EXPLOITATION OF A ROUGH APPROXIMATION OF THE OUTRANKING RELATION IN MULTICRITERIA CHOICE AND RANKING

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Abstract. Given a finite set A of actions evaluated by a family of criteria, we consider a preferential information in the form of a pairwise comparison table (PCT) including pairs of actions from a subset $B \subseteq A \times A$ described by graded preference relations on particular criteria and a comprehensive outranking relation. Using the rough set approach to the analysis of the PCT, we obtain a rough approximation of the outranking relation by a graded dominance relation. Decision rules derived from this approximation are then applied to a set $M \subseteq A$ of potential actions. As a result, we obtain a four-valued outranking relation on set M. The construction of a suitable exploitation procedure in order to obtain a recommendation for multicriteria choice and ranking is an open problem within this context. We propose an exploitation procedure that is the only one satisfying some desirable properties.

Keywords. Rough sets, multicriteria decision making, four-valued outranking, exploitation procedures.

1 Introduction

A rough set approach to multicriteria decision analysis has been proposed by Greco, Matarazzo and Slowinski (1996). This methodology operates on a pairwise comparison table (PCT) (Greco, Matarazzo and Slowinski, 1995), including pairs of actions described by graded preference relations on specific criteria and by a comprehensive preference relation. It builds up a rough approximation of the comprehensive preference relation using graded dominance relations. Furthermore, some decision rules in the "if ... then..." form are derived from the rough approximation of the preference relation. If the comprehensive preference relation is an outranking relation, the application of these decision rules to a set of actions gives a four-valued outranking relation (Tsoukias and Vincke, 1995, 1997), i.e. a binary relation which, with respect to any pair of actions (a,b), characterizes the proposition "a is at least as good as b" as true, contradictory, unknown or false. Finally, in order to obtain a recommendation (Roy, 1993) for the decision problem at hand, a suitable exploitation procedure of the four-valued outranking relation should be applied. This paper, which is a reduced version of Greco, Matarazzo, Slowinski and Tsoukias (1997), is focused on this exploitation procedure. More precisely, we consider multicriteria ranking and choice problems, and we propose an exploitation procedure, called scoring procedure, which we characterize by proving that it is the only one ensuring some desirable properties.

The paper is structured as follows. In section 2, we introduce the rough approximation of a preference relation and the generation of decision rules. In section 3, we describe the four-valued outranking relation. In section 4, we introduce the application of decision rules, showing how it defines a four-valued outranking relation. Furthermore, the scoring procedure is presented. Section 5 proposes a characterization of this scoring procedure. Section 6 groups conclusions.

2 Rough set analysis of a preferential information

2.1 Pairwise Comparison Table

In order to represent preferential information provided by the decision maker (DM) in form of a pairwise comparison of some actions, we shall use a pairwise comparison table, introduced in Greco, Matarazzo and Slowinski (1995).

Let A be a finite set of actions (feasible or not), considered by the DM as a basis for exemplary pairwise comparisons. Let also C be the set of criteria (condition attributes) describing the actions.

For any criterion $q \in C$, let T_q be a finite set of binary relations defined on A on the basis of the evaluations of actions of A with respect to the considered criterion q, such that $\forall (x,y) \in A \times A$ exactly one binary relation $t \in T_q$ is verified. More precisely, given the domain V_q of $q \in C$, if $v'_q, v''_q \in V_q$ are the respective evaluations of $x,y \in A$ by means of q and $(x,y) \in t$ with $t \in T_q$, then for each $w,z \in A$ having the same evaluations v'_q, v''_q by means of q, $(w,z) \in t$. For interesting applications it should be $card(T_q) \geq 2$, $\forall q \in C$.

Furthermore, let T_d be a set of binary relations defined on A (comprehensive pairwise comparisons) such that at most one binary relation $t \in T_d$ is verified, $\forall (x,y) \in A \times A$.

The *pairwise comparison table* (PCT) is defined as an information table $S_{PCT} = \langle B, C \cup \{d\}, T_C \cup T_d, g \rangle$, where $B \subseteq A \times A$ is a non-empty *sample of pairwise comparisons*, $T_C = \bigcup_{q \in C} T_q$, d is a decision corresponding to the comprehensive pairwise comparison (comprehensive preference binary relation), and $g: B \times (C \cup \{d\}) \to T_C \cup T_d$ is a total function such that $g[(x,y),q] \in T_q$ $\forall (x,y) \in A \times A$ and $\forall q \in C$, and $g[(x,y),d] \in T_d$, $\forall (x,y) \in B$. It follows that for any pair of actions

$(x,y)\in B$ one and only one binary relation $t\in T_d$ is verified. Thus, T_d induces a partition of B. In fact, information table S_{PCT} can be seen as decision table, since the set of considered criteria C and decision d are distinguished.

