific dimension).
Examples *(summarizing)*

ExA – Palio  →  finite set of alternatives ranking problem  
RA-MC (point 6)

ExB – nurses  →  finite set of alternatives assignment  
MC-GC (point 5)

ExC – saus.  →  infinite alternatives (vector space) clustering  
MC (point 3)

ExD – paths  →  finite (but large …) set of alternatives rating *OR* ranking  
MC (point 3)
ExD (paths): rating OR ranking

Let’s present a numerical instance.

<table>
<thead>
<tr>
<th>paths</th>
<th>km</th>
<th>acc(y)</th>
<th>linear.</th>
<th>beauty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 – 4</td>
<td>50</td>
<td>5.5</td>
<td>1.5</td>
<td>4</td>
</tr>
<tr>
<td>1 – 2 – 4</td>
<td>60</td>
<td>4.7</td>
<td>1.0</td>
<td>6</td>
</tr>
<tr>
<td>1 – 2 – 3 – 4</td>
<td>80</td>
<td>4.6</td>
<td>1.2</td>
<td>9</td>
</tr>
</tbody>
</table>

Ground info
- DM indicates as criteria: **time, risk, beauty**
- Risk is slightly more important than time
- These two are much more important than beauty

Rating
- Define the classes
- Define reference situations
- Put each path in its class

Ranking
- A common scale (utility)
- Compute the ranking (order.)
- Sensitivity analysis (w.r.t. the weights)
Rating
• Define k classes (k = 2, good-bad)
• Define reference situations (for each criteria)
• Put each path in its class (compare with ref.)

Protocols – Primitives – Ref.situations
• Define attributes related to DM indications:
  1) time = km * linear
  2) risk = acc / km
  3) beauty = beauty
• Attributes & Ref.situations:

<table>
<thead>
<tr>
<th>Path</th>
<th>Time</th>
<th>Risk</th>
<th>Beauty</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (14)</td>
<td>75</td>
<td>0.110</td>
<td>4</td>
</tr>
<tr>
<td>B (124)</td>
<td>60</td>
<td>0.078</td>
<td>6</td>
</tr>
<tr>
<td>C (1234)</td>
<td>96</td>
<td>0.057</td>
<td>9</td>
</tr>
</tbody>
</table>

→ weights: 0.40 0.50 0.10
→ ref.situations: 70 m. 0.100 6.5
**ExD: ranking**

**Ranking**
- Define a common scale (utility)
- Compute the ranking (max utility)
- Sensitivity analysis (w.r.t. weights)

**Utility (a common scale)**
- Linear scales for all the attributes:
  - time → [0, 100]
  - risk → [0, 0.200]
  - beauty → [0, 10]

**Evaluation matrix:**

<table>
<thead>
<tr>
<th>Path</th>
<th>U(time)</th>
<th>U(risk)</th>
<th>U(beauty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (14)</td>
<td>0.25</td>
<td>0.45</td>
<td>0.40</td>
</tr>
<tr>
<td>B (124)</td>
<td>0.40</td>
<td>0.61</td>
<td>0.60</td>
</tr>
<tr>
<td>C (1234)</td>
<td>0.04</td>
<td>0.71</td>
<td>0.90</td>
</tr>
</tbody>
</table>

**Weights** → 0.40 0.50 0.10

**Utilities**
- U(A) = 0.365
- **U(B) = 0.525**
- U(C) = 0.461

**Crucial question**
- What can we do when the number of paths is large?
- What info can be associated with each single arc?
Some basic tools
To decide …

A formal decision process needs tools:

i. to abstract

ii. to analyze

iii. to summarize

(and much more …)
Tools to abstract / 1

- 1736
- Konigsberg

- The 7 bridges
- A riddle

- Euler
- Graph theory

- The Euler model
- The answer (similar to …)
The death of Count Kinskij

- The count drank poisoned water (from one of his 7 lovers)
- All 7 lovers were in the castle the day of his death
- The murderer should have come to the castle twice (one for..., one for...), the others only once.
- Statements of the 7 women:

  - Alice saw B C E F
  - Barbie saw A C D E G
  - Clara saw A B D
  - Diana saw B C E
  - Elena saw A B D G
  - Francis saw A G
  - Gloria saw B E F

