### From part 1

- It is possible to treat the elements of the "decision space" \((\omega, c, d)\) in a coordinated way.
- If the elements are independent it is possible to eliminate them one-by-one, thus obtaining a final function or vector M1 of the (continuous or discrete) decision variables.
- On the contrary, if there is a dependence (i.e. the criteria depend on the states of nature) the elimination follows a forced path.
- Finally, if there is a mutual dependence you must proceed "along the diagonals" (by examining the behavior of the alternatives one by one).

<table>
<thead>
<tr>
<th>Example</th>
<th>Decision</th>
<th>Function/Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>ExA – Palio</td>
<td>(MC-RA, ranking)</td>
<td>6 – 2 (3) – 1</td>
</tr>
<tr>
<td>ExB – nurses</td>
<td>(MC-SC, assign)</td>
<td>5 – 2 or 3 – 1</td>
</tr>
<tr>
<td>ExC – saus.</td>
<td>(MC, cluster)</td>
<td>3 – 1</td>
</tr>
<tr>
<td>ExD – paths</td>
<td>(MC, rating-rank))</td>
<td>3 – 1</td>
</tr>
</tbody>
</table>

Now let's consider specific tools
Tools for «point 2» problems

(i) Perception
(ii) Experiments & dec. tree
(i) A real decision process: perception

- **Uncertainties** (non deterministic context, …)
- Complexity (problem dimension, non linearity, …)
- Several stakeholders (distributed decision power)
- Different rationalities (criteria and preferences)
- Different time horizons (often)
- Need of simulation models

What … if …

- The DM *perception of the problem*
Decision processes in a non-deterministic context

1. Math. programming
2. **Risk analysis**
3. Multi-objective (criteria)
4. Group choice
5, 6, 7, 8 \(\rightarrow\) \(\ldots\)

Information

- complete
- partial \([*]\)

Objectives

- one
- more

Dec. makers

- one
- more

\([*]\) \(\rightarrow\) non-deterministic context

perception & mental models
Two (opposite) theories

(a) Normative theory (prescriptive)

what the DM should do

(b) Cognitive theory (descriptive)

what the DM really does

experimental tests

When they are the same?

if the (single) DM has all the information (in a deterministic way) and has clearly in mind the criterion (one) of evaluation

ideal problem \(\rightarrow\) point 1
N-1° Principle of INVARIANCE

Equivalent (from the logical point of view) versions of the same problem **must** produce the same choice

**Examples**
- Change names or positions for the options
- Change measure units
- Add a constant value for all the results

**Counterexamples**
- Lotteries (cases A, B, C)
- Ellsberg paradox (1961)
Lotteries (case A and case B)

Better A1 or A2?
- Better ...

Better B1 or B2?
- Better ...
Lotteries (case C)

But notice that ...

Better C1 or C2?

C1 → sum of A1 and B2
C2 → sum of A2 and B1
Ellsberg

Now you have a second chance (after the ball is re-inserted)

White ball win

Better to take from A or B?

better ...

ambiguity aversion

Black ball win

Better to take from A or B?

better ...

ambiguity aversion?
Cognitive theory: a first principle

Principle of NON NEUTRALITY

The aggregation of (decisional) options is not a neutral operation!

Given the two preferences on A1 and B2, it is not guaranteed that their aggregation (C1) is the preferred one.

- Caution: do not combine too easily the options
- Normally, the ambiguity is avoided, “even if this is not rational” (Ellsberg)
N-2° Principle of **DOMINANCE**

If the DM prefers A with respect to B in every scenario (or context or state of nature) the choice must be A.