In this paper, we consider S_{PCT} related to the choice and ranking problems (Roy, 1985) and assume that the exemplary pairwise comparisons provided by the DM can be presented in terms of *graded preference binary relations*:

$$T_q = \{ P_q^h, h\in H_q\},$$

where $H_q = \{h\in Z: h\in[-p_q, r_q]\}$ and $p_q, r_q\in N$, $\forall q\in C$ and $\forall(x,y)\in A\times A$;

- $x P_q^h y$, $h>0$, means that action x is preferred to action y by degree h with respect to criterion q,

- $x P_q^h y$, $h<0$, means that action x is not preferred to action y by degree h with respect to criterion q,

- $x P_q^0 y$ means that x is similar (asymmetrically indifferent) to y with respect to criterion q.

Let us remark that the similarity represented by the binary relation P_q^0 has been introduced by Slowinski and Vanderpooten (1995,1996, 1998) in very general terms, i.e. without any specific reference to preference modeling. Let us remember that a *similarity* relation is only *reflexive* (i.e., with respect to P_q^0, we have $x P_q^0 x$ $\forall x\in A$ and $\forall q\in C$), relaxing therefore the properties of symmetry and transitivity. The abandon of the transitivity requirement is easily justifiable, remembering – for example – Luce's paradox of the cups of tea (1956). As for the symmetry, one should notice that yRx, which means "y is similar to x", is directional; there is a *subject* y and a *referent* x, and in general this is not equivalent to the proposition "x is similar to y", as maintained by Tversky (1977). This is quite immediate when the similarity relation is defined in terms of a percentage difference between evaluations of the actions compared on the attribute at hand, calculated with respect to the evaluation of the referent action. In terms of preference modeling, similarity relation, even if not symmetric, resembles indifference relation. Thus, in this context, we also call this similarity relation "asymmetric indifference".

Of course, $\forall x,y\in A$

$$[x P_q^h y,\ h\geq 0] \Leftrightarrow [y P_q^k x,\ k\leq 0].$$

Therefore, $\forall(x,y),(w,z)\in A\times A$ and $\forall q\in C$:

- if $x P_q^h y$ and $w P_q^k z$, $k\geq h\geq 0$, then w is preferred to z not less than x is preferred to y with respect to criterion q;

- if $x P_q^h y$ and $w P_q^k z$, $k\leq h\leq 0$, then w is not preferred to z not less than x is not preferred to y with respect to criterion q.

The set of binary relations T_d is defined analogously; however, $x P_d^h y$ means that x is comprehensively preferred to y by degree h.

2.2 Rough approximation of a preference relation

Let $H_P = \bigcap_{q \in P} H_q$ $\forall P \subseteq C$. Given $x,y \in A$, $P \subseteq C$ and $h \in H_P$, we say that x *positively dominates* y by degree h with respect to the set of criteria P iff $x P_q^f y$ with $f \geq h$, $\forall q \in P$. Analogously, $\forall x,y \in A$, $P \subseteq C$ and $h \in H_P$, x *negatively dominates* y by degree h with respect to the set of criteria P iff $x P_q^f y$ with $f \leq h$, $\forall q \in P$. Thus, $\forall h \in H_P$, every $P \subseteq C$ generates two binary relations (possibly empty) on A, which will be called *P-positive-dominance of degree h*, denoted by D_{+P}^h, and *P-negative-dominance of degree h*, denoted by D_{-P}^h, respectively. The relations D_{+P}^h and D_{-P}^h satisfy the following properties:

(P1) if $(x,y) \in D_{+P}^h$, then $(x,y) \in D_{+R}^k$, for each $R \subseteq P$ and for every $k \leq h$;

(P2) if $(x,y) \in D_{-P}^h$, then $(x,y) \in D_{-R}^k$, for each $R \subseteq P$ and for every $k \geq h$.

In the following, we consider a PCT where the decision d can have only two values on $B \subseteq A \times A$:

1) x outranks y, which will be denoted by xSy or $(x,y) \in S$,

2) x does not outrank y, which will be denoted by $xS^c y$ or $(x,y) \in S^c$,

where “x outranks y” means “x is at least as good as y” (Roy, 1985). Let us remember that the minimal property verified by the outranking relation S is reflexivity (see Roy, 1991; Bouyssou, 1996).

We propose to approximate the binary relation S by means of the D_{+P}^h binary dominance relations. Therefore, S is seen as a *rough binary relation* (see Greco, Matarazzo and Slowinski, 1995).

