Chapter 1

FROM CONCORDANCE / DISCORDANCE TO THE MODELLING OF POSITIVE AND NEGATIVE REASONS IN DECISION AIDING

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Abstract The principle of concordance / discordance was introduced by B. Roy in his very early work on Multiple Criteria Decision Analysis. Although such a principle is grounded by strong evidence from real life decision situations, the way in which it has been implemented in existing MCDA methods allows only for its partial and limited use. Indeed, the principle lacks a theoretical frame enabling a more general use in decision analysis. The paper presents a possible generalisation of this principle under the concepts of positive and negative reasons. For this purpose, a new formalism, (a four valued logic) is suggested. Under such a formalism the concordance test is seen as the evaluation of the existence of positive reasons supporting the sentence *"x is at least as good as y"*, while the discordance test can be viewed as the evaluation of the existence of negative reasons against the same sentence. A number of results obtained in preference modelling and aggregation shows the potentiality of this approach.

1. INTRODUCTION

Consider a Parliament. The government has the support of the majority of seats, although not a very strong one. Suppose now that a law on a very sensitive issue (such as education, religion, national defence, minority rights etc.) is introduced for discussion by the government. Several political, social and ethical issues are involved. Suppose finally that the opposition strongly mobilises, considering that the law is a major attack against "something". Massive demonstrations are organised, an aggressive media campaign is pursued etc.. It is quite reasonable that the government will try to find a compromise on some aspects of the law in order to improve its "acceptability". Note, however, that such a compromise concerns aspects argued by the minority and not the majority.

Which decision rule is the government using to choose an appropriate law proposal in such a situation? *A law proposal x is considered "better" than proposal y iff it meets the majority will and does not mobilise the minority aversion.* It should be observed that the minority is considered here as an independent decision power source. Such a "decision rule" is a regular practice in all mature democracies. Although the minority does not have the power to impose its political will, it has the possibility of expressing a "veto", at least occasionally. Such a "negative power" may not necessarily be codified somewhere, but is accepted. Actually, it is also a guarantee of the democratic game. When the present majority becomes a minority it will be able to use the same "negative power".

Consider now the Security Council of the United Nations. Here, a number of nations are officially endowed with a veto power such that resolutions taken with a majority of votes (even the highest ones) can be withdrawn if such a veto is used. We observe that in this case the decision rule *"x is better than y if it is the case for the majority and no veto is used against x"* is officially adopted. Again we observe that the countries having a veto power do not have a "positive power" (impose a decision), but only a "negative" one.

Finally, consider the very common situation where the faculty has to deliberate on the admission of candidates to a course (let's say a management course). Then consider two candidates: the first, x, having quite good grades, systematically better than the second, y, but with a very bad grade in management science; then candidate y, who is systematically worse than x, but has an excellent grade in management science. Several faculty members will claim that, although candidate y is not better than candidate x, it is also difficult to consider x better than y due to their inverse quality concerning the key class of the course, management science. The same faculty members will also claim that the two candidates cannot be considered indifferent because they are completely different. These members are intuitively adopting the same decision rule as in the previous two examples: *candidate x is better than candidate y iff (s)he has a majority of grades in (her)his favour and is not worse in* *a number of key classes.* For an extensive discussion on the question of grades in decision support see Bouyssou et al., 2000.

If we consider a class grade of a candidate as (her)his value on a criterion, the reader will observe that in the above decision rule there exist criteria having a "negative power". Such a "negative power" is not compensated by the "positive power" of the majority of criteria. It acts independently and only in a negative sense.

We could continue with several other real life examples going from vendor rating to bid selection and loan allowance. In all such cases it is frequent to find the intuitive decision rule: *alternative x is better than alternative y iff there is a majority of "reasons" supporting x wrt to y and there is no strong opposition to x wrt to y.*

In order to be more formal we will use a large preference relation of the type "x is at least as good as y" (denoted $S(x, y)$, also known as "outranking" relation) such that:

$$S(x, y) \iff C(x, y) \wedge \neg D(x, y) \tag{1.1}$$

where:
$C(x, y)$ means there is a majority of reasons supporting x wrt to y;
$D(x, y)$ means there is a strong opposition to x wrt to y;
$\wedge$ and $\neg$ being the conjunction and negation operators respectively.

We use the predicate $C(x, y)$ in order to verify a *concordance test* concerning x wrt to y and the predicate $D(x, y)$ in order to verify a *discordance test* concerning x wrt to y.

As we saw, this is a widely used empirical decision rule. The legitimate questions are: how can such a rule be used in a decision support method? Under which conditions can it be applied and what type of results should we expect? On which theoretical grounds can such a rule be formalised as a general principle?

In this paper we will try to contribute to the discussion on the above questions. Section 2 introduces, in general terms, the methods adopting the concordance / discordance principle in the area of Multiple Criteria Decision Analysis. Such methods are well known under the name of *outranking methods*. A critical discussion on a number of problems arising from such methods is introduced in this section. Then section 3 suggests a generalisation of the concordance /discordance principle under the *positive and negative reasons* approach. Such an approach suggests a general frame under which different problems of preference modelling and aggregation can be viewed. In this section we introduce a number of theoretical results based on the use of new formalisms extending the expressive power of first order languages. Several open questions are also introduced.

2. CONCORDANCE / DISCORDANCE IN MCDA

2.1 CRISP OUTRANKING RELATIONS

The use of the concordance / discordance principle in decision support methods dates back to the seminal paper of Roy, 1968, where it was first introduced, beginning the well known ELECTRE family of the Multiple Criteria Decision Analysis methods (see Roy, 1991).

The idea is simple. Consider formula 1.1.If we are able to associate a criterion on which alternatives can be compared to each "reason" (for the concept of criterion and of coherent family of criteria under this perspective, see Roy and Bouyssou, 1993; Vincke, 1992b), then $C(x, y)$ represents the existence of a "significant" coalition of criteria for which "x is at least as good as y" and $D(x, y)$ represents the existence of a "significant opposition" against this proposition. To give an example, due to Roy, 1968, we can use the following definitions:

$$C(x,y) \iff \frac{\sum_{j \in J^S_{xy}} w_j}{\sum_j w_j} \geq \gamma \tag{1.2}$$

$$D(x,y) \iff \exists g_j : g_j(y) - g_j(x) > v_j \tag{1.3}$$

where:

$g_j,\ j = 1, \ldots, n$ are the criteria, to be maximised;
w_j are importance coefficients associated to each criterion;
J^S_{xy} represents the set of criteria for which x is at least as good as y; more precisely, $J^S_{xy} = \{j \in \{1, \ldots, n\}, g_j(y) - g_j(x) \leq q_j\}$ where q_j is the indifference threshold attached to criterion g_j
γ is a majority threshold;
v_j is a veto threshold on criterion j.

In this case a sufficiently strong, let us say positive, coalition is any subset of criteria of which the sum of the importance coefficients is at least γ. A sufficiently strong, let's say negative, coalition is any single criterion provided it is endowed with veto power. The relation S is better known as an "outranking relation" (see also Ostanello, 1985; Vincke, 1999). A large part of the so-called "outranking methods" is based on this principle with a number of possible variations, since $C(x, y)$ and $D(x, y)$ can be defined using a large variety of formulas.

Besides such variations in the definition of C and/or D it should be noted that various sophistications of classical *concordance and non-discordance* rules have been proposed to extend their ability to discriminate or their descriptive power. In this respect, let us observe that:

- defining concordance and discordance tests in terms of all or nothing conditions is not always adequate (see *e.g.* Perny and Roy, 1992). As we shall see in the next subsection, in some situations, it is worthwhile to consider a "concordance" and a "discordance" index for each ordered pair of alternatives, opening the way to the establishment of a "valued outranking relation".