**Examples**
- I prefer to be missionary (with respect to engineer) in peace and prefer to be missionary (...) in war
- I prefer chicken with respect to beef (when there is nothing else) and I prefer chicken ... also when there is fish

**Counterexamples (see in next lessons)**
- Extraction (Tversky, Kahneman, 1986) → see the following sl.
- The possible choices in uncertainty conditions (with the DM risk attitude)
Extraction (in two conditions)

### room 1

<table>
<thead>
<tr>
<th>n. of balls</th>
<th>situation A</th>
<th>situation B</th>
</tr>
</thead>
<tbody>
<tr>
<td>90 white</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6 red</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>1 green</td>
<td>30</td>
<td>45</td>
</tr>
<tr>
<td>1 blue</td>
<td>-15</td>
<td>-10</td>
</tr>
<tr>
<td>2 yellow</td>
<td>-15</td>
<td>-15</td>
</tr>
</tbody>
</table>

**Better A or B?**

**better ...**

### room 2

<table>
<thead>
<tr>
<th>n. of balls</th>
<th>situat. C</th>
<th>situat. D</th>
<th>n. of balls</th>
</tr>
</thead>
<tbody>
<tr>
<td>90 white</td>
<td>0</td>
<td>0</td>
<td>90 white</td>
</tr>
<tr>
<td>6 red</td>
<td>45</td>
<td>45</td>
<td>7 red</td>
</tr>
<tr>
<td>1 green</td>
<td>30</td>
<td>-10</td>
<td>1 green</td>
</tr>
<tr>
<td>3 yellow</td>
<td>-15</td>
<td>-15</td>
<td>2 yellow</td>
</tr>
</tbody>
</table>

**Better C or D?**

**better ...**

*but C ≡ A and D ≡ B*
**Choice (in two conditions)**

<table>
<thead>
<tr>
<th></th>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>w5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Invest</strong></td>
<td>0</td>
<td>45</td>
<td>30</td>
<td>-15</td>
<td>-15</td>
</tr>
<tr>
<td><strong>Build</strong></td>
<td>0</td>
<td>45</td>
<td>45</td>
<td>-10</td>
<td>-15</td>
</tr>
<tr>
<td><strong>p(w)</strong></td>
<td>.90</td>
<td>.06</td>
<td>.01</td>
<td>.01</td>
<td>.02</td>
</tr>
</tbody>
</table>

**presentation 1**

Better or Build ?

<table>
<thead>
<tr>
<th></th>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>w5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Invest</strong></td>
<td>0</td>
<td>45</td>
<td>30</td>
<td>-15</td>
<td>-15</td>
</tr>
<tr>
<td><strong>Build</strong></td>
<td>0</td>
<td>45</td>
<td>45</td>
<td>-10</td>
<td>-15</td>
</tr>
<tr>
<td><strong>p(w)</strong></td>
<td>.90</td>
<td>.06</td>
<td>.01</td>
<td>.03</td>
<td></td>
</tr>
</tbody>
</table>

**presentation 2**

Better or Build ?
Cognitive theory: three more principles

C-2° Principle of EVIDENCE
The dominance among options should be obvious

C-3° Principle of ASYMMETRY
Possibility of losing K is more important than winning K

C-4° Principle of COMPACTNESS
An aggregated option (A) has an importance less than the sum of the importances of the single sub-options (A₁,A₂)

\[ \pi(A) < \pi(A₁) + \pi(A₂) \]
Normative theory: principles & (counter)examples

**N-3°**

**Principle of TRANSITIVITY**

If the decision prefers A over B and B over C, then A **must** be preferred over C

**Examples:**
- V. Rossi is better than Stoner, and Stoner is better than Melandri, so ...
- Buying emission units (Kyoto prot.) is better than cutting the production, and cutting the production is better than not respecting the emission constraints, so ...

**Counterexamples:**

A new car + accessories

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>ob1</td>
<td>50</td>
<td>55</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>ob2</td>
<td>50</td>
<td>55</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>ob3</td>
<td>50</td>
<td>55</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>ob4</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

B > A
C > B
D > C

D > A ?

or rather
the options are incomparable ?

standard 10.000€
+air cond. 1.000€
+alloy rims 1.000€
+...

(but finally ...)
Cognitive theory: progression vs. crash

**Principle of CRASH**

The DM is (relatively) indifferent to small progressive changes, but at some point (s)he becomes aware of the (large) gap and ...