The P-lower approximation of S, denoted by $\underline{P}\, S$, and the P-upper approximation of S, denoted by $\overline{P}\, S$, are defined, respectively, as:

$$\underline{P}\, S = \bigcup_{h \in H_P} \left\{ \left(D_{+P}^h \cap B \right) \subseteq S \right\},$$

$$\overline{P}\, S = \bigcap_{h \in H_P} \left\{ \left(D_{+P}^h \cap B \right) \supseteq S \right\}.$$

Taking into account property (P1) of the dominance relations D_{+P}^h, $\underline{P}\, S$ can be viewed as the dominance relation D_{+P}^h which has the largest intersection with B included in the outranking relation S, and $\overline{P}S$ as the dominance relation D^h_{+P} including S which has the smallest intersection with B.

Analogously, we can approximate S^c by means of the D^h_{-P} dominance relations:

$$\underline{P}S^c = \bigcup_{h \in H_P} \left\{ \left(D^h_{-P} \cap B \right) \subseteq S^c \right\},$$

$$\overline{P}S^c = \bigcap_{h \in H_P} \left\{ \left(D^h_{-P} \cap B \right) \supseteq S^c \right\}.$$

Taking into account property (P2), the interpretation of $\underline{P}S^c$ and $\overline{P}S^c$ is similar to the interpretation of $\underline{P}S$ and $\overline{P}S$.

2.3 Decision rules

We can derive a generalized description of the preferential information contained in a given PCT in terms of decision rules.

We will consider the following kinds of decision rules:

1) D_{++}-decision rule, being a statement of the type: $x D^h_{+P} y \Rightarrow xSy$;

2) D_{+-}-decision rule, being a statement of the type: *not* $x D^h_{+P} y \Rightarrow xS^c y$;

3) D_{-+}-decision rule, being a statement of the type: *not* $x D^h_{-P} y \Rightarrow xSy$;

4) D_{--}-decision rule, being a statement of the type: $x D^h_{-P} y \Rightarrow xS^c y$.

Speaking about decision rules we will simply understand all the four kinds of decision rules together.

If there is at least one pair $(w,z) \in B$ such that $w D^h_{+P} z$ and wSz, and there is no $(v,u) \in B$ such that $v D^h_{+P} u$ and $vS^c u$, then $x D^h_{+P} y \Rightarrow xSy$ is accepted as a D_{++}-decision rule. A D_{++}-decision rule $x D^h_{+P} y \Rightarrow xSy$ will be called *minimal* if there is not any other rule $x D^k_{+R} y \Rightarrow xSy$ such that $R \subseteq P$ and $k \leq h$. Let us observe that, since each decision rule is an implication, a minimal decision rule represents an implication such that there is no other implication with an antecedent at least of the same weakness and a consequent of at least the same strength. The other rules can be characterized analogously.

Theorem 2.1. (Greco, Matarazzo, Slowinski, 1996). If

1) x D^h_{+P} y$\Rightarrow$xSy is a minimal D_{++}-decision rule, then $\underline{P}S = D^h_{+P} \cap B$,

2) x D^h_{-P} y$\Rightarrow$xS^{c}y is a minimal D_{--}-decision rule, then $\underline{P}S^c = D^h_{-P} \cap B$,

3) *not* x D^h_{+P} y$\Rightarrow$xS^{c}y is a minimal D_{+-}-decision rule, then $\overline{P}S = D^h_{+P} \cap B$,

4) *not* x D^h_{-P} y$\Rightarrow$xSy is a minimal D_{-+}-decision rule, then $\overline{P}S^c = D^h_{-P} \cap B$.

3 Four-valued outranking

The basic idea of the four-valued outranking model of preferences (Tsoukias and Vincke, 1995, 1997) is connected with the search of "positive reasons" and "negative reasons" (xSy and xS^{c}y) supporting a hypothesis of the truth of a comprehensive outranking relation for an ordered pair (x,y) of actions. The combination of presence and absence of the positive and the negative reasons creates four possible situations for the outranking:

1) *true outranking*, denoted by xS^{T}y, iff there exist sufficient positive reasons to establish xSy and there do not exist sufficient negative reasons to establish xS^{c}y;
2) *contradictory outranking*, denoted by xS^{K}y, iff there exist sufficient positive reasons to establish xSy and sufficient negative reasons to establish xS^{c}y;
3) *unknown outranking*, denoted by xS^{U}y, iff there do not exist sufficient positive reasons to establish xSy and there do not exist sufficient negative reasons to establish xS^{c}y;
4) *false outranking*, denoted by xS^{F}y, iff there do not exist sufficient positive reasons to establish xSy and there exist sufficient negative reasons to establish xS^{c}y.

