What is a Decision Problem?

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Outline

Problems

- Patients triage in emergency room;
- Identification of classes of similar DNA sequences;
- Star ratings of hotels;
- Waste collection vehicle routing;
- Vendor rating and bids assessment;
- Optimal mix of sausages;
- Chip-set lay out;
- Airplanes priority landing;
- Tennis tournament scheduling ...

What is a decision problem?

Consider a set A established as any among the following:

- an enumeration of objects;
- a set of combinations of binary variables (possibly the whole space of combinations);
- a set of profiles within a multi-attribute space (possibly the whole space);
- a vector space in $\mathbb{R}^n$.

Technically:

A Decision Problem is a partitioning of A under some desired properties.

What is important?

What does really matter?

In designing, choosing, applying, implementing, understanding, explaining, justifying, a method?

What are the primitives?

And what is the derived information and the expected outcomes?

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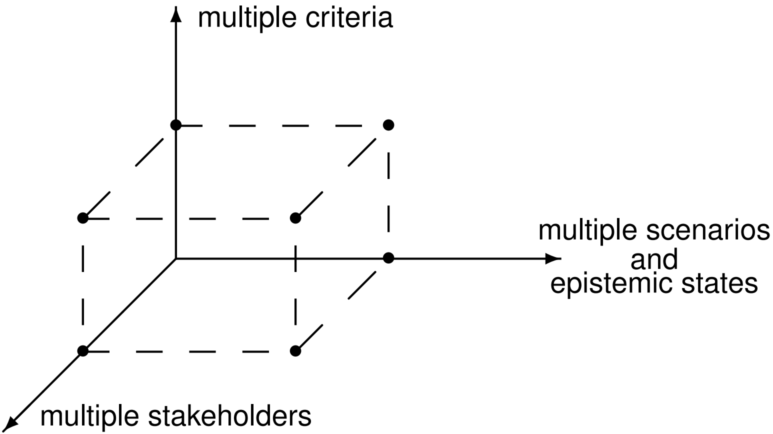
And what is the derived information and the expected outcomes?

Why is not straightforward?

- multiple opinions
- multiple values
- multiple likelihoods

+ algorithmic aspects

Partitioning? How?



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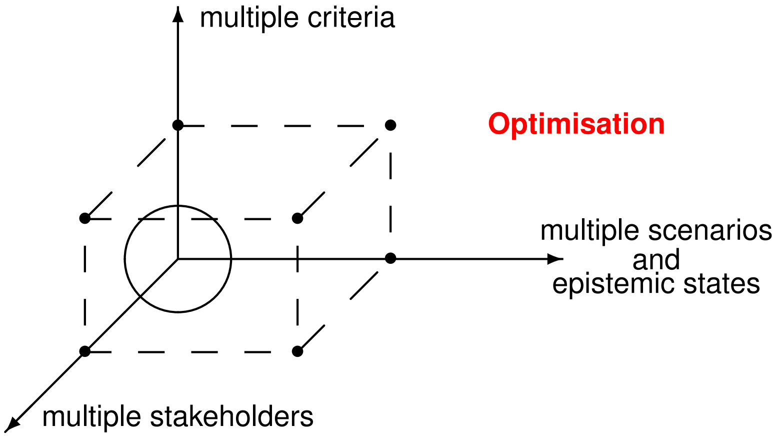