- representing the strength of coalitions of criteria by an additive and/or decomposable measure is not necessarily adequate. As shown in Fargier and Perny, 2001; Grabisch and Perny, 2001, in some situations, preferences require non-additivity to be representable by a concordance rule.

Readers aware of social choice theory will recognise in the above formula a variation of a Condorcet type majority rule. From such a perspective it should be noted that:

- the binary relation S defined in this way can only be guaranteed to be reflexive (on this point see Bouyssou, 1996);

- in other terms the relation S is not an ordering relation (neither completeness nor transitivity can be guaranteed) and, from an operational point of view, can be of little help on its own;

- from the above reasons it appears necessary, once the relation S is established, to use a so-called "exploitation procedure", which is an algorithm that transforms such a relation into an ordering relation (at least a partial order).

Concerning "exploitation procedures" and more generally outranking methods, see Vanderpooten, 1990; Vincke, 1992a; Bouyssou, 1992a; Bouyssou, 1992b; Bouyssou and Perny, 1992; Bouyssou and Pirlot, 1997; Marchant, 1996; Pirlot, 1995, for a more detailed and formal discussion. We are not going to further analyse the so-called outranking methods, although we will briefly discuss three remarks corresponding to important research directions concerning such methods.

1. It is clear that the importance parameters and the concordance and discordance thresholds are strongly related (see also Roy and

Mousseau, 1996). Actually both concepts are used in order to establish what the coalitions of criteria enabling to confirm the sentence "x is at least as good as y" are or to confirm its negation. In fact, consider a three criteria setting where the importance parameters are fixed at $w_1 = 0.45$, $w_2 = 0.35$ and $w_3 = 0.2$. If we fix the concordance threshold at 0.7, it is equivalent to claiming that only criteria c_1 and c_2 can form a winning positive coalition (except for unanimity), therefore both c_1 and c_2 are strictly necessary for such coalitions. If we fix the concordance threshold to 0.6, it is equivalent to claiming that the winning positive coalitions now include (except for the previous ones) the one formed by c_1 and c_3. Only criterion c_1 is strictly necessary now. Therefore, such parameters are just convenient numerical representations of a more complex issue concerning the "measurement" of the strength of each coalition of criteria with respect to the sentence "x is at least as good as y".

2. All MCDA methods based on the use of "outranking relations" are based on a two step procedure: the first establishing the outranking relation itself through any of its many variants and the second transforming the outranking relation into an ordering relation. Up to now, there is no way of establishing whether a specific formula of outranking must correspond to a specific form of "exploitation procedure". Any combination of the two steps appears legitimate provided it satisfies the requirements of the decision process and the client's concerns.

3. Recently Bouyssou and Vincke, 1997; Bouyssou et al., 1997; Pirlot, 1997; Bouyssou and Pirlot, 1999; Greco et al., 2001, showed that the precise way by which the outranking relations are defined can be seen as an instance of non transitive, non additive conjoint measurement. More has to be done in this direction, but a unifying frame with other approaches in MCDA is now possible and has to be thoroughly investigated.

2.2 FUZZY OUTRANKING RELATIONS

The concordance test defined by (1.2) relies on a simple definition of the set J^S_{xy} of criteria concordant with the assertion *x is at least as good as y*. It supposes implicitly that we are able to decide clearly whether a criterion is concordant with the assertion or not. As recalled above, a criterion g_j is considered concordant with respect to proposition $S(x, y)$ iff the score difference $g_j(y) - g_j(x)$ does not exceed a (indifference) threshold q_j. However, fixing a precise value for q_j is not easy and the concordance test (1.2) can be artificially sensitive to modifications of criterion values, especially when the scale of criterion g_j is continuous (see Perny and Roy, 1992 and Perny, 1998 for a precise discussion on this topic). A useful solution to overcome this difficulty was proposed by B. Roy a long time ago (see Roy, 1978). The idea was to define a concordance index $C_j(x, y)$, valued in the unit interval, and defined from the quantities $g_j(x)$ and $g_j(y)$ for each criterion g_j and each pair (x, y). By convention, $C_j(x, y) = 1$ means that the criterion g_j is fully concordant with the assertion $S(x, y)$ whereas $C_j(x, y) = 0$ means that criterion g_j is definitely not concordant with this assertion. There is, of course, the possibility of considering intermediate values between 1 and 0 which makes the construction more expressive. It leaves room for a continuum of intermediary situations between concordance and non-concordance. As an example, we recall the definition of concordance indices proposed by Roy in the Electre III method (Roy, 1978) :

$$C_j(x, y) = \frac{p_j(g_j(x)) - \min\{g_j(y) - g_j(x), p_j(g_j(x))\}}{p_j(g_j(x)) - \min\{g_j(y) - g_j(x), q_j(g_j(x))\}} \tag{1.4}$$

where:

- q_j is an indifference threshold, which is a real-valued function such that, for any pair of alternatives (x, y), $q_j(g_j(x))$ is the maximal value of a score difference of type $g_j(y) - g_j(x)$ that could be compatible with indifference between x and y;

- p_j is a preference threshold, which is a real-valued function such that, for any pair of alternatives (x, y), $p_j(g_j(y))$ is the minimal positive value of a score difference of type $g_j(x) - g_j(y)$ that could be compatible with the preference of x over y. The condition $\forall z \in \mathbb{R},\ q_j(z) < p_j(z)$ is assumed.

Such a concordance index is pictured in figure 1.1.

Note that similar ideas apply to defining concordance indices with respect to a strict preference $P(x, y)$ (see Brans and Vincke, 1985) and to indifference $I(x, y)$ (Perny, 1998). In any case, the concordance index can be interpreted as the membership degree of criterion j to the concordant coalition J^S_{xy} (or J^P_{xy}, J^I_{xy}).

Coming back to outranking relations, the concordant coalition with respect to $S(x, y)$ must be seen as a *fuzzy subset* of $\{1, \ldots, n\}$ characterised by membership function $\mu_{J^S_{xy}}(j) = C_j(x, y)$. Thus, the concordance test must be modified to take this sophistication into account. Two main ideas were suggested by Roy:

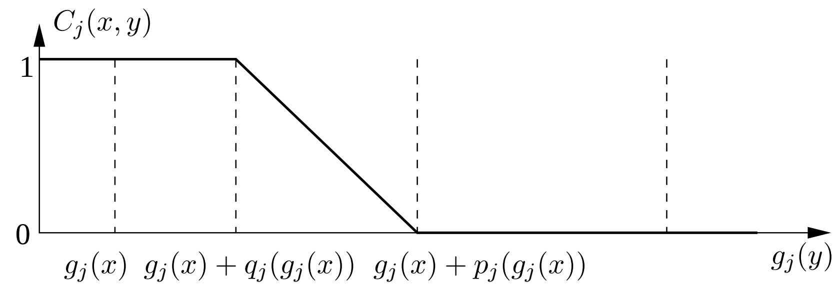

Figure 1.1 Valued outranking indices in Electre III

1. adapting the Electre I concordance test (1.2) so as to use concordance indices. A simple solution derived from the Electre IS method (Roy and Skalka, 1984) is given by the following concordance test:

$$C(x,y) \iff \frac{\sum_{j \in J^S_{xy}} w_j C_j(x,y)}{\sum_j w_j} \geq \gamma \qquad (1.5)$$