Cognitive theory: estimation

**Principle of OVER/UNDER-ESTIMATION**

There is an inclination to

- over-estimate events with small probability
- under-estimate events with high probability (except in case of certainty)

Asymmetry in dealing with subjective probability
A famous example: the frame effect

- Avian influenza (possible death)
- Group at risk: 600 people

Protocol A
- 200 people will survive

Protocol B
- with $p = \frac{1}{3}$, 600 will survive
- with $p = \frac{2}{3}$, nobody will survive

Better A or B?

Protocol A
- 400 people will die

Protocol B
- with $p = \frac{1}{3}$, nobody will die
- with $p = \frac{2}{3}$, 600 will die

Better A or B?

- Aversion to the risk in case of winnings (better A)
- Propensity for risk in case of losses (better B)
(ii) Experiments: axioms of probability theory

A1 - Probability \( p(e) \) of an event (e): value between 0 (impossible) and 1 (certain)

A2 - Complementary probability (the event does not occur): \( 1 - p(e) \)

A3 - For events \( (e_1, e_2, \ldots, e_k) \) that are mutually exclusive: \( p(e_1 \text{ OR } \ldots \text{ OR } e_k) = p(e_1) + \ldots + p(e_k) \)

A4 - For 2 independent events \( (e_1, e_2) \): \( p(e_1 \text{ AND } e_2) = p(e_1) \times p(e_2) \)

A5 - For 2 non-independent events \( (e_1, e_2) \):

\[
p(e_1/e_2) = \frac{p(e_1 \text{ AND } e_2)}{p(e_2)} = \frac{p(e_2/e_1) \times p(e_1)}{p(e_2)}
\]

conditional probability

(Bayes, 1763)

Example follows
Probabilities before and after the experiment (Bayes)

$\omega_1 = \text{good weather}$
$\omega_2 = \text{bad weather}$

Before experiment:

$\begin{array}{cc}
\omega_1 & \omega_2 \\
.80 & .20 \\
\end{array}$

$p(\omega)$

$\begin{array}{ccc}
y_1 & y_2 & y_3 \\
.55 & .25 & .20 \\
\end{array}$

$p(y)$

$\begin{array}{cc}
\omega_1 & \omega_2 \\
.50 & .05 \\
.20 & .05 \\
.10 & .10 \\
\end{array}$

$p(\omega, y)$

After experiment:

$\begin{array}{cc}
\omega_1 & \omega_2 \\
.91 & .09 \\
.80 & .20 \\
.50 & .50 \\
\end{array}$

$p(\omega/y)$

$\begin{array}{ccc}
y_1 & y_2 & y_3 \\
.63 & .25 & .12 \\
.25 & .25 & .50 \\
\end{array}$

$p(y/\omega)$

This case does not make much sense.
Uncertainty: the expected value

- If the probability distribution of $\omega$ is available …
- … consider the logic of the expected value (to be maximized)
- In the example $\rightarrow p(\omega_1) = 0.3, \ p(\omega_2) = 0.7$

\[
\begin{array}{cccc}
\omega_1 & x_1 & x_2 & x_3 & x_4 \\
5 & 8 & 1 & 10 \\
6 & 3 & 9 & 2 \\
\end{array}
\]

\[
L_5 = 5.7 4.5 6.6 4.4
\]

- Sometimes also variance is considered (to be minimized)

The expected value logic removes the dependence from the state of nature (if probabilities are available)

\[
\sigma_j^2 = \sum_i \left[ f(\omega_i, x_j) - \bar{f}(x_j) \right]^2 \cdot p(\omega_i)
\]

$i$: state of nature  
$j$: alternative  
$\bar{f}(x_j)$: expected value of alternative $j$

\[
0.21 5.25 13.44 13.44
\]
An example: oil extraction

• A potentially rich area
• States of nature $\omega_1$: oil; $\omega_2$: no oil
• Possible actions $x_1$: buy taking full advantage
  $x_2$: rent for 50 years
  $x_3$: rent for 10 years
  $x_4$: do nothing

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>100</td>
<td>80</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-30</td>
<td>-6</td>
<td>-4</td>
<td>0</td>
</tr>
</tbody>
</table>