The solution

The death of Count Kinskij

Women statements

\[
\begin{align*}
A &\Rightarrow E & D &\Rightarrow C \\
A &\Rightarrow E & G &\Rightarrow F \\
A &\Rightarrow B & G &\Rightarrow F
\end{align*}
\]

\(\neg (so \ A \ lies)\)

Impossible!
Graph theory & decision problems

- **General reports**
  - [http://en.wikipedia.org/wiki/Graph_theory](http://en.wikipedia.org/wiki/Graph_theory)

- **Applications**
  - [http://www....](http://www....)
  - [http://www....](http://www....)

- **A famous problem – TSP**
  - [http://www-e.uni-magdeburg.de/mertens/TSP/index.html](http://www-e.uni-magdeburg.de/mertens/TSP/index.html)
  - [http://www.tsp.gatech.edu/index.html](http://www.tsp.gatech.edu/index.html)
Tools for analysis / 1

- Branch & Bound

- Branching rules → build a tree
- A lot of (small) subproblems
- Bounding rules → cut the tree
- A lot of applications
Tools for analysis / 2

- **Sudoku** (Corriere della Sera, 3 Sept. 2010)

```
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>4</th>
<th></th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>?</td>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>
```

- What number in position ? ... or ...
- what position for number 4 in the upper right square?
**Sudoku** (Corriere della Sera, 3 Sept. 2010)

What in position $X$? $\rightarrow$ 2 or 9 (and not 8 because …)

**branch:** (a) $\rightarrow X = 2$ but if $X = 2$ …

**branch:** (b) $\rightarrow X = 9$ in this case …
Sudoku (Corriere della Sera, 3 Sept. 2010)

What in position $Y$? → 5 or 9

situations to be explored are (b1) $Y = 5$, and (b2) $Y = 9$

branch: (b1) $\rightarrow Y = 5$  (in this case 9 is not ...)
**Step 53 (of b1)**

Stop!

*Found a unique solution, so:*

*no branch (b2):*
The solution (visualization)

- Branching rules
- A lot of (easier) subproblems
- Ending rules
Who is the best boxeur in the world of all times?

Indicators:
- strength
- speed
- n. of victories
- years of premiership
- ...

We need a common frame to compare the alternatives!
All the tools helping the paths from \( k \) to 1

Examples of tools

- Bayes theorem
- Experiments & dec. trees
- Pairwise comparison
- Eigenvectors
- Peer evaluation
- Linear algebra (Frobenius)
- ...

Cases / models follow (in part 2)
Going to the origin
From point $k$ to point 1

How?
- What is (are) the path(s) ?
- What are the the conditions ?
- What are the tools ?

Let’s start with:
- a numerical example (the nurse problem)
- a comparison between three situations
ExB – nurses / 1 (assignment)

Alternatives (candidates) → C1, C2, C3, C4, C5
Dec. makers → P (chief surgeon), V (vice-surgeon)
Destinations → dept.MG / dept.ON / dept.PE / no assumption.
Evaluation → age + form (specialisation) + interview (motivations)

Learning protocols → a set of questions for P and V (answers follow)
P: (i) more experience = better work; (ii) specialisation is an important element for working in dept.ON; (iii) experience and motivations are key factors for working in any department.
V: (i) 30 years are the ideal age to work in PE (so he is against the entry of people over 40 in that dept.); (ii) the experience and specialisation are key factors to work in MG; (iii) motivations are not significant, except for dept.ON.

Primitives → rules for modelling the problem (by learning protocols)
P: (i) his preferences are increasing with the age of the candidates; (ii) coherence specialization-dept. is important (especially for ON); (iii) the weight of first and third attribute is significant.
V: (i) in PE the ideal age is 30 years, he is contrary to ages over-40; (ii) age (as a proxy of experience) and specialization are important in MG; (iii) motivations are important in dept.ON.