By such definitions it is possible to apply the rough approximations of outranking relations S and S^c defined on B, in order to build a preference model on M$\times$M, where M$\subseteq$A, which could further be exploited to get a recommendation (choice or ranking) with respect to a set of actions from M. In other words, we are able to move from a descriptive model of decision maker's preferences expressed on B to a prescriptive model on M$\subseteq$A.

4 Application of decision rules and definition of a final recommendation

Given a set D of decision rules, obtained in the way described in section 2, and two actions v,u$\in$A,

1) if x D^h_{+P} y$\Rightarrow$xSy is a D_{++}-decision rule and v D^h_{+P} u, then we conclude that vSu,

2) if *not* x D^h_{+P} y$\Rightarrow$xS^cy is a D_{+-}-decision rule and *not* v D^h_{+P} u, then we conclude that vS^cu,

3) if *not* x D^h_{-P} y$\Rightarrow$xSy is a D_{-+}-decision rule and *not* v D^h_{-P} u, then we conclude that vSu,

4) if x D^h_{-P} y$\Rightarrow$xS^cy is a D_{--}-decision rule and v D^h_{-P} u, then we conclude that vS^cu.

According to the four-valued logic, from the application of the decision rules to the pair of actions (x,y)$\in$A$\times$A there may arise one of the four following states:

- *true outranking*, denoted by xS^Ty: this is the case when there exists at least one D_{++}-decision rule and/or at least one D_{-+}-decision rule stating that xSy, and no D_{--}-decision rule or D_{+-}-decision rule stating that xS^cy;

- *false outranking*, denoted by xS^Fy: this is the case when there exists at least one D_{--}-decision rule and/or at least one D_{+-}-decision rule stating that xS^cy, and no D_{++}-decision rule or D_{-+}-decision rule stating that xSy;

- *contradictory outranking*, denoted by xS^Ky: this is the case when there exists at least one D_{++}-decision rule and/or at least one D_{-+}-decision rule stating that xSy, and at least one D_{--}-decision rule and/or at least one D_{+-}-decision rule stating that xS^cy;

- *unknown outranking*, denoted by xS^Uy: this is the case when there is no D_{++}-decision rule or D_{-+}-decision rule stating that xSy, and no D_{--}-decision rule or D_{+-}-decision rule stating that xS^cy.

Theorem 4.1. (Greco, Matarazzo, Slowinski, 1996) The application of all the decision rules obtained for a given S_{PCT} on any pair of actions (v,u)$\in$A$\times$A results in the same outranking relation as obtained by the application of the minimal decision rules only.

From Theorem 4.1, we conclude that the set of all decision rules is completely characterized by the set of the minimal rules. Therefore, only the latter ones are presented to the DM and applied in the decision problem at hand.

In order to define a recommendation with respect to the actions of M$\subseteq$A, we can calculate a particular score based on the outranking relations S and S^c obtained from the application of these rules to the actions of M.

$\forall$M$\subseteq$A and $\forall$x$\in$M, let

- $M^{++}(x)$ ={y$\in$M-{x}: there is at least one D_{++} -decision rule and/or at least one D_{-+}-decision rule stating that xSy},

- $M^{+-}(x)=\{y\in M-\{x\}$: there is at least one D_{++}-decision rule and/or at least one D_{-+}-decision rule stating that $ySx\}$,
- $M^{-+}(x)=\{y\in M-\{x\}$: there is at least one D_{+-}-decision rule and/or at least one D_{--}-decision rule stating that $yS^{c}x\}$,
- $M^{--}(x)=\{y\in M-\{x\}$: there is at least one D_{+-}-decision rule and/or at least one D_{--}-decision rule stating that $xS^{c}y\}$.

To each $x\in M$ we assign a *score*

$$S(x,M) = S^{++}(x,M) - S^{+-}(x,M) + S^{-+}(x,M) - S^{--}(x,M)$$

where $S^{++}(x,M)=card[M^{++}(x)]$, $S^{+-}(x,M)=card[M^{+-}(x)]$, $S^{-+}(x,M)=card[M^{-+}(x)]$, $S^{--}(x,M)=card[M^{--}(x)]$.

We can use this score to work out a recommendation in the ranking and choice problems. For the ranking problem, S(x,M) establishes a total preorder on M. For choice problems, the final recommendation is $x^{*}\in M$ such that $S(x^{*},M)=\max_{x\in M} S(x,M)$. We call these exploitation procedures *scoring procedures*.

5 A characterization of the scoring procedure

The use of a score-based procedure in presence of a four-valued outranking relation is a problem which goes beyond the exploitation of rough approximations (see Tsoukias and Vincke, 1997). For this reason we start with some general remarks concerning the use of such procedures.