**Experiment** (a sample drill):
• $y_1$ = probably there is oil
• $y_2$ = analysis not clear
• $y_3$ = probably there is no oil

\[
p(\omega) = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix} \quad \omega_1 \quad \omega_2
\]

\[
p(\omega,y) = \begin{bmatrix} 0.18 & 0.24 & 0.08 \\ 0.02 & 0.16 & 0.32 \end{bmatrix}
\]

\[
p(y) = \begin{bmatrix} 0.20 & 0.40 & 0.40 \end{bmatrix}
\]

\[
p(\omega/y) = \begin{bmatrix} 0.90 & 0.60 & 0.20 \\ 0.10 & 0.40 & 0.80 \end{bmatrix}
\]

different w.r.t. $p(\omega)$
Decision tree: construction

- The complete tree

- Left part (before [1]) related to the experiment and to its outcome

- Right part (after [1]) related to the choice $x_j$ and to the state of nature $\omega_i$
The final outcome: a strategy

Node labels
- ○ → expected value
- □ → best option

Conclusion:
- do the experiment
- select the following strategy →
  - if $y_1 \rightarrow x_1$
  - if $y_2 \rightarrow x_1$
  - if $y_3 \rightarrow x_2$
Tools for «point 3» problems

(i) Pairwise comparison
(ii) Choquet integral
(i) Pairwise comparison

To obtain the vector $w$ of the weights it is possible to do a set of pairwise comparisons, thus obtaining a matrix $A$.

**Example**

<table>
<thead>
<tr>
<th></th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>1</td>
<td>1/3</td>
<td>2/3</td>
</tr>
<tr>
<td>$w_2$</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$w_3$</td>
<td>3/2</td>
<td>1/2</td>
<td>1</td>
</tr>
</tbody>
</table>

$w_2$ is 3 times more important than $w_1$.

$w_2$ is 2 times more important than $w_3$.

... (etc.)

Matrix $A$ is:

- **positive** $\Rightarrow a_{ij} > 0$
- **reciprocal** $\Rightarrow a_{ij} = 1/a_{ji}$
- **consistent** $\Rightarrow a_{ik} = a_{ij} \cdot a_{jk}$
From matrix A to vector w

If A is consistent

By definition is:

\[ a_{ij} = \frac{a_{ik}}{a_{jk}} \]

\[ a_{ij} = \frac{w_i}{w_j} \]

\[ a_{ik} / a_{jk} = \frac{w_i}{w_j} \]

All the columns represent (with a coeff. of proportionality)
the vector w  easy case!
**Eigenvalues**

- **Matrix A consistent**
  - columns proportional
  - rank of the matrix = 1
  - only one eigenvalue $\lambda_{\text{max}} \neq 0$

- **Matrix A positive**
  - trace = sum of eigenvalues

- **Elements of the diagonal = 1 (all)**
  - trace = n
Vector of the weights

\[ A \cdot x \]

with \( a_{ij} = a_{ik}/a_{jk} = x_i/x_j \) (it is independent by \( k \))

\[ \forall i \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} \frac{x_i}{x_j} x_j = n x_i \]

\[ A \cdot x = n x \]

\( x \) is the main eigenvector

each column is proportional to the eigenvector

vector \( w \) of the weights:

\( w \) is the main eigenvector normalized (sum = 1)
Supporting (spanning) tree

The minimum number of pairwise comparisons is n-1 but only if they are «spanning» the graph.

\[ a_{13} = a_{12} \cdot a_{23} \]
\[ a_{14} = a_{12} \cdot a_{24} \]
\[ a_{34} = a_{32} \cdot a_{24} = \left( \frac{1}{a_{23}} \right) \cdot a_{24} \]
If matrix A is not consistent?

Inconsistencies: \( a_{ik} \neq a_{ij} \cdot a_{jk} \)

**Example**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1/6</td>
<td>1/3</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Matrix is consistent

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>1/3</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Matrix is not consistent

We must estimate the main eigenvector (and the error)

If the consistency error is “small” OK (if no ...)
What about the cons. error $\mu$?