Data:

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<th>motiv.</th>
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<tbody>
<tr>
<td>C1</td>
<td>45</td>
<td>general</td>
<td>+</td>
</tr>
<tr>
<td>C2</td>
<td>31</td>
<td>maternity</td>
<td>++</td>
</tr>
<tr>
<td>C3</td>
<td>37</td>
<td>orthopedics</td>
<td>=</td>
</tr>
<tr>
<td>C4</td>
<td>25</td>
<td>no special..</td>
<td>+++</td>
</tr>
<tr>
<td>C5</td>
<td>35</td>
<td>dentist</td>
<td>– (no)</td>
</tr>
</tbody>
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Primitives \(\rightarrow\) rules for modelling the problem (by learning protocols)

**P:** (i) his preferences are increasing with the age of the candidates; (ii) coherence specialization-dept. is important (especially for ON); (iii) the weight of first and third attribute is significant.

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**Fitting with MG:**

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<th>motiv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>0.8</td>
<td>1.0</td>
<td>0.3</td>
<td>1.0</td>
<td>1.0</td>
<td>0.5</td>
</tr>
<tr>
<td>C2</td>
<td>0.3</td>
<td>0.6</td>
<td>0.7</td>
<td>0.3</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>C3</td>
<td>0.6</td>
<td>0.4</td>
<td>0.1</td>
<td>0.6</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>C4</td>
<td>0.2</td>
<td>0.0</td>
<td>0.9</td>
<td>0.0</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>C5</td>
<td>0.5</td>
<td>0.1</td>
<td>0.0</td>
<td>0.5</td>
<td>0.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

(for dec. maker P) (for dec. maker V)

**Rules \(\rightarrow\) see Primitives**
ExB – nurses / 3

A three dimensional matrix (fitting candidates-dept.MG) and two possible paths

Path (1): negotiation (agreement) among decision makers to identify shared candidate-attribute values (numerical example uses the average between P and V values), thus obtaining the matrix on the left. A second phase is an aggregation via multi attribute analysis. Referring to our cube, the approach brings the problem from point 5 to point 3 and then to 1.

Path (2): each decision maker produces the vector of his fitting levels between candidates and dept MG: in the example surgeon P uses weights (0.4, 0.2, 0.4) consistent with his statements in the learning phase, while V ignores the column Mot and gives equal weight to the other two. This approach brings the problem from point 5 to point 4 and then to point 1.

Both paths produce a vector with 5 values indicating the fitness the other candidates with MG. With the same procedure we obtain the fitness of all the candidates with the other depts (ON and PE) or not.

The final situation is therefore an fitness matrix to permit the best fit between candidates and depts (or their refusal).
Comparison: a problem at point 2 of the cube

<table>
<thead>
<tr>
<th>State of n.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>90</td>
<td>60</td>
<td>50</td>
<td>80</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>30</td>
<td>80</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>10</td>
<td>50</td>
<td>90</td>
<td>10</td>
</tr>
</tbody>
</table>

$\omega_i =$ i-th state of nature
## From point 2 to point 1

<table>
<thead>
<tr>
<th>State of n.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>$p_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>90</td>
<td>60</td>
<td>50</td>
<td>80</td>
<td>.50</td>
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<tr>
<td>$\omega_2$</td>
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<td>80</td>
<td>20</td>
<td>20</td>
<td>.10</td>
</tr>
<tr>
<td>$\omega_3$</td>
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<td>50</td>
<td>90</td>
<td>10</td>
<td>.40</td>
</tr>
<tr>
<td>Exp. val.</td>
<td>52</td>
<td>58</td>
<td>63</td>
<td>46</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**
- Use of different approaches (also without prob.)
- More info on probability using experiments
Comparison: a problem at point 3 of the cube

<table>
<thead>
<tr>
<th>Criteria</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_1$</td>
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<td>60</td>
<td>50</td>
<td>80</td>
</tr>
<tr>
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<tr>
<td>$c_3$</td>
<td>10</td>
<td>50</td>
<td>90</td>
<td>10</td>
</tr>
</tbody>
</table>