2. interpreting the concordance test in a multi-valued logic. This is the option used in Electre III (Roy, 1978). This amounts to defining the level to which the concordance test is fulfilled. Consistently with the previous propositions the truth value $c(x,y) \in [0,1]$ returned by the concordance test can be defined, for example, by:

$$c(x,y) = \frac{\sum_{j \in J^S_{xy}} w_j C_j(x,y)}{\sum_j w_j} \qquad (1.6)$$

The same ideas apply to the discordance test whose role is to check whether some criteria are strongly conflicting with the proposition $S(x,y)$. The classical test, with the veto threshold, is not always convenient. This is particularly true when the criterion scale is continuous; it does not seem appropriate to declare that a given criterion g_j should have a right of veto over $S(x,y)$ when $g_j(y) - g_j(x) > v_j(g_j(x))$ but should entirely lose this right as soon as the inequality no longer holds. A continuous transition seems preferable. For this reason, discordance indices measuring the extent to which criterion j is strongly opposed to a statement $S(x,y)$ were introduced in (Roy, 1978).

$$D_j(x,y) = \min\left(1, \max\left(0, \frac{g_j(y) - g_j(x) - p_j(g_j(x))}{v_j(g_j(x)) - p_j(g_j(x))}\right)\right) \quad (j = 1, \ldots, n)$$

where $p_j(x) < v_j(x)$ for all x.

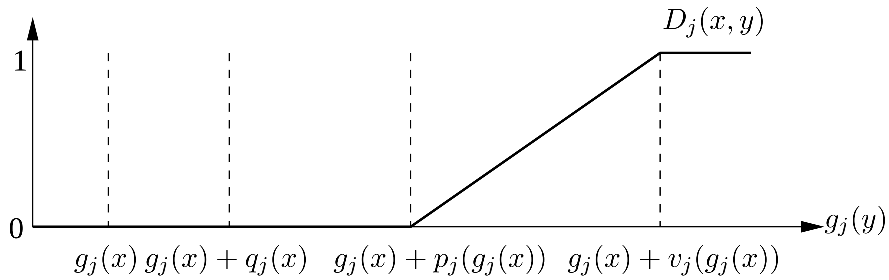

Figure 1.2 The discordance index $D_j(x,y)$ in the Electre III method

Thus, the discordant coalition can also be seen as a fuzzy subset of $\{1,\ldots,n\}$ characterised by the membership function $\mu_{J^D_{xy}}(j) = D_j(x,y)$. The discordance test must therefore be modified to take this sophistication into account. Consistently with the concordance tests introduced above, two main ideas can be put forward:

1. adapting the Electre I and Electre III concordance test (1.2) so as to use discordance indices. A simple solution inspired by the Electre III method (Roy, 1978) is given by the following discordance test:

$$D(x,y) \iff 1 - \prod_{j \in J^{D\gamma}_{xy}} (1 - D_j(x,y)) > \delta \tag{1.7}$$

   where $J^{D\gamma}_{xy} = \{j \in \{1,\ldots,n\}, D_j(x,y) > \gamma\}$ and $\gamma, \delta \in (0,1)$ are the overall concordance and discordance thresholds respectively. Note that the test is defined in such a way that the presence of at least one fully discordant criterion g_j (such that $D_j(x,y) = 1$) is sufficient to make the discordant test positive;

2. interpreting the discordance test in a multi-valued logic. This is the option implicitly used in Electre III (Roy, 1978). This amounts to defining the level to which the discordance test is fulfilled. Consistent with the previous proposition the truth value $d(x,y) \in [0,1]$ returned by the discordance test can be defined, for example, by:

$$d(x,y) = 1 - \prod_{j \in J^{D\gamma}_{xy}} (1 - D_j(x,y)) \tag{1.8}$$

Note that this formulation avoids possible discontinuities due to the use of the cutting threshold δ.

When the concordance test is (1.5) and the discordance test is (1.7) the construction of the outranking relation S is obviously defined by equation (1.1). When the concordance test is (1.6) and the discordance test is (1.8) the equation (1.1) must be interpreted in a multivalued logic. This leads to defining the overall outranking index $s(x, y) \in [0, 1]$ for any pair (x, y) of alternatives as a non-decreasing function of $c(x, y)$ and a non-increasing function of $d(x, y)$. As an example, B. Roy uses the following equality in Electre III:

$$s(x, y) = c(x, y)(1 - d(x, y)) \tag{1.9}$$

The reader is referred to Perny and Roy, 1992 and Perny, 1998 for a more general and systematic construction of outranking relations in the framework of fuzzy set theory.

2.3 PROBLEMS

The use of the so-called outranking methods in MCDA is now largely acknowledged and several empirical validations may be found in the literature (see Roy and Bouyssou, 1993; Vincke, 1992b; Bouyssou et al., 2000). It is nevertheless possible to note a number of significant open questions.

- The definition of "outranking" makes use of a concordance and a non-discordance test, which both have to be verified in order to establish that the outranking relation holds. If any of the two tests fails for a given ordered pair of alternatives, the conclusion is that the outranking relation does not hold for this ordered pair. However, the reader can note that there is a big semantic difference between a situation where a majority of criteria supports that "x is at least as good as y", but there is a veto and a situation where there is neither majority nor veto.

  In other words, when comparing two alternatives x and y the use of the concordance / discordance principle introduces four different epistemic situations:

  - concordance and non-discordance;
  - concordance and discordance;
  - non-concordance and non-discordance;
  - non-concordance and discordance.

but only two valuations are possible (either the outranking relation holds or it does not).

- The definition of the overall outranking relation, at least as it usually appears in outranking methods, implicitly imposes that the criteria to be aggregated should at least be weak orders.

  If the preference models of the criteria to be aggregated are Pseudo-orders (preference structures allowing a numerical representation using thresholds), there is no way to use such a specific information in the establishment of the outranking relation. Only in the case where the outranking relation is a fuzzy binary relation is it possible to use the specific information included in pseudo orders when these are represented as fuzzy relations themselves (see Roy, 1991). If the preference models of the criteria under aggregation are partial orders then it is possible that the absence of preference or indifference at the single criterion level could lead to a "non outranking", not due to conflicting preferences, but due to ignorance. There is however no way to distinguish such situations.

- As already mentioned by Vincke, 1982, each preference aggregation step leads to a result which is (from a relational point of view) poorer than the original information. This is obvious, since the aggregation procedure eliminates some information. Moreover, as already reported by Bouyssou, 1996, an outranking relation is not necessarily a complete relation (not even a partial order). From this point of view, there is a problem if such an approach has to be used in presence of a hierarchy of criteria. If at each layer we keep the result of the aggregation as it is and then we aggregate at the next layer, we will very soon obtain an (almost) empty relation. On the other hand, if at each layer, after aggregation, we transform the outranking relation into a weak order (so that we can correctly apply the aggregation procedure again), we introduce a bias in each aggregation step the consequences of which are unknown. While in usual situations of decision support the use of an exploitation procedure can be discussed with the client, this is not possible in a hierarchical aggregation problem and the above problem can become severe.

From the above discussion it is clear that the principle of concordance / discordance, as it is applied in the so-called outranking methods, can be used locally (only in preference aggregation). On the other hand, it cannot be applied for broader classes of modelling purposes since it lacks a sufficient abstraction level. Besides the above criticism, it should be noted that there is no single-criterion preference model based on the principle of concordance / discordance.