We want also to stress that such procedures are not the only possibility when four-valued outranking relations have to be exploited. Moreover, the reader may notice that the use of the score, as defined in this paper, conceals the difference between uncertainty due to contradictions (contradictory outranking S^{K}) and uncertainty due to lack of information (unknown outranking S^{U}) contributing in the same manner to the score S(x,M). However, in our opinion, any exploitation procedure results in a loss of information since it reduces the rich form of knowledge contained in the outranking relations (S and S^{c}) to a poorer one which is the final choice or ranking. In favor of the scoring procedure play its intuitive nature (it is easy to understand by decision makers), its clear and straightforward characterization (as it will be demonstrated in the following) and its easiness in implementation. In other words, we sacrifice some richness of the information to the easiness of use.

5.1 Some previous results

The scoring procedure proposed in the previous section can be considered as an extension to the four-valued logic of the well-known Copeland ranking and choice method (see Goodman, 1954; Fishburn, 1973).

These procedures have been characterized by Rubinstein (1980) and Henriet (1985) and, with respect to valued binary relations, by Bouyssou (1992a and b).

In this subsection we remember synthetically the results of Bouyssou, while in the following subsection we extend them to the four-valued outranking relation.

A valued (binary) outranking relation on A is a function R associating an element of [0,1] with each ordered pair of actions $(a,b)\in A\times A$, with $a\neq b$. Let $\mathit{R}(A)$ be the set of all valued binary relations on A and 2^A the set of all non-empty subsets on A. A *ranking method* (RM), denoted by $\geq$, is a function assigning a ranking $\geq(M,R)$ on $M\subseteq A$ to any valued relation $R\in \mathit{R}(A)$ and to any (non-empty) $M\subseteq A$. A *choice function* (CF) on A is a function

$$C: 2^A\times \mathit{R}(A)\rightarrow 2^A$$

such that $C(M,R)\subseteq M$, for each $M\in 2^A$ and $R\in \mathit{R}(A)$.

The following properties of ranking and choice exploitation procedure are considered (Bouyssou, 1992a and b):

1) *strong monotonicity*: an exploitation procedure is strongly monotonic iff it responds in the right direction to a modification of R. More formally,

1a) RM $\geq$ is strongly monotonic iff $\forall a,b\in M\subseteq A$ and $\forall R\in \mathit{R}(A)$

$$a\geq(M,R)b \Rightarrow a >(M,R')b,$$

where $>(M,R)$ is the asymmetric part of $\geq(M,R)$ and R' is identical to R except that $R(a,c)<R'(a,c)$ or $R(c,a)>R'(c,a)$ for some $c\in M-\{a\}$;

1b) a CF C is strongly monotonic iff $\forall R\in \mathit{R}(A)$ and all $M\in 2^A$

$$a\in C(M,R) \Rightarrow \{a\} = C(M,R')$$

where R' is defined as previously.

2) *neutrality*: an exploitation procedure is neutral iff it does not discriminate between actions just because of their labels. More formally,

2a) a RM $\geq$ is neutral iff for all permutations σ on A, $\forall R\in \mathit{R}(a)$ and $\forall a,b\in M\subseteq A$

$$a\geq(M,R)b \Leftrightarrow \sigma(a)\geq(\sigma(M),R^\sigma)\sigma(b)$$

where R^σ is defined by $R^\sigma(\sigma(a), \sigma(b))=R(a,b)$ $\forall a,b\in A$;

2b) a CF C is neutral iff for all permutations σ on A, $\forall R\in \mathit{R}(a)$ and $\forall M\in 2^A$

$$a\in C(M,R) \Leftrightarrow \sigma(a)\in C(\sigma(M),R^\sigma).$$

3) *independence of circuits*: a circuit of length q in a digraph is an ordered collection of arcs $(u_1, u_2, ..., u_q)$ such that for i=1, 2, ...,q, the initial extremity of u_i is the final extremity of u_{i-1} and the final extremity of u_i is the initial extremity of u_{i+1}, where u_0 is interpreted as u_q and u_{q+1} as u_1. A circuit is elementary iff each node being the extremity of one arc in the circuit is the extremity of exactly two arcs in the circuit. A transformation on an elementary circuit consists of adding the same quantity to the value of all the arcs in the circuit. A transformation on an elementary circuit is admissible if all the transformed valuations are still between 0

and 1. An exploitation procedure is independent of circuits iff its results do not change after an admissible transformation of R. More formally,

3a) a RM $\geq$ is independent of circuits iff $R,R'\in \mathcal{R}(A)$, R' is obtained from R through an admissible transformation on an elementary circuit of length 2 or 3 and $\forall a,b\in M\subseteq A$

$$a\geq(M,R)b \Rightarrow a\geq(M,R')b;$$

3b) a CF C is independent of circuits iff $\forall M\in 2^A$ and $\forall R,R'\in \mathcal{R}(A)$, such that R' is obtained from R through an admissible transformation on an elementary circuit of length 2 or 3 on M,

$$C(M,R)=C(M,R').$$

The property of independence of circuits makes an explicit use of the cardinal properties of the valuations R(a,b). This is not the case of the neutrality and monotonicity (Bouyssou, 1992a and b).