$c_i = i$-th criterion
### From point 3 to point 1

<table>
<thead>
<tr>
<th>Criteria</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
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<tbody>
<tr>
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<td></td>
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<td>50</td>
<td>80</td>
<td>.50</td>
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<tr>
<td>$c_2$</td>
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<td>80</td>
<td>20</td>
<td>20</td>
<td>.10</td>
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<tr>
<td>$c_3$</td>
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<td>50</td>
<td>90</td>
<td>10</td>
<td>.40</td>
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<td>52</td>
<td>58</td>
<td>63</td>
<td>46</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**  
- check the independence of criteria  
- identify the DM preference structure with …
Comparison: a problem at point 4 of the cube

<table>
<thead>
<tr>
<th>Dec. Mak.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d₁</td>
<td>90</td>
<td>60</td>
<td>50</td>
<td>80</td>
</tr>
<tr>
<td>d₂</td>
<td>30</td>
<td>80</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>d₃</td>
<td>10</td>
<td>50</td>
<td>90</td>
<td>10</td>
</tr>
</tbody>
</table>

\[ d_i = \text{i-th decision maker} \]
### From point 4 to point 1

<table>
<thead>
<tr>
<th>Dec. Mak.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>$\pi_i$</th>
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</thead>
<tbody>
<tr>
<td>↓</td>
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<td></td>
<td></td>
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<tr>
<td>$d_1$</td>
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<td>60</td>
<td>50</td>
<td>80</td>
<td>.50</td>
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<tr>
<td>$d_2$</td>
<td>30</td>
<td>80</td>
<td>20</td>
<td>20</td>
<td>.10</td>
</tr>
<tr>
<td>$d_3$</td>
<td>10</td>
<td>50</td>
<td>90</td>
<td>10</td>
<td>.40</td>
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<tr>
<td>Soc. ch.</td>
<td>52</td>
<td>58</td>
<td>63</td>
<td>46</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**
- a shared scale (between DM’s)?
- what method for power indices?
What about the mixed cases?

From 2 to 1 → estimation of **probabilities**

From 3 to 1 → estimation of **weights**

From 4 to 1 → estimation of **powers**

What about the path from 5 (or …) to 1?

*The mixed case occurs when:*

i. the result depends on several criteria & DM’s & st. of nature

ii. the path is not unique (see the following examples)
ExA: from M3 to M1

Matrix dimensions:
M3(ω, c, x) → M1(x)
Palio di Siena *(more)*

- RA (risk an.) → $\omega_1$ (dry), $\omega_2$ (windy), $\omega_3$ (rainy)
- MC (m. criteria) → $c_1$ (horse), $c_2$ (jockey)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$p_1$ ?</td>
</tr>
<tr>
<td>$c_1$</td>
<td>8</td>
<td>5</td>
<td>1</td>
<td>7</td>
<td>4</td>
<td>$w_1$ ?</td>
</tr>
<tr>
<td>$c_2$</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>2</td>
<td>5</td>
<td>$w_2$ ?</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td></td>
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<td></td>
<td></td>
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<td>$p_2$ ?</td>
</tr>
<tr>
<td>$c_1$</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>$w_1$ ?</td>
</tr>
<tr>
<td>$c_2$</td>
<td>4</td>
<td>6</td>
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<td>3</td>
<td>5</td>
<td>$w_2$ ?</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$p_3$ ?</td>
</tr>
<tr>
<td>$c_1$</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>$w_1$ ?</td>
</tr>
<tr>
<td>$c_2$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>3</td>
<td>8</td>
<td>$w_2$ ?</td>
</tr>
</tbody>
</table>
No relation between RA and MC \((prob \ & \ weights \ are \ independent)\)

- RA \(\rightarrow\) \(p_1 = 0.5, \ p_2 = 0.1, \ p_3 = 0.4\)
- MC \(\rightarrow\) \(w_1\) (horse) = 0.7, \(w_2\) (jockey) = 0.3

- We suppose prob \ & \ weights are independent
Two (symmetric) ways

- You can move from $M_3 = f(\omega, c, x)$ to $M_1 = f(x)$:
  1. passing through $M_2 = f(c, x) \rightarrow$ probab. $p_1$, $p_2$, $p_3$
  2. passing through $M_2 = f(\omega, x) \rightarrow$ weights $w_1$ e $w_2$