3. POSITIVE AND NEGATIVE REASONS

The discussion in the previous section cannot conceal the fact that the concordance / discordance principle is based on a solid empirical ground. When comparing two alternatives under one or more criteria we are often led to consider what is “for” and what is “against” a preference among the two alternatives separately. Quite often “for” is not the complement of what is “against” and vice-versa. It is quite difficult to justify a preference by just saying “there is nothing against it”. When decisions have been elaborated, the concordance / discordance principle is in fact deeply rooted in common sense. We therefore claim that it is not the principle itself that has to be argued, but the way in which it has been implemented up to now.

We will hereafter present a general approach trying to improve the abstraction level of such a principle. The idea is very simple. When comparing two alternatives consider the “positive reasons” (which may support a preference) and the “negative reasons” (which may be against the preference) independently. If these “positive” and “negative” reasons can be modelled in a formal way, such an approach will lead to a general preference model which can be used at any moment of the decision aiding process: single criterion preference modelling, preference aggregation, measurement, classification etc.. For this purpose it will be necessary to introduce a specific formalism. The following is based on results published in Tsoukiàs and Vincke, 1995; Tsoukiàs and Vincke, 1997; Tsoukiàs and Vincke, 1998; Tsoukiàs and Vincke, 2001; Perny and Tsoukiàs, 1998; Ngo The and Tsoukiàs, 2001.

3.1 THE FORMALISM

Hereafter, we briefly present the basic concepts of the logic formalism we use in the paper. The basic property of such a logic is to explicitly represent situations of hesitation due either to lack of information (missing or uncertain) or to excess information (ambiguous or contradictory). A detailed presentation of the DDT logic can be found in Tsoukiàs, 1996.A detailed presentation of the continuous extension of DDT introduced at the end of the subsection can be found in Perny and Tsoukiàs, 1998.

The DDT Logic. The DDT logic, which is a four-valued first order language, is based on a net distinction between the “negation” (which represents the part of the universe verifying the negation of a predicate) and the "complement" (which represents the part of the universe which does not verify a predicate) since the two concepts do not necessarily coincide. The four truth values represent four epistemic states of an agent towards a sentence (α) that is:

- α *is true (t):* there is evidence that it is true and there is no evidence that it is false;
- α *is false (f):* there is no evidence that it is true and there is evidence that it is false;
- α *is unknown (u):* there is neither evidence that it is true nor that it is false;
- α *is contradictory (k):* there is both evidence that it is true and that it is false.

The logic is based on a solid algebraic structure which is a Boolean algebra on a bilattice of the set of its truth values (k and u are incomparable on one dimension of the bilattice and t and f are incomparable on the other dimension of the bilattice). The logic extends the one introduced by Belnap, 1977 and uses results from Ginsberg, 1988; Fitting, 1991.

The logic introduced deals with uncertainty. A set $\mathcal{A}$ may be defined, but the membership of an object a to the set may be unsure either because the information is not sufficient or because the information is contradictory.

In order to distinguish between these two principal sources of uncertainty the knowledge of the *"membership"* of a in $\mathcal{A}$ and of the *"non-membership"* of a in $\mathcal{A}$ are evaluated independently since they are not necessarily complementary. Under this perspective from a given knowledge we have two possible entailments, one, positive, about membership and one, negative, about non-membership. Therefore, any predicate is defined by two sets, its positive and its negative extension in the universe of discourse. Since the negative extension does not necessarily correspond to the complement of the positive extension of the predicate we can expect that the two extensions possibly overlap (due to the independent evaluation) and that there exist parts of the universe of discourse that do not belong to either of the two extensions. The four truth values capture these situations.

Under such a logic, for any well formed formula α, we may use the following sentences:

- $\neg\ \alpha$ (not α, the negation);

- $\not\sim \alpha$ (perhaps not α, the weak-negation);
- $\sim \alpha$ (the complement of α, $\sim \alpha \equiv \neg \not\sim \neg \not\sim \alpha$);
- $\triangle\alpha$ (presence of truth for α);
- $\triangle\neg\alpha$ (presence of truth for $\neg\alpha$);
- $\mathbf{T}\alpha$ (the true extension of α);
- $\mathbf{K}\alpha$ (the contradictory extension of α);
- $\mathbf{U}\alpha$ (the unknown extension of α);
- $\mathbf{F}\alpha$ (the false extension of α).

Between $\mathbf{T}\alpha$, $\mathbf{K}\alpha$, $\mathbf{U}\alpha$, $\mathbf{F}\alpha$ on the one side and $\triangle\alpha$ and $\triangle\neg\alpha$ on the other side the following hold:

$$\mathbf{T}\alpha \iff \triangle\alpha \wedge \neg\triangle\neg\alpha \tag{1.10}$$
$$\mathbf{K}\alpha \iff \triangle\alpha \wedge \triangle\neg\alpha \tag{1.11}$$
$$\mathbf{U}\alpha \iff \neg\triangle\alpha \wedge \neg\triangle\neg\alpha \tag{1.12}$$
$$\mathbf{F}\alpha \iff \neg\triangle\alpha \wedge \triangle\neg\alpha \tag{1.13}$$

A continuous extension. The DDT logic introduced above distinguishes four possible interpretations of a formula α, namely "true", "false", "contradictory", "unknown", all defined from the two conditions $\triangle\alpha$ and $\triangle\neg\alpha$ reflecting the presence of truth for α and $\neg\alpha$ respectively. However, this presence of truth cannot always be thought of as an all or nothing concept. Following the example of Concordance and Discordance concepts, introducing intermediary states between the "full presence of truth" and the "full absence of truth" can be useful. We can imagine a continuum of situations between these extremal situations, enabling to differentiate a multitude of information states between $\triangle\alpha$ and $\neg\triangle\alpha$, and $\triangle\neg\alpha$ and $\neg\triangle\neg\alpha$. For this reason, conditions $\triangle\alpha$ and $\triangle\neg\alpha$ will be represented by real values $b(\alpha)$ and $b(\neg\alpha)$ respectively, chosen in the unit interval in order to reflect the "strength" or the "credibility" of the two arguments. From these two values, a degree of truth, contradictory, unknown and false can be defined in the same spirit as what has been done in equations (1.10–1.13). As an example, we mention here a possible solution proposed and justified by Perny and Tsoukiàs, 1998 (for an alternative approach see Fortemps and Słowiński, 2001):

$$t(\alpha) = \min(b(\alpha),\ 1 - b(\neg\alpha)) \tag{1.14}$$
$$k(\alpha) = \max(\ b(\alpha) + b(\neg\alpha) - 1,\ 0) \tag{1.15}$$
$$u(\alpha) = \max(1 - b(\alpha) - b(\neg\alpha),\ 0) \tag{1.16}$$
$$f(\alpha) = \min(1 - b(\alpha),\ b(\neg\alpha)) \tag{1.17}$$

and therefore :

$$\begin{aligned} t(\alpha) + k(\alpha) &= b(\alpha) \\ f(\alpha) + k(\alpha) &= b(\neg\alpha) \\ t(\alpha) + u(\alpha) &= 1 - b(\neg\alpha) \\ f(\alpha) + u(\alpha) &= 1 - b(\alpha) \end{aligned}$$

Using these equations, any formula α is represented by the truth matrix $v(\alpha)$:

$$v(\alpha) = \begin{pmatrix} t(\alpha) & k(\alpha) \\ u(\alpha) & f(\alpha) \end{pmatrix} \tag{1.18}$$

with $t(\alpha)+k(\alpha)+u(\alpha)+f(\alpha) = 1$ for any proposition α. Thus, the set of all possible values is represented by the continuous bi-lattice represented in figure 1.3.