$A*x = \lambda_{max}*x$  \hspace{1cm} \forall i \hspace{1cm} \sum_{j=1}^{n} a_{ij}x_j = \lambda_{max}x_i$ (row i-th of the matrix)

$\forall i \hspace{1cm} \lambda_{max} = \sum_{j=1}^{n} a_{ij} \frac{x_j}{x_i} \sigma_{ij}$ \hspace{1cm} $\Rightarrow \lambda_{max} = \sum_{j=1}^{n} \sigma_{ij}$

Sum of the rows $\rightarrow$  

$\sum_{i=1}^{n} \lambda_{max} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$  

$n\lambda_{max} = n + \sum_{1\leq i<j\leq n} \left( \sigma_{ij} + \frac{1}{\sigma_{ij}} \right)$  

$\sum_{1\leq i<j\leq n} \left( \sigma_{ij} + \frac{1}{\sigma_{ij}} \right) = n(\lambda_{max} - 1)$

Divide the result by $n(n-1)$ and subtract 1

$\mu \Rightarrow \frac{\sum_{1\leq i<j\leq n} \left( \sigma_{ij} + \frac{1}{\sigma_{ij}} \right)}{n(n-1)} - 1 = \frac{n(\lambda_{max} - 1)}{n(n-1)} - 1 \Rightarrow \mu = \frac{\lambda_{max} - n}{n - 1}$

If $A$ consistent: $\lambda_{max} = n \rightarrow \mu = 0$
What happens if the attributes (objectives or criteria) are not mutually independent?

OR

if it is not possible to demonstrate their independence?

The Choquet integral
You have to help a Palio bookmaker. His evaluation concerning the contrada’s chance to win are based on two attributes: values (utilities) of horse and jockey. The situation (utilities) of the four contrada are in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Onda</th>
<th>Bruco</th>
<th>Torre</th>
<th>Selva</th>
</tr>
</thead>
<tbody>
<tr>
<td>horse</td>
<td>100</td>
<td>0</td>
<td>45</td>
<td>30</td>
</tr>
<tr>
<td>jockey</td>
<td>0</td>
<td>100</td>
<td>45</td>
<td>65</td>
</tr>
<tr>
<td>average</td>
<td>50</td>
<td>50</td>
<td>45</td>
<td>47.5</td>
</tr>
</tbody>
</table>

**Bookmaker perception:**

- **a. same weight**
- **b. contrada Torre is the favourite**

Which weight is it possible to assign to the two attributes?
In the utility space ...

- No couple of weights \((\alpha, \beta)\) determines the victory of Torre, the contrada indicated by the bookmaker as the best one.
- It's necessary to change the model ...
MAUT modifications

- Association of a unique value $U$ (utility) to each alternative (among the $n$, finite or infinite, possible alternatives): $U$ expresses the overall satisfaction with respect to the $m$ attributes $t_1, t_2, \ldots t_m$ considered.

- It is necessary to obtain the utility function $U$ on the base of the utilities of each attribute.

- Both comments are true, but it is necessary to take care of:
  (i) synergies,
  (ii) redundance
Example: a student grant

You have to help the commission for an Erasmus grant. The evaluation is based on three attributes, the results of the student in **M** (mathematics), **F** (physics), **L** (literature). The situation is the following.

<table>
<thead>
<tr>
<th></th>
<th>Colorni</th>
<th>Luè</th>
<th>Noce</th>
<th>Lia</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>M</strong> - mathematics</td>
<td>9</td>
<td>5</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td><strong>F</strong> - physics</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td><strong>L</strong> - literature</td>
<td>5</td>
<td>10</td>
<td>7.5</td>
<td>8</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>7.33</strong></td>
<td>7</td>
<td>7.16</td>
<td>7</td>
</tr>
<tr>
<td><strong>Minimum</strong></td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td><strong>Maximum</strong></td>
<td>9</td>
<td>10</td>
<td>7.5</td>
<td>8</td>
</tr>
</tbody>
</table>

The commission (decision maker) says that:
1. criteria **M** and **F** have the same importance (weight)
2. criteria **M** and **F** are more relevant than **L** (1.5 time)
3. criteria **M** and **F** are redundant (a student good in **M** is also ...)
4. students are favorite if they are balanced (synergy **M-L** and **F-L**)
Case of criteria not mutually independent