Given $R\in \mathcal{R}(A)$ and $M\subseteq A$, a net flow $S_{NF}(x,M,R)$ can be associated to each $x\in M$ as follows:

$$S_{NF}(x,M,R) = \sum_{b\in M-\{x\}} (R(x,b) - R(b,x)).$$

More specifically, the RM $\geq$ such that

$$a\geq(M,R)b \text{ iff } S_{NF}(a,M,R)\geq S_{NF}(b,M,R)$$

is called *net flow ranking method*, and the CF C such that

$$C(M,R) = \{a\in M: S_{NF}(a,M,R)\geq S_{NF}(b,M,R)\ \forall b\in M\}.$$

is called *net flow choice method*.

Theorem 5.1. (Bouyssou 1992a). The net flow method is the only RM that is neutral, strongly monotonic and independent of circuits.

Theorem 5.2. (Bouyssou 1992b). The net flow method is the only CF that is neutral, strongly monotonic and independent of circuits.

5.2 Properties of the exploitation procedures for the four-valued outranking

In order to characterize the scoring procedure we consider a four-valued outranking relation as a function R_{4V} associating an element of $\{S^T, S^U, S^K, S^F\}$ with each ordered pair of actions $(a,b)\in A\times A$. Now, RM $\geq$ and CF C are defined analogously for a four-valued outranking relation, i.e., for $\mathcal{R}_{4v}(A)$ being the set of all possible four-valued relations on A, RM $\geq$ is a function assigning a ranking $\geq(M,R_{4v})$ on $M\subseteq A$ to any $R_{4v}\in \mathcal{R}_{4v}(A)$ and to any $M\subseteq A$, and CF C on A is a function

$$C: 2^A\times \mathcal{R}_{4v}(A)\rightarrow 2^A$$

such that $C(M,R_{4v})\subseteq M$, for each $M\in 2^A$ and each $R_{4v}\in \mathcal{R}_{4v}(A)$.

Moreover, the property of neutrality maintains the same formulation as in the exploitation procedure for the valued outranking relation, i.e.

- a RM $\geq$ is neutral iff for all permutations σ on A, $\forall M\subseteq A$, $\forall R_{4v}\in \mathcal{R}_{4v}(A)$ and $\forall a,b\in M$

$$a\geq(M,R_{4v})b \Leftrightarrow \sigma(a) \geq(\sigma(M),R^{\sigma}_{4v})\sigma(b)$$

- a CF C is neutral iff for all permutations σ on A, $\forall M\in 2^A$ and $\forall R_{4v}\in \mathcal{R}_{4v}(A)$

$$a\in C(M,R_{4v}) \Leftrightarrow \sigma(a)\in C(\sigma(M),R^{\sigma}_{4v})$$

where for any permutation σ and $\forall a,b\in A$, R^{σ}_{4v} is defined by

$$R^{\sigma}_{4v}(\sigma(a),\sigma(b))=R_{4v}(a,b).$$

Instead, the strong monotonicity and the independence of circuits properties have a formal definition which is slightly different from the previous definition and requires some new concepts.

A *4v-transformation* on the pair $(a,b)\in A\times A$ consists of changing the outranking relation S^X into the outranking relation S^Y, where $S^X, S^Y\in\{S^T, S^U, S^K, S^F\}$, and it is denoted by

$$aS^Xb \rightarrow aS^Yb.$$

Let us denote by $S^X \rightarrow S^Y$ the class of all the transformations $aS^Xb \rightarrow aS^Yb$ with $(a,b)\in A\times A$ and $S^X, S^Y\in\{S^T, S^U, S^K, S^F\}$.

Let T be the set of all 4v-transformations on the pairs $(a,b)\in A\times A$. We introduce an equivalence binary relation E on T. More specifically,

$$[aS^Xb \rightarrow aS^Yb]\ E\ [aS^Wb \rightarrow aS^Zb]$$

means that the transformation $[aS^Xb \rightarrow aS^Yb]$ has the same "strength" as the transformation $[aS^Wb \rightarrow aS^Zb]$, where $S^X,S^Y,S^W,S^Z\in\{S^T, S^U, S^K, S^F\}$.