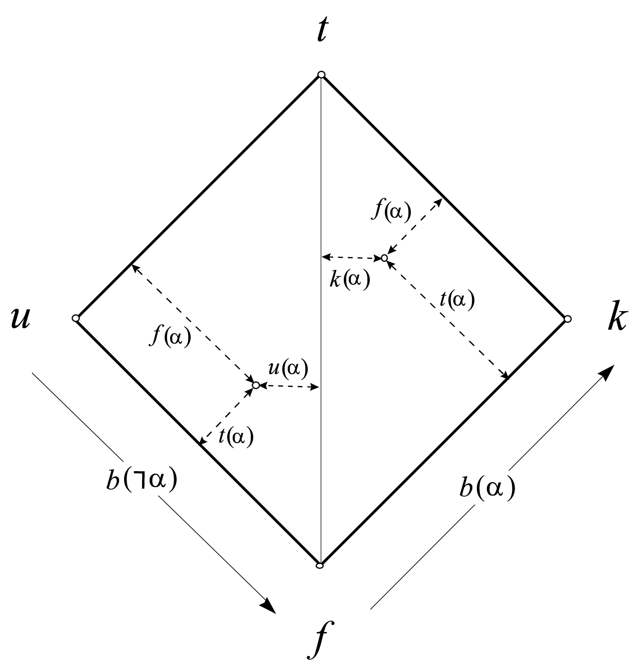

Figure 1.3 The continuous bi-lattice

Note that, by construction, there is a one-to-one correspondence between the points of this bi-lattice, and the matrices defined by equations (1.18) and (1.14–1.17).

3.2 APPLICATIONS IN PREFERENCE MODELLING

We can now use the formalism introduced above for preference modelling and decision support purposes. Given a set A and a binary relation S modelling the concept "at least as good as", we are allowed to write formulas of the type:

- $\triangle S(x, y)$: there is (presence of) truth in claiming that x is at least as good as y;
- $\triangle \neg S(x, y)$: there (is presence) of truth in claiming that x is not at least as good as y;
- $\neg \triangle S(x, y)$: there is no (presence of) truth in claiming that x is at least as good as y;
- $\neg \triangle \neg S(x, y)$: there is no (presence of) truth in claiming that x is not at least as good as y;

Clearly, from equations (1.10–1.13), we obtain:

$$\mathbf{T}S(x,y) \iff \triangle S(x,y) \ \wedge \ \neg\triangle\neg S(x,y) \tag{1.19}$$
$$\mathbf{K}S(x,y) \iff \triangle S(x,y) \ \wedge \ \triangle\neg S(x,y) \tag{1.20}$$
$$\mathbf{U}S(x,y) \iff \neg\triangle S(x,y) \ \wedge \ \neg\triangle\neg S(x,y) \tag{1.21}$$
$$\mathbf{F}S(x,y) \iff \neg\triangle S(x,y) \ \wedge \ \triangle\neg S(x,y) \tag{1.22}$$

which enable to establish the true, contradictory, unknown and false extensions of the relation S respectively.

Combining such extensions with the extensions of the inverse relation S^{-1} we obtain the **PC** preference structure (see Tsoukiàs and Vincke, 1995; Tsoukiàs and Vincke, 1997) where ten different basic preference relations can be defined $(P, H, K, I, J, U, R, T, V, L)$ enabling to clearly distinguish different types of hesitation and incomparability when two alternatives are compared. In such a way we are able, for instance, to distinguish two situations of hesitation between strict preference ($\mathbf{T}P(x,y) \equiv \mathbf{T}S(x,y) \ \wedge \ \mathbf{F}S^{-1}(x,y)$) and indifference ($\mathbf{T}I(x,y) \equiv \mathbf{T}S(x,y) \ \wedge \ \mathbf{T}S^{-1}(x,y)$), one where S^{-1} is contradictory ($\mathbf{T}H(x,y) \equiv \mathbf{T}S(x,y) \ \wedge \ \mathbf{K}S^{-1}(x,y)$) and one where S^{-1} is unknown ($\mathbf{T}K(x,y) \equiv \mathbf{T}S(x,y) \ \wedge \ \mathbf{U}S^{-1}(x,y)$). We are also able to distinguish the situation where both S and S^{-1} are unknown (ignorance) from the situation where both S and S^{-1} are false (conflict). The reader will find more details on the semantics of the ten basic relations in Tsoukiàs and Vincke, 1997.

We are now able to use such results in order to model generalised concordance and discordance conditions on a single criterion. The concept is very simple: we associate the concordance condition the formula "presence of truth in x is at least as good as y ($\triangle S(x,y)$)" and the discordance condition to the formula "presence of truth in x is not at least as good as y ($\triangle\neg S(x,y)$)". We will show how such an approach applies to the problem of comparing alternatives represented through intervals.

Consider two alternatives x and y whose values are known under the form of an interval: $[l(x), r(x)], [l(y), r(y)]$; $\forall x,\ l(x) < r(x)$, $l(x)$ and $r(x)$ representing the left and right extremes respectively. It is well known that it is possible to compare x to y using the interval order preference structure such that:
$P(x,y)$ if $r(x) > l(x) > r(y) > l(y)$,
$I(x,y)$ otherwise (see figure 1.4).

However, we may intuitively consider that the case where $r(x) > r(y) > l(x) > l(y)$ represents a situation of hesitation between preference and indifference. Moreover, we may wish to distinguish the case $r(x) > r(y) > l(y) > l(x)$ from the case $r(y) > r(x) > l(x) > l(y)$ since the two intervals are inversely included (see also figure 1.4). Such a preference structure was first studied in Tsoukiàs and Vincke, 2001; Ngo The et al., 2000, under the name of PQI interval order (Q representing the hesitation between P and I).

We are able to show that using the **PC** preference structure and the positive / negative reasons approach we can model such situations of hesitation due to the presence of an interval representation in a positive way.

Definition 1 *(see Tsoukiàs and Vincke, 2001) A PQI preference structure on a finite set A is a PQI interval order iff there exist two real valued functions l and r, such that $\forall\ x, y \in A$:*
i) $r(x) > l(x)$;
ii) $P(x,y) \iff r(x) > l(x) > r(y) > l(y)$;
iii) $Q(x,y) \iff r(x) > r(y) > l(x) > l(y)$;
iv) $I(x,y) \iff r(x) > r(y) > l(y) > l(x)$ *or* $r(y) > r(x) > l(x) > l(y)$.