- It is based on the definition of two elements:
  - a capacity (fuzzy measure)
  - a sum (Choquet integral)

- **Capacity**:
  - if $M = \{1, \ldots, m\}$ is the attributes (criteria) set
  - **capacity** is a function $\mu : 2^M \rightarrow [0, 1]$ such that
    \[
    \mu(\emptyset) = 0 ; \mu(M) = 1 ; \mu(A) \leq \mu(B) \text{ if } A \text{ is included in } B
    \]

- **Choquet integral**:
  - $\mu$ is a capacity $M = \{1, \ldots, m\}$ and $f$ is the function that represents the results (utility) of the alternatives among the different criteria
  - the Choquet integral $C_\mu$ is the sum (with $i=1,\ldots,m$)
    \[
    C_\mu = [f(\sigma_1) - f(\sigma_0)]*\mu(A_1) + \ldots + [f(\sigma_m) - f(\sigma_{m-1})]*\mu(A_m)
    \]
    with $A_i=\{\sigma_i, \sigma_{i+1}, \ldots, \sigma_m\}$ and $\sigma_i$ permutation with $f(\sigma_i)$ ascending
Representation (lattice)

Capacity:

$\mu(\Phi) = 0.00$
$\mu(M) = 0.45$
$\mu(F) = 0.45$
$\mu(L) = 0.30$
$\mu(MF) = 0.50$
$\mu(ML) = 0.90$
$\mu(FL) = 0.90$
$\mu(MFL) = 1.00$

Capacity $\mu(A_i)$ takes into account the commission indications? (synergies and redundancies)
Results

- There are n candidates \((n=4: \text{Colorni, Luè, Noce, Lia})\)
- For each it is necessary to calculate \(C_\mu\) (Choquet integral)
- For each it should be necessary to define a permutation
- It is better to use a graphic scheme (see next slide)
- Each candidate has an ascending order of results
- It is possible to represent it as a path between \(\Phi\) and MFL
- To each node an increment \(\Delta\) is associated (added value)
- To each node a weight is associated (weight is the capacity)
- \(C_\mu\) value is calculated with a weighed sum
Student Colorni

\[ C_\mu (\text{Colorni}) = 5 \times 1.0 + 3 \times 0.5 + 1 \times 0.45 = 6.95 \]
Student Luè

\[ C_\mu (\text{Luè}) = 5 \times 1.0 + 1 \times 0.9 + 4 \times 0.30 = 7.10 \]
Student Noce

$C_\mu (\text{Noce}) = 7 \times 1.0 + 0 \times 0.9 + 0.5 \times 0.3 = 7.15$
\[ C_\mu (\text{Lia}) = 5 \times 1.0 + 3 \times 0.9 + 0 \times 0.3 = 7.7 \]
Final result

- Colorni seemed to be the best candidate
- But we weren't considering the redundancies
- The best one is Lia, thanks to the synergy
- A graph is used for calculating \((lattice \ 2^M)\)
- Increment represents the added value

In this way it is possible to take into account:

- **synergies** → given \(\mu_{ij} > \mu_i + \mu_j\)
- **redundancies** → given \(\mu_{ij} < \mu_i + \mu_j\)
Palio di Siena *(more)*

**Bookmaker’s perception:**
(i) same weights to the attributes  
(ii) Torre is the favourite

The couple horse-jockey makes contrada Torre the favorite for the bookmaker. The weights have to be given: to the 2 attributes and to the combination of these → **how?**

<table>
<thead>
<tr>
<th>Horse</th>
<th>Onda</th>
<th>Bruco</th>
<th>Torre</th>
<th>Selva</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jockey</td>
<td>100</td>
<td>0</td>
<td>45</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>100</td>
<td>45</td>
<td>65</td>
</tr>
</tbody>
</table>

\[
\mu_i = \ldots \\
\mu_j = \ldots \\
\mu_{ij} = \ldots \\
\]

**Synergy** → \( \mu_i + \mu_j < \mu_{ij} \)
The “horse/jockey” factor

C_µ (O) = ..., C_µ (B) = ..., C_µ (T) = ..., C_µ (S) = ...
Tools for «point 4» problems

(i) Two approaches
(ii) Peer evaluation
(i) The two approaches to group decision

- Research of the critical points
- Proposing new/mitigating/compensative measures (from “dividing” to “enlarging the cake”)
- Do "win-win» solutions exist? (game can be not a zero-sum game)

Possible conflict ...
How to manage it?
Create information / 1

Analytic support: calculation of the indices of conflict, based on the distances between decision makers.