<table>
<thead>
<tr>
<th></th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>.35</td>
<td>.07</td>
<td>.28</td>
</tr>
<tr>
<td>$w_2$</td>
<td>.15</td>
<td>.03</td>
<td>.12</td>
</tr>
<tr>
<td></td>
<td>.50</td>
<td>.10</td>
<td>.40</td>
</tr>
</tbody>
</table>
From M3 to M1: the two ways

(i) Through $\hat{M}_2$: (point 3):

$\hat{M}_2 \rightarrow$

\[
\begin{array}{cccccc}
A & B & C & D & E & Wi \\
--- & --- & --- & --- & --- & --- \\
5.4 & 5.0 & 2.7 & 5.5 & 5.2 & .7 \\
4.4 & 6.0 & 8.0 & 2.5 & 6.2 & .3 \\
\end{array}
\]

$\hat{M}_2 \rightarrow$

\[
\begin{array}{cccccc}
A & B & C & D & E \\
--- & --- & --- & --- & --- \\
5.10 & 5.30 & 4.29 & 4.60 & 5.50 & M1 \\
\end{array}
\]

(ii) Through $\tilde{M}_2$ (point 2):

$\tilde{M}_2 \rightarrow$

\[
\begin{array}{cccccc}
\omega_1 & \omega_2 & \omega_3 \\
--- & --- & --- \\
A & B & C & D & E & pi \\
--- & --- & --- & --- & --- & --- \\
6.8 & 5.3 & 3.4 & 5.5 & 4.3 & .5 \\
2.6 & 5.3 & 3.5 & 3.7 & 7.1 & .1 \\
3.6 & 5.3 & 5.6 & 3.7 & 6.6 & .4 \\
\end{array}
\]

$\tilde{M}_2 \rightarrow$

\[
\begin{array}{cccccc}
\omega_1 & \omega_2 & \omega_3 \\
--- & --- & --- \\
A & B & C & D & E \\
--- & --- & --- & --- & --- \\
5.10 & 5.30 & 4.29 & 4.60 & 5.50 & M1 \\
\end{array}
\]
Relation between RA and MC \textit{(weights depend on probab.)}

- RA $\rightarrow$ $p_1 = 0.5$, $p_2 = 0.1$, $p_3 = 0.4$
- MC $\rightarrow$ $w_1$ and $w_2$ different (column by column)
- \textit{So, in this situation the path through M2 is impossible}
From M3 to M1 passing (only) through M2

\[ \begin{array}{cccccc}
A & B & C & D & E \\
\omega_1 & 6.8 & 5.3 & 3.4 & 5.5 & 4.3 \\
\omega_2 & 3.0 & 5.5 & 4.5 & 3.5 & 6.5 \\
\omega_3 & 4.2 & 5.6 & 6.2 & 3.4 & 7.2 \\
\end{array} \]

\( \omega_1 \omega_2 \omega_3 \)

**point 2 of the cube**
Mutual relation between RA and MC

- Correlation between probab. & weights → p1, p2, p3 are connected with w1, w2
- *This situation is displayed in the picture*

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>p2</th>
<th>p3</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>.18</td>
<td>.10</td>
<td>.22</td>
</tr>
<tr>
<td>W2</td>
<td>.08</td>
<td>.38</td>
<td>.04</td>
</tr>
<tr>
<td></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
No way (using the edges)

- It is impossible “to reduce” the problem by eliminating elements one by one (using the edges)
- *Then, what can we do?*
The situation *(considering, for instance, alternative A)*

<table>
<thead>
<tr>
<th></th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(\omega_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_1)</td>
<td>8</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(c_2)</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(p_1)</th>
<th>(p_2)</th>
<th>(p_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w_1)</td>
<td>.18</td>
<td>.10</td>
<td>.22</td>
</tr>
<tr>
<td>(w_2)</td>
<td>.08</td>
<td>.38</td>
<td>.04</td>
</tr>
</tbody>
</table>

- Total score for alternative A → \(8 \times 0.18 + 2 \times 0.10 + \ldots\)
Straight from M3 to M1 *(using the diagonal)*