Theorem 1 *(see Tsoukiàs and Vincke, 2001) A PQI preference structure on a finite set A is a PQI interval order iff there exists a partial order I_l such that:*
i) $I = I_l \cup I_r \cup I_0$ *where* $I_0 = \{(x,x),\ x \in A\}$ *and* $I_r = I_l^{-1}$;
ii) $(P \cup Q \cup I_l)P \subset P$;
iii) $P(P \cup Q \cup I_r) \subset P$;

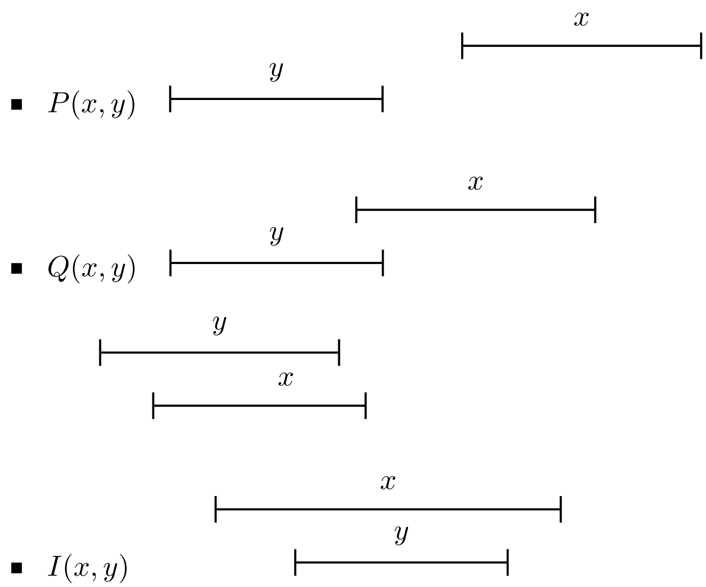

Figure 1.4 *PQI* Interval Orders

iv) $(P \cup Q \cup I_l)Q \subset P \cup Q \cup I_l$;
v) $Q(P \cup Q \cup I_r) \subset P \cup Q \cup I_r$;

On this basis we can present the following characterisation result:

Definition 2 *(see Ngo The and Tsoukiàs, 2001) A PC preference structure, having characteristic relation S on a finite set A, is a PQI interval order iff $\exists : l, r : A \mapsto \mathcal{R}$, such that $\forall\, x, y \in A$:*
i) $r(x) > l(x)$;
ii) $\Delta S(x, y) = r(x) \geq l(y) \ \wedge \ [r(x) < r(y) \ \vee \ l(x) \geq l(y)]$
iii) $\Delta \neg S(x, y) = l(x) < l(y) \ \wedge \ r(x) < r(y)$

Theorem 2 *(see Ngo The and Tsoukiàs, 2001) Definitions 1 and 2 are equivalent.*

The consequence of this theorem is that:
- $P = \mathbf{T}P \iff \mathbf{T}S \ \wedge \ \mathbf{F}S^{-1}$
- $Q = \mathbf{T}H \iff \mathbf{T}S \ \wedge \ \mathbf{K}S^{-1}$
- $I = \mathbf{T}I \cup \mathbf{T}K \cup \mathbf{T}K^{-1}$
$\iff (\mathbf{T}S \ \wedge \ \mathbf{T}S^{-1}) \ \vee \ (\mathbf{T}S \ \wedge \ \mathbf{U}S^{-1}) \ \vee \ (\mathbf{U}S \ \wedge \ \mathbf{T}S^{-1})$

Theorem 3 *(see Ngo The and Tsoukiàs, 2001) A PC preference structure is a PQI interval order iff*
- *i)* $I = I_0 = \{(x,x), x \in A\}$*;*
- *ii)* $\forall x, y,\ \mathbf{T}S(x,y) \ \vee\ \mathbf{T}S(y,x)$*;*
- *iii)* $\forall x, y, z,\ \mathbf{F}S(x,y) \ \wedge\ \neg\mathbf{T}S(y,z) \ \Rightarrow\ \mathbf{F}S(x,z)$*;*
- *iv)* $\forall x, y, z,\ \mathbf{U}S(x,y) \ \wedge\ (\mathbf{K}S \ \vee\ \mathbf{F}S)(y,z) \ \Rightarrow\ \neg\mathbf{T}S(x,z)$*;*
- *v)* $\forall x, y, z,\ \mathbf{U}S(x,y) \ \wedge\ \mathbf{U}S(y,z) \ \Rightarrow\ \mathbf{U}S(x,z)$*;*
- *vi)* $\forall x, y, z,\ \mathbf{K}S(x,y) \ \wedge\ \mathbf{F}S(y,z) \ \Rightarrow\ \mathbf{F}S(x,z)$*;*
- *vii)* $\forall x, y, z,\ \mathbf{K}S(x,y) \ \wedge\ \mathbf{K}S(y,z) \ \Rightarrow\ (\mathbf{K}S \ \vee\ \mathbf{F}S)(x,z)$*;*
- *viii)* $\forall x, y, z,\ \mathbf{K}S(x,y) \ \wedge\ \mathbf{U}S(y,z) \ \Rightarrow\ (\mathbf{K}S \ \vee\ \mathbf{U}S)(x,z)$*;*
- *ix)* $\forall x, y, z,\ \mathbf{U}S(x,y) \ \wedge\ \mathbf{U}S(z,y) \ \Rightarrow\ \neg\mathbf{F}S(x,z) \ \wedge\ \neg\mathbf{F}S(z,x)$*;*

Consider again the two alternatives represented by the intervals. Consider the examples presented in figure 1.4. It could be claimed that the second case of $Q(x,y)$ is more an indifference than an ambiguous preference. In fact the two intervals are almost included in one another. The intuitive reasoning is that hesitation between preference and indifference begins only when one interval is "sufficiently to the left" of the other (and ends, as usually, when it is completely to the left of the other). Such a reasoning corresponds to the use of an "intermediate point" for each interval which we denote $m(x)$, such that $\forall x\ r(x) > m(x) > l(x)$.

We can give the following definition to such a structure.

Definition 3 *(see Vincke, 1988; Tsoukiàs and Vincke, 1998) A PQI preference structure on a finite set A is a double threshold order iff there exist three real valued functions l, m and r, such that* $\forall\ x, y \in A$*:*
i) $r(x) > m(x) > l(x)$*;*
ii) $P(x,y) \iff l(x) > r(y)$*;*
iii) $Q(x,y) \iff r(y) > l(x) > m(y)$*;*
iv) $I(x,y) \iff m(y) > l(x)$ *and* $m(x) > l(y)$*.*

Theorem 4 *(see Vincke, 1988; Tsoukiàs and Vincke, 1998) A PQI preference structure on a finite set A is a double threshold order iff:*
- $\forall x, y, z, w\ Q(x,y) \ \wedge\ I(y,z) \ \wedge\ Q(z,w) \ \rightarrow\ P(x,w) \ \vee\ Q(x,w)$
- $\forall x, y, z, w\ Q(x,y) \ \wedge\ I(y,z) \ \wedge\ P(z,w) \ \rightarrow\ P(x,w)$
- $\forall x, y, z, w\ P(x,y) \ \wedge\ I(y,z) \ \wedge\ P(z,w) \ \rightarrow\ P(x,w)$
- $\forall x, y, z, w\ P(x,y) \ \wedge\ Q^{-1}(y,z) \ \wedge\ P(z,w) \ \rightarrow\ P(x,w)$

This is a well-known preference structure, known under the name of double threshold order, which was first studied in Roy and Vincke, 1984 (see also Roy and Vincke, 1987; Vincke, 1988). Pseudo-orders described in section 2 are particular cases of double threshold orders.

We can again introduce the positive / negative reasons approach as follows.