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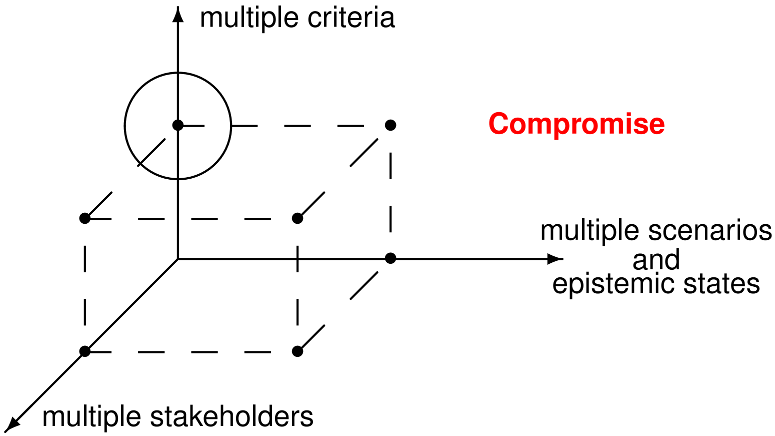
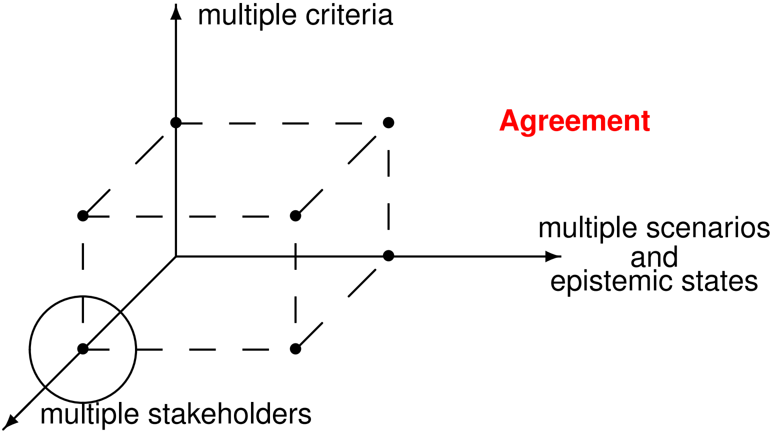


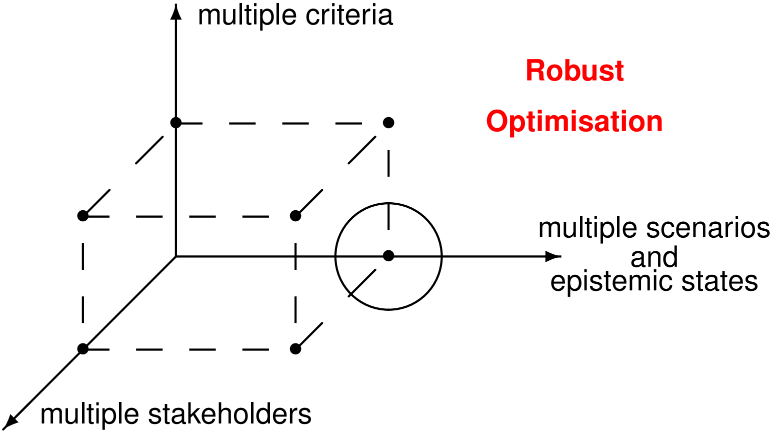
Compromise

Partitioning? How?



Agreement

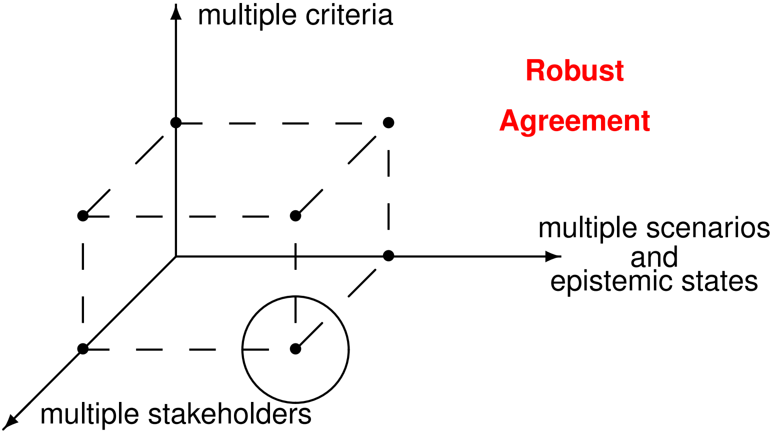
Partitioning? How?



Robust

Optimisation

Partitioning? How?



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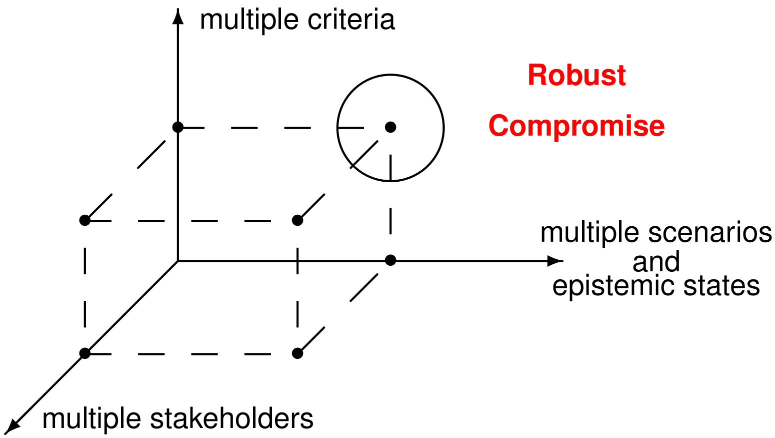