- Impacts (*numbers of impacts may not coincide*):
  - distance of each player from the average value of each impact

- Utility funct. → examination of those which do not coincide

- **Weights**: construction of distance D matrix

  \[ D = [d_{ij}], \quad \text{with } d_{ij} \geq 0 \quad (\text{symmetric}?) \]

Distance matrix among [weights vectors of] decision makers
Create information / 2

Individual indices of conflict:
- sum for rows = distance of the row player from the others
- sum for columns = distance of others from the column player

Global indices of conflict:
- number $d_{ij} \neq 0 \rightarrow$ number of different vectors of weights
- average $d_{ij} \rightarrow$ average distance among weight vectors
- max $d_{ij} \rightarrow$ maximum level of conflict among two players

Barycentric solution:
- vector at the minimum distance from the vectors of the others
The two approaches

\[ M_3 (A,C,D) \]

\[ M_2 (A,C) \] → \[ M_1 (A) \] → \[ M_2 (A,D) \] → \[ M_3 (A,C,D) \]

DISTILLATION

MAASTRICHT

\( A \) = alternatives

\( C \) = criteria

\( D \) = dec. makers

\( M_i \rightarrow \ i \) is the matrix dimension

multicriteria analysis

ranking vectors
Distillation

**Problem definition**

Player 1

- Problem definition
- Evaluation matrix
- Utility function
- Weights

Agreement

Player 2

- Problem definition
- Evaluation matrix
- Utility function
- Weights

Agreement

Final decision
Distillation: compromise research

- Cooperative approach: trust building

- Decision makers move to barycentric position
  - synchronous method → together
  - a-synchronous method → the first is the most critical decision maker

- For each step:
  - information about global conflict (global conflict index)
  - information about the most critical decision maker (individual conflict index)
Distillation: to the barycenter

Weighted barycentric vector

We calculate the distances between the components of the vectors of the weights of each decision maker and the components of the weighted barycentric vector.

Weights dec. maker 1

Weights dec. maker 2

... (others)
Maastricht

Decision maker 1

Multi criteria analysis (or other ...):
Sorting creation 1

Decision maker 2

... 

Decision maker n

Multi criteria analysis (or other ...):
Sorting creation n

Conflict management

Final decision
(ii) Peer evaluation

Have all the decision makers the same importance?

Weights determined a priori:
- a meta-decision maker exists;
- he has a weight proportional to the number of people that he represents.

Weights determined by the group itself → cross-check, peer evaluation:
- (1) average method
- (2) eigenvector method
  - player can assign a weight to himself,
  - player must assign weights just to other players.
1. Average method

Player **Di** can vote to himself

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>(w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>0.80</td>
<td>0.10</td>
<td>0.10</td>
<td>0.333</td>
</tr>
<tr>
<td>D2</td>
<td>0.05</td>
<td>0.70</td>
<td>0.30</td>
<td>0.350</td>
</tr>
<tr>
<td>D3</td>
<td>0.15</td>
<td>0.20</td>
<td>0.60</td>
<td>0.317</td>
</tr>
</tbody>
</table>

Vectors of the weights expressed by each player (column sum = 1)

Average vectors of the weights of the players

Inconsistent \(\rightarrow\) players have the same relevance!
### 2. Eigenvector method

Player Di can vote to himself

<table>
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<td>0.10</td>
</tr>
<tr>
<td>D2</td>
<td>0.05</td>
<td>0.70</td>
<td>0.30</td>
</tr>
<tr>
<td>D3</td>
<td>0.15</td>
<td>0.20</td>
<td>0.60</td>
</tr>
</tbody>
</table>