Definition 4 *(see Tsoukiàs and Vincke, 1998) A* **PC** *preference structure is a double threshold order* **iff** *there exist three real valued functions* l*,* m *and* r*, such that* $\forall\, x, y \in A$*:*
1. $r(x) > m(x) > l(x)$
2. $\forall x, y \;\; \triangle S(x, y) \iff r(x) > l(y)$
3. $\forall x, y \;\; \triangle\neg S(x, y) \iff l(y) > m(x)$

Theorem 5 *(see Tsoukiàs and Vincke, 1998) Definitions 3 and 4 are equivalent.*

The consequence of this theorem is that:
- $P = \mathbf{T}P \iff \mathbf{T}S \;\wedge\; \mathbf{F}S^{-1}$
- $Q = \mathbf{T}H \iff \mathbf{T}S \;\wedge\; \mathbf{K}S^{-1}$
- $I = \mathbf{T}I \iff \mathbf{T}S \;\wedge\; \mathbf{T}S^{-1}$

Theorem 6 *(see Tsoukiàs and Vincke, 1998) A binary relation* S *characterises a* **PC** *preference structure which is a double threshold order* **iff**
1. $\forall x, y \;\; (\mathbf{T}S(x, y) \;\vee\; \mathbf{T}S(y, x)) \;\wedge\; \neg\mathbf{U}S(x, y) \;\wedge\; \neg\mathbf{U}S(y, x)$.
2. $\forall x, y, z, w \;\; \mathbf{T}S(x, y) \;\wedge\; \mathbf{T}S(z, w) \;\rightarrow\; \mathbf{T}S(x, w) \;\vee\; \mathbf{T}S(z, y)$.
3. $\forall x, y, z, w \;\; \mathbf{T}S(x, y) \;\wedge\; \mathbf{K}S(z, w) \;\rightarrow\; \mathbf{T}S(x, w) \;\vee\; \neg\mathbf{F}S(z, y)$.
4. $\forall x, y, z, w \;\; \mathbf{K}S(x, y) \;\wedge\; \mathbf{K}S(z, w) \;\rightarrow\; \neg\mathbf{F}S(x, w) \;\vee\; \neg\mathbf{F}S(z, y)$.

The previous results illustrate the potential role of $\triangle S$ and $\triangle\neg S$ in preference modelling, as a medium between criterion values and preference relations. Firstly, they can be used to derive a compact representation of complex preference structures (PQI interval orders, double threshold orders) using intervals of criterion value. Conversely, note that they provide a more expressive language to compare alternatives described by imprecise criterion values. As a last illustration, we assume that any $x \in A$ is represented by an interval $[l(x), r(x)]$ (representing the set of plausible values for $g(x)$, for a given criterion function g) and we introduce the following non-conventional preference structure:

$$\triangle S(x, y) \iff r(x) \geq l(y) \tag{1.23}$$
$$\triangle\neg S(x, y) \iff r(y) > l(x) + v \tag{1.24}$$

where v represents the maximal difference of of type $g(y) - g(x)$ which is compatible with $S(x, y)$ (a kind of veto threshold). Such a construction can be seen as an alternative to definitions 2 and 4 aiming at defining a preference structure in which negative arguments remain very close to the original ideas of discordance and veto. Using equations (1.19–1.22) we obtain:

$$\mathbf{T}S(x,y) \iff r(x) \geq l(y) \text{ and } r(y) \leq l(x) + v \tag{1.25}$$
$$\mathbf{K}S(x,y) \iff r(x) \geq l(y) \text{ and } r(y) > l(x) + v \tag{1.26}$$
$$\mathbf{U}S(x,y) \iff r(x) < l(y) \text{ and } r(y) \leq l(x) + v \tag{1.27}$$
$$\mathbf{F}S(x,y) \iff r(x) < l(y) \text{ and } r(y) > l(x) + v \tag{1.28}$$

Note that the four belief states attached to the predicate $S(x,y)$ correspond to four complementary and significantly distinct situations of the two intervals. Interestingly, $\mathbf{T}S(x,y)$ corresponds to a situation where outranking is intuitively justified whereas this is just the contrary for $\mathbf{F}S(x,y)$. Similarly, $\mathbf{K}S(x,y)$ corresponds to a natural conflicting situation due to the simultaneous possibilities of:

- finding two possible values for x and y such that x is better than y,
- finding two possible values for x and y such that y is much better than x

Finally $\mathbf{U}S(x,y)$ seems also justified because on the one hand, there is no possibility for x to receive a better evaluation than y, but on the other y cannot be strongly better than x.

Using the continuous bi-lattice. Suppose now that we want to compare alternatives described by fuzzy intervals of criterion values. For the sake of simplicity we denote by X a generic alternative identified to a fuzzy interval of the real line and characterised by the possibility distribution μ_X, taking its values in the unit interval. For any fuzzy interval X, we define:

$$\begin{aligned}
S(X) &= \{x \in \mathbb{R}, \mu_X(x) > 0\} \quad \text{(The support of } X\text{)} \\
C(X) &= \{x \in \mathbb{R}, \mu_X(x) = 1\} \quad \text{(The core of } X\text{)} \\
l^-(X) &= \inf\{x \in S(X)\} \\
l^+(X) &= \inf\{x \in C(X)\} \\
r^-(X) &= \sup\{x \in C(X)\} \\
r^+(X) &= \sup\{x \in S(X)\}
\end{aligned}$$

Note that, since $C(X) \subseteq S(X)$, we have: $l^-(x) \leq l^+(x) \leq r^-(x) \leq r^+(x)$. Moreover, since X is a fuzzy interval, μ_X is necessary increasing on $[l^-(x), l^+(x)]$ and decreasing on $[r^-(x), r^-(x)]$. Thus, we can define $L(X)$ and $R(X)$, the left and right boundaries of X, as fuzzy subsets of X characterised by the following possibility distributions:

$$\mu_{L(X)}(x) = \begin{cases} \mu_X(x) & \text{if } x \in [l^-(x), l^+(x)] \\ 0 & \text{otherwise.} \end{cases}$$
$$\mu_{R(X)}(x) = \begin{cases} \mu_X(x) & \text{if } x \in [r^-(x), r^+(x)] \\ 0 & \text{otherwise.} \end{cases}$$

Considering two fuzzy intervals X and Y, we want to evaluate the outranking $S(X,Y)$. For this, we need to extend the comparison models introduced above to classical intervals. Let us observe that the comparison of two non-fuzzy intervals x and y was based on the relative positions of the borders $l(x)$, $r(x)$, $l(y)$, $r(y)$. Hence, in the fuzzy case, we have to compare the fuzzy borders $L(X)$, $R(X)$, $L(Y)$, $R(Y)$. Interpreting a fuzzy interval X (resp. Y) of the real line as the fuzzy set of possible values for an alternative x (resp y), we define the quantity $\geq(X,Y)$ (resp $<(X,Y)$) as the necessity of the event $x \geq y$ (resp. $x < y$, see Dubois and Prade, 1988; Perny and Roubens, 1998). We obtain:

$$\geq(X,Y) = 1 - \sup_{(x,y)\in S(X)\times S(Y):\ x<y} \min\{\mu_X(x), \mu_Y(y)\}$$
$$<(X,Y) = 1 - \sup_{(x,y)\in S(X)\times S(Y):\ x\geq y} \min\{\mu_X(x), \mu_Y(y)\}$$

These equations suggest a natural extension of $\Delta S(x,y)$ and $\Delta\neg S(x,y)$ compatible with fuzzy intervals. For example, equations (1.23–1.24) can be extended to the case of fuzzy intervals by:

$$b(S(X,Y)) = \geq(R(X), L(Y))$$

$$b(\neg S(X,Y)) = <(R(X), L(Y)+v)$$

where $L(Y)+v$ is the fuzzy set defined by:

$$\forall x \in \mathbb{R},\quad \mu_{L(Y)+v}(x) = \mu_{L(Y)}(x-v)$$

Thus, we obtain a basis for computing $t(S(X,Y))$, $k(S(X,Y))$, $u(S(X,Y))$, $f(S(X,Y))$ using equations (1.10–1.13). These four values make it possible to evaluate the level of confidence we may have in the outranking $S(X,Y)$ as well as the level of conflict between the pros and cons. It provides a compact representation of the relative position of X and Y while keeping high descriptive possibilities.

