What is a Decision Problem?

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Outlines

1. Motivations
2. Methods
3. Problem Statements
4. Questions

What is a Decision Problem?
Problems

- Patients triage in emergency room;
- Identification of classes of similar DNA sequences;
- Star ratings of hotels;
- Waste collection vehicle routing;
- Vendor rating and bids assessment;
- Optimal mix of sausages;
- Chip-set lay out;
- Airplanes priority landing;
- Tennis tournament scheduling ...
What is a decision problem?

Consider a set $A$ established as any among the following:

- an enumeration of objects;
- a set of combinations of binary variables (possibly the whole space of combinations);
- a set of profiles within a multi-attribute space (possibly the whole space);
- a vector space in $\mathbb{R}^n$.

Technically:

A Decision Problem is a partitioning of $A$ under some desired properties.
What is important?

What does really matter?
In designing, choosing, applying, implementing, understanding, explaining, justifying, a method?

What are the primitives?
And what is the derived information and the expected outcomes?
What is important?

What does really matter?
In designing, choosing, applying, implementing, understanding, explaining, justifying, a method?

What are the primitives?
And what is the derived information and the expected outcomes?
Why is not straightforward?

- multiple opinions
- multiple values
- multiple likelihoods
- algorithmic aspects
Partitioning? How?

multiple scenarios and epistemic states

multiple criteria

multiple stakeholders
Motivations
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Partitioning? How?

- multiple scenarios
- Optimisation
- multiple criteria
- multiple stakeholders
- multiple scenarios and epistemic states

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Partitioning? How?

- Multiple scenarios
- Multiple criteria
- Multiple epistemic states
- Multiple stakeholders

Compromise

What is a Decision Problem?
Partitioning? How?

- multiple criteria
- multiple scenarios and epistemic states
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Agreement
Partitioning? How?

- Robust Optimisation
- Multiple scenarios and epistemic states
- Multiple criteria
- Multiple stakeholders

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Partitioning? How?

- Agreed
- Compromise

What is a Decision Problem?
Partitioning? How?

multiple stakeholders →

multiple criteria →

multiple scenarios and epistemic states

MESS
Behind a criterion other criteria may be considered in a hierarchy of criteria (objectives);

Behind a stakeholder other actors may have to be considered, that precise stakeholder being a speaker for a community;

Behind a state of the nature other uncertainties may have to be considered;

Any combination of the above may in reality occur as complex as possible.
Claim 1
All the previously mentioned problems boil down in aggregating some ordering relations applied on the set $A$.

Claim 2
Establishing the set $A$ is on its turn a decision problem. We explore one step of the recursion without any loss of generality.

Claim 3
From an algorithmic point of view a decision problem boils down to an optimisation algorithm.
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Primitives

- The set $A$.
- The description of the elements of $A$.
- Preference (ordering) statements about $A$ and its subsets.
- Preference statements are of two types: relative and absolute ones.
Critical Issues

- The set of alternatives
- Problem statement
- Differences of preferences
- Hierarchy/Separability/Indipendence
- Positive and Negative Reasons
Partitioning? For what?

Practically we partition $A$ in $n$ classes. These can be:

<table>
<thead>
<tr>
<th>Ordered</th>
<th>Rating</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-defined wrt</td>
<td>Defined only through</td>
<td></td>
</tr>
<tr>
<td>some external norm</td>
<td>pairwise comparison</td>
<td></td>
</tr>
<tr>
<td>Not Ordered</td>
<td>Assigning</td>
<td>Clustering</td>
</tr>
</tbody>
</table>

Two special cases:
- there are only two classes (thus complementary);
- the size (cardinality) of the classes is also predefined.
What is a ranking problem?

Primitive

The primitive is a binary relation on \( A \): \( \succeq \subseteq A \times A \) to be read “at least as good as”.