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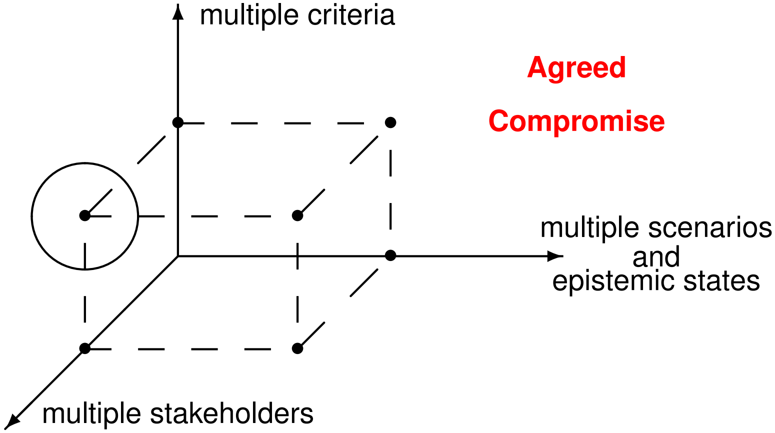

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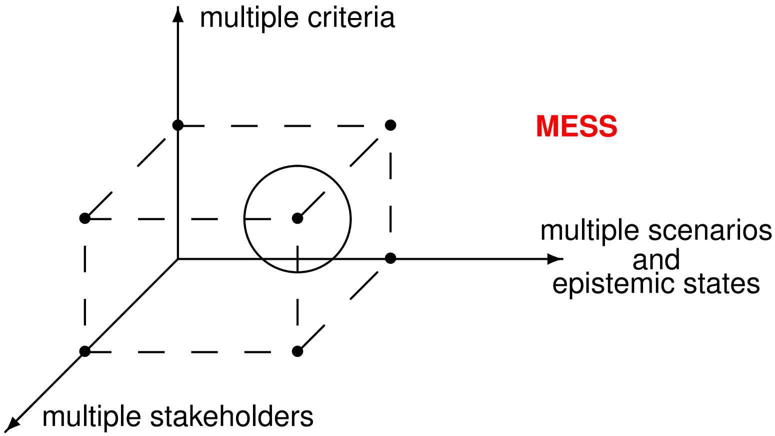

Is that all?

- Behind a criterion other criteria may be considered in a hierarchy of criteria (objectives);
- Behind a stakeholder other actors may have to be considered, that precise stakeholder being a speaker for a community;
- Behind a state of the nature other uncertainties may have to be considered;
- Any combination of the above may in reality occur as complex as possible.

Claims

Claim 1

All the previously mentioned problems boil down in aggregating some ordering relations applied on the set A.

Claim 2

Establishing the set A is on its turn a decision problem. We explore one step of the recursion without any loss of generality.

Claim 3

From an algorithmic point of view a decision problem boils down to an optimisation algorithm.

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Primitives

- The set A.
- The description of the elements of A.
- Preference (ordering) statements about A and its subsets.
- Preference statements are of two types: relative and absolute ones.

Critical Issues

- The set of alternatives
- Problem statement
- Differences of preferences
- Hierarchy/Separability/Indipendence
- Positive and Negative Reasons

Partitioning? For what?

Practically we partition A in n classes. These can be:

	Pre-defined wrt some external norm	Defined only through pairwise comparison
Ordered	Rating	Ranking
Not Ordered	Assigning	Clustering

Two special cases:
- there are only two classes (thus complementary);
- the size (cardinality) of the classes is also predefined.

What is a ranking problem?

Primitive

The primitive is a binary relation on A: $\succeq \subseteq A \times A$ to be read "at least as good as".

Result

The result is a partitioning of A in $[A_1], \cdots [A_n]$ such that:
$[A_j] \geq [A_i] \Leftrightarrow j \geq i$ and
$\forall x \in [A_j], y \in [A_i] : x \succeq' y$

Discussion 1

What is a choice problem?

We partition A in two classes $[A_1] \geq [A_2]$. Thus $[A_1] = \sup_A(\succeq')$.

What is an optimisation problem?