Similarly, the PQI interval order structure (see definition 2) might be extended by defining:

- $b(S(X,Y))$ from quantities $\geq(R(X), L(Y))$, $<(R(X), R(Y))$ and $\geq(L(X) \geq L(Y))$,

– $b(\neg S(x,y))$ from $< (L(X), L(Y))$ and $< (R(X), R(Y))$.

Note however that such a generalisation is not straightforward, due to the complexity of conditions *ii*) used in the definition. This is left for further investigation.

3.3 APPLICATION IN MULTIPLE CRITERIA AGGREGATION

Moving to multiple criteria we should bear in mind that each criterion is now equipped with a **PC** preference structure (which also contains all classic preference structures) where positive and negative reasons are explicitly considered. Consider formula 1.1. Instead of computing whether $S(x,y)$ holds or not, we explicitly compute $\triangle S(x,y)$ (equivalent to the concordance concept) and $\triangle\neg S(x,y)$ (equivalent to the discordance concept). To give an example we could write:

$$\triangle S(x,y) \iff \mu_1(J_{xy}^{\triangle S}) \geq \gamma \quad (1.29)$$

$$\triangle\neg S(x,y) \iff \mu_2(J_{xy}^{\triangle\neg S}) \geq \delta \quad (1.30)$$

where:
- $J_{xy}^{\triangle S}$ is the set of criteria $g_j \in G$ for which $\triangle S_j(x,y)$ holds;
- $J_{xy}^{\triangle\neg S}$ is the set of criteria $g_j \in G$ for which $\triangle\neg S_j(x,y)$ holds;
- μ_1 and μ_2 represent two functions ($\mu_1, \mu_2 : 2^G \mapsto \mathbb{R}$ "measuring" the "strength" or "importance" of criteria coalitions;
- γ and δ being thresholds representing a "security level" for the decision maker.

For the same reasons as those mentioned in the preference modelling section, it is worth considering a fuzzy extension of the aggregation method proposed above. A natural proposition for this derives directly from fuzzy concordance and discordance indices introduced in section 2.2. The most natural is:

$$b(S(x,y)) = c(x,y) \quad (1.31)$$

$$b(\neg S(x,y)) = d(x,y) \quad (1.32)$$

More generally, we propose (see also Perny and Tsoukiàs, 1998) a fuzzy extension of (1.29–1.30) by setting:

$$b(S(x,y)) = \mu_1(J_{xy}^{\triangle S})$$

$$b(\neg S(x,y)) = \mu_2(J_{xy}^{\triangle\neg S})$$

The following remarks can be made.

1. In the previous examples, for a given (x, y), $\triangle S$ ($\triangle\neg S$) is evaluated wrt the criteria for which $\triangle S_j$ ($\triangle\neg S_j$) holds. It could be the case that the criteria for which $\neg\triangle\neg S_j$ ($\neg\triangle S_j$) holds could also be considered. It is necessary to study what the consequences of such choices are, but a priori it is up to the will and intuition of the decision-maker and the analyst to define the most appropriate formula.

2. Functions μ_1 and μ_2 should "measure" the importance of each coalition. We are not necessarily limited to the use of additive representations. Moreover, neither μ_1 and μ_2 have to be of the same shape nor should they use the same information. It is sufficient to consider that a couple of importance parameters is associated to each criterion , one applying when the criterion belongs to a "positive" coalition, the other applying when the criterion belongs to a "negative" coalition. This is already the case with the veto concept. A veto condition implies that a criterion becomes very important, but only in negative terms. The "positive" importance of such a criterion remains the same.

3. It is now possible to have an homogeneous preference model for all levels of modelling and decision support. We have positive and negative reasons for each single criterion, we have the same when such criteria are aggregated and if we have to go up a hierarchy of criteria and/or decision makers we have an elegant way of keeping positive and negative reasons distinguished until the final level. Under such a perspective we can not totally prevent the problem of losing information as the aggregations are repeated, but we have better control of the process and more clear justifications for the final recommendation.

What can we obtain as a final result? The two relations $\triangle S$ and $\triangle\neg S$ are just two binary relations for which nothing, except reflexivity, can be imposed. What happens if a final result in the form of an ordering relation is expected? As an example, in the following we will briefly present three procedures (suggested in Tsoukiàs, 1997; the first also studied and axiomatised in Greco et al., 1998).

1. Given the two binary relations $\triangle S$ and $\triangle\neg S$ compute the following score for each alternative:

$$\begin{aligned}\sigma(x) &= |\{y:\ \triangle S(x,y)\}| + |\{y:\ \triangle\neg S(y,x)\}| \\ &\quad -|\{y:\ \triangle S(y,x)\}| - |\{y:\ \triangle\neg S(x,y)\}|\end{aligned}$$

then order the alternatives on the decreasing value of σ.

Such a score generalises the "net flow" procedure introduced by Brans and Vincke, 1985). It has the nice property of introducing three possible situations for any couple (x, y):
- strict preference (when $\sigma(x) = 2$ and $\sigma(y) = -2$);
- weak preference (when $\sigma(x) = 1$ and $\sigma(y) = -1$);
- indifference (when $\sigma(x) = 0$ and $\sigma(y) = 0$).

2. Given the two binary relations $\triangle S$ and $\triangle\neg S$ compute, for all alternatives, the following scores:

$$\begin{aligned}\sigma^+(x) &= |\{y:\ \triangle S(x,y)\}| + |\{y:\ \triangle\neg S(y,x)\}| \\ \sigma^-(x) &= |\{y:\ \triangle S(y,x)\}| + |\{y:\ \triangle\neg S(x,y)\}|\end{aligned}$$

construct two pre-orders, the first one based on the decreasing value of σ^+, the second one based on the increasing value of σ^- and intersect the two resulting pre-orders. The intersection is a partial pre-order generalising the classical ranking procedure based on leaving and entering flows (see Brans and Vincke, 1985).

3. Construct the pre-ordering relation $\succeq^*$ obtained by the transitive closure of the relation $\succeq$ defined by:

$$\succeq (x,y) \iff \triangle S(x,y) \wedge \triangle\neg S(y,x)$$

4. Generalise such operations to the fuzzy case.

It should be noted that very little research has been carried out insofar as the formal properties of such procedures are concerned (with the exception of the first one). Some preliminary propositions can be found in Perny and Tsoukiàs, 1998 that are valid both in the crisp and fuzzy cases; some alternative options have been proposed in Fortemps and Słowiński, 2001.

4. CONCLUSIONS

Following the example of concordance and discordance concepts introduced by B. Roy in multiple criteria aggregation methods, we have presented some elements of a new and non-conventional approach to preference modelling. The main originality of this approach is to consider positive and negative arguments independently with respect to a given assertion about preferences. Hence, the various possible combinations of these two independent views on preferences provide a richer set of situations than usual. This potential richness is represented by a lattice of truth values from which a new multi-valued logic is constructed.

In the context of preference modelling, the increased expressive power of the resulting language should be useful. Among others, it enables the description of the possible belief states of a decision-maker facing a complex situation due to imprecise evaluations or conflicting criteria. Due to the high expressivity of this language, the construction of preference models, the definition of aggregation procedures and the conception of choices, ranking or sorting procedures must be revisited. Throughout the paper, examples are given suggesting possible options to initiate some work in this direction. We think it is worth continuing this preliminary investigation.

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