We define the following *equivalence classes* for E:

1. $E^0=(S^T\rightarrow S^T)\cup(S^F \rightarrow S^F)\cup(S^U \rightarrow S^U)\cup(S^K \rightarrow S^K)\cup(S^K \rightarrow S^U)\cup(S^U \rightarrow S^K)$, i.e. the class of the transformations from an outranking S^X to an outranking S^Y of the same strength;
2. $E^1=(S^U \rightarrow S^T)\cup(S^K \rightarrow S^T)\cup(S^F \rightarrow S^U)\cup(S^F \rightarrow S^K)$, i.e. the class of the transformations from an outranking S^X to an outranking S^Y having a greater strength;
3. $E^{-1}=(S^T \rightarrow S^U)\cup(S^T \rightarrow S^K)\cup(S^U \rightarrow S^F)\cup(S^K \rightarrow S^F)$, i.e. the class of the transformations from an outranking S^X to an outranking S^Y having a weaker strength;

4. $E^2=(S^F \rightarrow S^T)$, i.e. the class of the transformation from an outranking S^X to an outranking S^Y having a far greater strength (from total absence of outranking to sure presence of outranking);

5. $E^{-2}=(S^T \rightarrow S^F)$, i.e. the class of the transformation from an outranking S^X to an outranking S^Y having a far weaker strength (from sure presence of outranking to total absence of outranking).

Within the context of a four-valued outranking relation,

1'a) a RM $\geq$ is strongly monotonic iff $\forall M \subseteq A$ and $\forall a,b \in M$

$$a \geq (M,R_{4v}) b \Rightarrow a > (M,R'_{4v}) b$$

where $>(M,R_{4v})$ is the asymmetric part of $\geq(M,R_{4v})$ and R'_{4v} is identical to R_{4v} except that R'_{4v} is obtained from R_{4v} by means of a 4v-transformation $aS^Xc \rightarrow aS^Yc$ with $(S^X \rightarrow S^Y) \subset E^1 \cup E^2$ or $cS^Xa \rightarrow cS^Ya$ with $(S^X \rightarrow S^Y) \subset E^{-1} \cup E^{-2}$ for some $c \in M-\{a\}$;

1'b) CF C is strongly monotonic iff $\forall M \in 2^A$ and $R_{4v} \in \mathcal{R}_{4v}(A)$

$$a \in C(M, R_{4v}) \Rightarrow \{a\} = C(M, R'_{4v})$$

where R'_{4v} is defined as previously.

A 4v-transformation on an elementary circuit consists of performing a 4v-transformation of the same equivalence class in the arcs of the circuit. A 4v-transformation on an elementary circuit is admissible if all the transformed outranking relations belong to the set $\{S^T, S^U, S^K, S^F\}$; e.g., if we have aS^Tb, bS^Uc, cS^Ta, an admissible transformation on the elementary circuit $\{(a,b), (b,c), (c,a)\}$ is aS^Ub, bS^Fc, cS^Ka. Let us point out that the elementary transformation on the arcs are $aS^Tb \rightarrow aS^Ub$, $bS^Uc \rightarrow bS^Fc$, $cS^Ta \rightarrow cS^Ka$. Therefore a RM $\geq$ is independent of circuits if $R_{4v}, R'_{4v} \in \mathcal{R}_{4v}(A)$, R'_{4v} being obtained from R_{4v} through an admissible transformation on an elementary circuit and

$$a \geq (M,R_{4v}) b \Rightarrow a \geq (M,R'_{4v}) b.$$

Analogously, a CF C is independent of elementary circuits iff, under the same hypotheses, $\forall M \in 2^A$ and $\forall R_{4v}, R'_{4v} \in \mathcal{R}_{4v}(A)$

$$C(M,R_{4v}) = C(M,R'_{4v}).$$

Let us remark that the four-valued outranking R_{4v} expresses some possible preference situations without using any numerical evaluation. Therefore, the property of independence of circuits makes no use of cardinal properties of the relations, similarly to the property of neutrality and monotonicity.