A choice problem for which:
- $\succeq = \succeq'$
- $x \succeq y \iff f(x) \geq f(y)$.
- Thus $[A_1] = \max_A f(x)$

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Discussion 2

Why is $\succeq'$ different from $\succeq$?

Generally speaking $\succeq$ is not an ordering relation since preferences can be partial and or inconsistent. If we have to proceed with some operational procedure we need to transform $\succeq$ to an ordering relation $\succeq'$.

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How do we learn $\succeq$?
What properties should $\succeq'$ fulfill?

What is a clustering problem?

Primitive

The primitive is a set of binary relations on A: $\approx_l \subseteq A \times A$ to be read “similar to”.

Result

The result is a partitioning of A in $[A_1], \cdots [A_n]$ such that:
$\exists \approx_l : \forall x, y \in [A_j] \;\; x \approx y$ and
$\forall x \in [A_j],\; y \in [A_i] : \neg(x \approx y)$

Discussion 1

Indiscernibility.

In case $\approx_l$ are equivalence relations then the partitioning of A results in constructing the indiscernibility relation on A. However, this is not generally the case and $[A_j] = \sup_A(\approx_l)$.

In other terms we try to maximise similarity within classes (clusters) and minimise similarity among classes (clusters).

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Discussion 2

Distances.

If $\approx_l$ are nested similarity relations with nice properties then we can establish a metric:
- $s(x, y)$: how similar is x to y?
- $d(x, y)$: how distant is x from y?
Then $[A_y] = \{x | \max_A F(s(x, y))\}$,
F being a measure of the overall similarity of the elements of $[A_y]$ with respect to y.

What properties should F and the metrics fulfill?

What is a rating problem?

Primitive

The primitive is a binary relation on A: $\succeq \subseteq A \times P \cup P \times A$
to be read "at least as good as".
P being the set of external "norms" characterising the ordered classes $C_1 \rhd \cdots \rhd C_n$

Result

The result is to assign each element of A in a C_j such that:
$x \in C_j \;\Leftrightarrow\; x \succeq' p_j, p_{j+1}, \cdots p_n$ and $p_1 \cdots p_{j-1} \succeq' x$

Discussion 1

Constraint Satisfaction

If $\forall x, y \in A \cup P \;\; x \succeq y \;\Leftrightarrow\; f(x) \geq f(y)$.
Then $x \in C_j \;\Leftrightarrow\; f(p_{j-1}) \geq f(x) \geq f(p_j)$.
This is a Constraint Satisfaction Problem.

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What is an assigning problem?

Primitive

The primitive is a set of binary relations on A: $\approx_l \subseteq A \times P \cup P \times A$ to be read "similar to".
P being the set of external "norms" characterising the classes $C_1 \cdots C_n$

Result

The result is to assign each element of A in a C_j such that:
$x \in C_j \;\Leftrightarrow\; \exists \approx_l : x \approx_l p_j$

Discussion 1

Constraint Satisfaction

If $\forall x, y \in A \cup P \;\; x \approx_I y \;\Leftrightarrow\; f(x) = f(y)$.
This is once again a Constraint Satisfaction Problem.

Basic Claim

- **Any unsupervised decision problem is an optimisation problem.**
- **Any supervised decision problem is a constraint satisfaction problem.**

Since any constraint satisfaction problem can be seen as an optimisation problem, we can definitely focus only to the later ones

When in reality we just know that:

$$
\begin{array}{rcccl}
w\ {}_1 z & {}_1 & x & {}_1 y & {}_1 t \\
w & {}_2 & y & {}_2 x & {}_2 t \ {}_2 z \\
w\ {}_3 t & {}_3 & x & {}_3 y & {}_3 z \\
z & {}_4 & y & {}_4 x & {}_4 t \ {}_4 w \\
 & & \vdots & &
\end{array}
$$

The Problem

Suppose we have n ordering relations $\succeq_1 \cdots \succeq_n$ on the set A. We are looking for an overall ordering relation $\succeq$ on A "representing" the different orders.

$$\begin{tikzcd}
\succeq_i (x,y) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & f_i(x), f_i(y) \arrow[d, leftrightarrow] \\
\succeq (x,y) \arrow[r, leftrightarrow] & F(x,y)
\end{tikzcd}$$

Two fundamental questions

1. **How do we consider preferences and differences of preferences along a single criterion/dimension?**
2. **How do we consider preferences and differences of preferences among several different criteria/dimensions?**

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