- **A** → $8 \times 0.18 + 2 \times 0.10 + 3 \times 0.22 + 4 \times 0.08 + 4 \times 0.38 + 5 \times 0.04 = 4.34$
- **B** → $5 \times 0.18 + 5 \times 0.10 + 5 \times 0.22 + 6 \times 0.08 + 6 \times 0.38 + 6 \times 0.04 = 5.50$
- **C** → $1 \times 0.18 + 2 \times 0.10 + 5 \times 0.22 + 9 \times 0.08 + 7 \times 0.38 + 7 \times 0.04 = 5.14$
- **D** → $7 \times 0.18 + 4 \times 0.10 + 4 \times 0.22 + 2 \times 0.08 + 3 \times 0.38 + 3 \times 0.04 = 3.96$
- **E** → $4 \times 0.18 + 8 \times 0.10 + 6 \times 0.22 + 5 \times 0.08 + 5 \times 0.38 + 8 \times 0.04 = 5.46$

So we get the ranking vector M1 …
Examples (*going to the origin*)

ExA – Palio  →  different ways
              probably not symmetric

ExB – nurses →  two ways
              symmetric (path1 & path 2)

ExC – saus.  →  one way / clustering
              vector space

ExD – paths  →  one way / ranking or rating
              finite/large n. of alternatives
Sausages – Data matrix

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>un. of measure</th>
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<tbody>
<tr>
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<td>57</td>
<td>63</td>
<td>58</td>
<td>60</td>
<td>55</td>
<td>65</td>
<td>grammes</td>
</tr>
<tr>
<td>i2</td>
<td>1</td>
<td>9</td>
<td>3</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>% of fat</td>
</tr>
<tr>
<td>i3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>certification</td>
</tr>
<tr>
<td>i4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>pork meat</td>
</tr>
<tr>
<td>i5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>awards</td>
</tr>
</tbody>
</table>

- i1: values between 55 and 65 → normalized range \([55,65]\)
- i2: values between 1 and 9 → normalized range \([0,10]\)
- \(i3, i4, i5: \text{binary values} \rightarrow \text{Hamming distance (def.)}\)
- We are looking for \(K = 2\) (homogeneous) groups

Two phases: (1) distance matrix, (2) clustering algorithm

Variables → \(m = 2\) continuous + 3 discrete variables, for the \(i\)-th component \((i=1,\ldots,m)\). Alternative → a point \(x(x_1,\ldots,x_m)\) in the \(m\)-dimension space (a mix of components). Primitive → definition of the distance from \(x_h\) and \(x_k\) and checking of the “discordance conditions”
Phase 1: from dissimilarity to distance

### i1

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<td>--</td>
<td>.5</td>
<td>.1</td>
<td>.4</td>
<td>.2</td>
<td>.7</td>
<td>.3</td>
</tr>
<tr>
<td>B</td>
<td>.5</td>
<td>--</td>
<td>.6</td>
<td>.1</td>
<td>.3</td>
<td>.2</td>
<td>.8</td>
</tr>
<tr>
<td>C</td>
<td>.1</td>
<td>.6</td>
<td>--</td>
<td>.5</td>
<td>.3</td>
<td>.8</td>
<td>.2</td>
</tr>
<tr>
<td>D</td>
<td>.4</td>
<td>.1</td>
<td>.5</td>
<td>--</td>
<td>.2</td>
<td>.3</td>
<td>.7</td>
</tr>
<tr>
<td>E</td>
<td>.2</td>
<td>.3</td>
<td>.3</td>
<td>.2</td>
<td>--</td>
<td>.5</td>
<td>.5</td>
</tr>
<tr>
<td>F</td>
<td>.7</td>
<td>.2</td>
<td>.8</td>
<td>.3</td>
<td>.5</td>
<td>--</td>
<td>1.0</td>
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<tr>
<td>G</td>
<td>.3</td>
<td>.8</td>
<td>.2</td>
<td>.7</td>
<td>.5</td>
<td>1.0</td>
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</table>

### i2

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<tbody>
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<td>.87</td>
<td>.37</td>
<td>.12</td>
<td>.87</td>
</tr>
<tr>
<td>B</td>
<td>1.0</td>
<td>--</td>
<td>.75</td>
<td>.12</td>
<td>.62</td>
<td>.87</td>
<td>.12</td>
</tr>
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### i3-4-5