Vectors of the weights expressed by each player (column sum = 1)

\[ \mathbf{S} \mathbf{w} = \mathbf{w} \]

\[ \mathbf{S} \mathbf{w} = \lambda \mathbf{w} \quad \text{(principal eigenvalue = 1)} \]

Vectors of the weights of the players

\[
\begin{align*}
\text{w} & = \begin{bmatrix} 0.330 \\ 0.363 \\ 0.307 \end{bmatrix}
\end{align*}
\]
A paradox

Rule (now) → player $D_i$ can not give a weight to himself

Possible solutions:
- each player has to vote at least for two players;
- 0 can not be used (weights $\geq$ predetermined $\varepsilon$).

L’assioma del cazzone

http://en.wikipedia.org/wiki/Perron-Frobenius_theorem

D1  D2  D3  D4
D1  0   0   0   0
D2  0   0   0   1
D3  0   0   0   0
D4  1   1   1   0

D1 and D3 have no preferences (> 0), so D1 and D3 p. of view are not relevant !!!
Veto → United Nations Security Council

15 members, 5 can veto
(USA, Russia, China, France, Great Britain)

Rule → a resolution is approved if:
   (i) gets at least 9 votes,
   (ii) there is no veto (from 1 of the 5).

How to determinate the weight of the members
(USA, Russia, China, … , D1, D2, … , D9, D10)
and coalition threshold in order to simulate
UN Security Council working process  ???
Service design
Green Move

Objective: design & test an electric car-sharing system in Milan

Coordinating a 2½ years project financed by the Lombardia Region (5 millions €), involving 8 research centers of Politecnico di Milano

Outcome:
- the design of a full scale service
- a trial with a limited number of docking stations in Milan

Switch from “buy a vehicle” paradigm to “buy mobility services”
How to design/formalize the service?

Problem characteristics:

- **different actors** and stratification of governance levels, e.g. public administration (municipality, region), associations, …
- **uncertainty** in the definition of the variables, e.g. future policies for urban mobility, travel demand estimation for a non-existing service
- **conflicting criteria**, e.g. costs vs territorial coverage (such as in BikeMI)
- structuring the problem itself is an issue, e.g. definition of the configuration options to be evaluated is a key issue (Hull and Tricker, 2005; Kelly et al., 2008; Jones et al., 2009)

How to formalize the complexity?
Casual maps & MCDA

Integration of causal maps with multi-criteria analysis:

- a powerful way of capturing decision-makers’ views
- widely used in problem-structuring
  \cite{Rosenhead2004}
- model the \textbf{effects of a link}
  (qualitative or quantitative methods)
- definition of \textbf{aggregation rules}
  qualitative $\rightarrow$ experts
  quantitative models (e.g. demand analysis)
A (partial) map for GM

Parameters defining the service

Evaluation criteria

Partial map for the design of a vehicle sharing service
Parameters

1. Type of vehicle (EV, hybrid, ICE)
2. Service area
3. Capillarity and intermodality
4. Spatial flexibility → 1w-2w
5. Flexibility of service → temporal flexibility of booking
6. Fare:
   6.1. modes (hourly, km)
   6.2. function type (concave, convex, linear)
   6.3. level (high, medium, low)
7. Economic incentives (parking, congestion tax, LPT)
8. Incentives for service (areas with traffic restrictions, lanes ...)
9. Re-allocation model
10. Mechanisms for promotion and marketing
Indicators

After the setting up of different options (alternatives), they can be evaluated and compared thanks to adequate indicators.

**Evaluation and performance indicators** (to measure the achievement of the objectives of stakeholders):

- Net Present Value
- $\Delta$ km traveled on the network
- $\Delta$ greenhouses gases
- $\Delta$ polluting emission
- Modal shift to sustainable mobility
- Number of users
- Connection in the social network
- Space occupied by parking
- Accessibility indicators
- ....
Flow chart

Parameters → Cause–effect chain → Indicators → Objectives → Stakeholders

- LPT company
- Local authorities
Conclusions (part 2)
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Thank you