5.3 An extension of the previous results to the four-valued outranking

To extend the results of Bouyssou (1992a and b), we associate an element of {0, 1/2, 1} with each $(a,b)\in A\times A$ introducing the valued outranking binary relation $\hat{R}_{4v}: A\times A\rightarrow[0,1]$ by stating:

$$\hat{R}_{4v}(a,b)=\begin{cases} 0 & \text{if } \; aS^F b \\ 1/2 & \text{if } \; aS^U b \;\text{ or }\; aS^K b \\ 1 & \text{if } \; aS^T b. \end{cases}$$

This is a reduction to the [0,1] interval of the lattice of the four truth values, where the values S^U and S^K are incomparable (no numerical value is used there). Such a reduction could be judged arbitrary, but the following result shows that $\hat{R}_{4v}$ satisfies some desirable properties, allowing us to say that $\hat{R}_{4v}$ is the only valued relation which faithfully represents R_{4v}. Let us consider F: $\{S^T, S^U, S^K, S^F\}\rightarrow[0,1]$. From each $R_{4v}\in \mathcal{R}_{4v}(A)$ we can obtain one $R\in \mathcal{R}(A)$ by stating $R(a,b)= F(R_{4v}(a,b))$ $\forall(a,b)\in A\times A$.

Let us consider the following properties $\forall(a,b),(c,d)\in A\times A$:

R1) $F(R_{4v}(a,b))=1$ iff $aS^T b$,

R2) $F(R_{4v}(a,b))=0$ iff $aS^F b$,

R3) $F(R^1_{4v}(a,b))-F(R^2_{4v}(a,b))=F(R^3_{4v}(c,d))-F(R^4_{4v}(c,d))$ iff $aS^X b$ according to R^1_{4v}, $aS^Y b$ according to R^2_{4v}, $cS^W d$ according to R^3_{4v}, $cS^Z d$ according to R^4_{4v} and

$$[aS^X b \rightarrow aS^Y b] \; E \; [cS^W d \rightarrow cS^Z d].$$

Property R1) says that $\forall(a,b)\in A\times A$ the transformation of the four-valued outranking R_{4v} into the valued outranking R should give the maximum value, i.e., $R(a,b)=1$, iff $aS^T b$. Analogously, property R2) says that, $\forall(a,b)\in A\times A$, the same transformation should give the minimum value, i.e., $R(a,b)=0$, iff $aS^F b$. Finally, property R3) says that, if 4v-transformations $S^X\rightarrow S^Y$ and $S^W\rightarrow S^Z$ are of the same strength, then we should have $F(S^X)-F(S^Y)= F(S^W)-F(S^Z)$.

Theorem 5.3. (Greco, Matarazzo, Slowinski, Tsoukias, 1997) Properties R1), R2) and R3) are satisfied if and only if

$$F(R_{4v}(a,b)) = \hat{R}_{4v}(a,b).$$

Lemma 5.1. (Greco, Matarazzo, Slowinski, Tsoukias, 1997) The following relation between the overall score S(x,M) and the net flow $S_{NF}(x,M,\hat{R}_{4v})$ holds:

$$S(x,M) = 2\, S_{NF}(x,M,\hat{R}_{4v}), \;\; \forall M\subseteq A \text{ and } \forall x\in M.$$

Lemma 5.1 shows that the overall score S(x,M) is a strictly positive monotonic transformation of the net flow $S_{NF}(x,M,\hat{R}_{4v})$. Therefore, we conclude that the ranking and the choice obtained from S(x,M) are the same as those obtained from $S_{NF}(x,M,\hat{R}_{4v})$.

Lemma 5.2. (Greco, Matarazzo, Slowinski, Tsoukias, 1997) Given $R_{4v},R'_{4v}\in\mathcal{R}_{4v}(A)$, if R'_{4v} is obtained from R_{4v} by an admissible 4v-transformation on an elementary circuit, then $\hat{R}'_{4v}$ is obtained from $\hat{R}_{4v}$ by an admissible transformation on an elementary circuit.

Due to Lemmas 5.1 and Lemma 5.2, Theorems 5.1 and 5.2 imply, respectively, the following two theorems (Greco, Matarazzo, Slowinski, Tsoukias, 1997).

Theorem 5.4. With respect to a four-valued outranking relation established by a set of decision rules, the scoring procedure based on S(x,M) is the only RM which is neutral, strongly monotonic and independent of circuits.

Theorem 5.5. With respect to a four-valued outranking relation established by a set of decision rules, the scoring procedure based on S(x,M) is the only CF which is neutral, strongly monotonic and independent of circuits.

6 Conclusions

We have been using the rough set approach to the analysis of preferential information concerning multicriteria choice and ranking problems. This information is given by a decision maker as a set of pairwise comparisons among some reference actions using the outranking relation. The outranking relation is approximated by means of a special form of dominance relation and decision rules are derived from these approximations. They represent the preference model of the decision maker. In result of application of these rules to a new set of potential actions, we get a four-valued outranking relation.

In this paper, we dealt with the problem of obtaining a recommendation from the above four-valued outranking relation. With this aim we proposed an exploitation procedure for ranking and choice problems based on a specific net flow score. Furthermore, we proved that this procedure is the only one which is neutral, strongly monotonic and independent of circuits.

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