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This matrix gives a “measure” of the distance between the objects (ph. 1).

\[
\begin{align*}
\text{d}_{ij} &= 0.2 \times i1 + \\
&\quad 0.4 \times i2 + \\
&\quad 0.4 \times i3-4-5 = \ldots
\end{align*}
\]
Phase 2: k-median algorithm \((with\ k=2)\)

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Initialization: \(\bar{A}, D\)

Cluster A → \{A, C, E, F\}
Cluster D → \{B, D, G\}

Stop
Phase 2: average distance algorithm

A  B  C  D  E  F  G
A  0  63  27  70  46  46  54
B  63  0  69  47  44  52  21
C  27  69  0  48  24  61  56
D  70  47  48  0  51  63  54
E  46  44  24  51  0  47  43
F  46  52  61  63  47  0  63
G  54  21  56  54  43  63  0

ACEF  BDG
ACEF  37.1  58.9
BDG  58.9  30.5

A  B  C  D  E  F  G
A  0  58.5  27.0  70.0  46.0  46.0
BG  58.5  10.5  53.0  50.5  57.5
C  27.0  62.5  0  48.0  24.0  61.0
D  70.0  50.5  48.0  0  51.0  63.0
E  46.0  43.5  24.0  51.0  0  47.0
F  46.0  57.5  61.0  63.0  47.0  0

ACEF  BDG
ACEF  37.1  56.6  61.3
BDG  56.6  10.5  50.5  0
D  61.3  50.5  0

ACEF  BG  D
ACEF  37.1  55.7  59.7  50.0
BG  55.7  10.5  50.5  57.5
D  59.7  50.5  0  63.0
F  50.0  57.5  63.0  0

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### Fase 2: minimum distance algorithm

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#### Minimum Distance Algorithm:

**Step 1:**
- Initial distances matrix:
  - A 0 54 27 70 46 46 54
  - B 54 0 56 47 43 52
  - C 27 56 0 48 24 61
  - D 70 47 48 0 51 63
  - E 46 43 24 51 0 47
  - F 46 52 61 63 47 0

**Step 2:**
- Update distances after selecting the next node: D
  - A 0 54 27 70 46 46
  - B 54 0 56 47 43 52
  - C 27 56 0 48 24 61
  - D 70 47 48 0 51 63
  - E 46 43 24 51 0 47
  - F 46 52 61 63 47 0

**Step 3:**
- Update distances after selecting the next node: E
  - A 0 54 27 70 46 46
  - B 54 0 56 47 43 52
  - C 27 56 0 48 24 61
  - D 70 47 48 0 51 63
  - E 46 43 24 51 0 47
  - F 46 52 61 63 47 0

**Step 4:**
- Update distances after selecting the next node: F
  - A 0 54 27 70 46 46
  - B 54 0 56 47 43 52
  - C 27 56 0 48 24 61
  - D 70 47 48 0 51 63
  - E 46 43 24 51 0 47
  - F 46 52 61 63 47 0

**Final distances matrix:**

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Conclusions (part 1)
So ...

- It is possible to process the elements of the "decision space" \((\omega, c, d)\) in a coordinated way.

- If the elements are independent, it is possible to eliminate them one-by-one, thus obtaining a final function or vector \(M1\) of (continuous or discrete) decision variables.

- On the contrary, in case of dependency (i.e. criteria depend on the state of nature), the elimination follows a forced path.

- At last, in case of mutual dependency, you must proceed "along the diagonals" (by examining the behavior of the alternatives one by one).
Examples (more)

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<tr>
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<th>Tool/Method</th>
<th>Steps</th>
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<td>ExA – Palio</td>
<td>(MC-RA, ranking)</td>
<td>6 → 2 (*) → 1</td>
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<td>ExB – nurses</td>
<td>(MC-SC, assign)</td>
<td>5 → 2 or 3 → 1</td>
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<tr>
<td>ExC – saus.</td>
<td>(MC, cluster)</td>
<td>3 → 1</td>
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<tr>
<td>ExD – paths</td>
<td>(MC, rate/ranki)</td>
<td>3 → 1</td>
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(*) the path through point 2 is more realistic …

Now let’s consider specific tools