Result

The result is a partitioning of \( A \) in \([A_1], \cdots [A_n]\) such that:

\[ [A_j] \geq [A_i] \iff j \geq i \text{ and } \forall x \in [A_j], y \in [A_i] : x \preceq' y \]
What is a choice problem?
We partition $A$ in two classes $[A_1] \succeq [A_2]$. Thus $[A_1] = \sup_A(\succeq')$.

What is an optimisation problem?
A choice problem for which:
- $\succeq = \succeq'$
- $x \succeq y \iff f(x) \geq f(y)$.
- Thus $[A_1] = \max_A f(x)$
What is a choice problem?

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Why is $\succeq'$ different from $\succeq$?

Generally speaking $\succeq$ is not an ordering relation since preferences can be partial and or inconsistent. If we have to proceed with some operational procedure we need to transform $\succeq$ to an ordering relation $\succeq'$. 
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How do we learn $\succeq$?
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How do we learn $\succeq$?
What properties should $\succeq'$ fulfill?
What is a clustering problem?

**Primitive**

The primitive is a set of binary relations on $A$: $\approx \subseteq A \times A$ to be read “similar to”.

**Result**

The result is a partitioning of $A$ in $[A_1], \cdots [A_n]$ such that:

$\exists \approx : \forall x, y \in [A_j] \ x \approx y$ and

$\forall x \in [A_j], \ y \in [A_i] : \neg(x \approx y)$
Indiscernibility.

In case $\equiv_I$ are equivalence relations then the partitioning of $A$ results in constructing the indiscernibility relation on $A$. However, this is not generally the case and $[A_j] = \sup_A(\equiv_I)$.

In other terms we try to maximise similarity within classes (clusters) and minimise similarity among classes (clusters).
Indiscernibility.

In case $\approx_l$ are equivalence relations then the partitioning of $A$ results in constructing the indiscernibility relation on $A$. However, this is not generally the case and $[A_j] = \sup_A(\approx_l)$.

In other terms we try to maximise similarity within classes (clusters) and minimise similarity among classes (clusters).
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Discussion 2

Distances.

If \( \approx_l \) are nested similarity relations with nice properties then we can establish a metric:
- \( s(x, y) \): how similar is \( x \) to \( y \)?
- \( d(x, y) \): how distant is \( x \) from \( y \)?

Then \( [A_y] = \{ x | \max_A F(s(x, y)) \} \),
\( F \) being a measure of the overall similarity of the elements of \( [A_y] \) with respect to \( y \).

What properties should \( F \) and the metrics fulfill?
What is a rating problem?

**Primitive**

The primitive is a binary relation on $A$: $\succeq \subseteq A \times P \cup P \times A$ to be read “at least as good as”. $P$ being the set of external “norms” characterising the ordered classes $C_1 \succ \cdots \succ C_n$.

**Result**

The result is to assign each element of $A$ in a $C_j$ such that:

$x \in C_j \iff x \succeq^' p_j, p_{j+1}, \ldots p_n$ and $p_1 \cdots p_{j-1} \succeq^' x$
**Discussion 1**

**Constraint Satisfaction**

If $\forall x, y \in A \cup P \ x \succeq y \iff f(x) \geq f(y)$.  
Then $x \in C_j \iff f(p_{j-1}) \geq f(x) \geq f(p_j)$.

This is a Constraint Satisfaction Problem.

**Why is $\succeq'$ different from $\succeq$?**

Generally speaking $\succeq$ is not an ordering relation since preferences can be partial and or inconsistent. If we have to proceed with some operational procedure we need to transform $\succeq$ to an ordering relation $\succeq'$. 
Discussion 1

Constraint Satisfaction

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What is an assigning problem?

**Primitive**
The primitive is a set of binary relations on \( A \): \( \approx_1 \subseteq A \times P \cup P \times A \) to be read “similar to”.
\( P \) being the set of external “norms” characterising the classes \( C_1 \cdots C_n \).

**Result**
The result is to assign each element of \( A \) in a \( C_j \) such that:
\[ x \in C_j \iff \exists \approx_i: x \approx_i p_j \]
Constraint Satisfaction

If \( \forall x, y \in A \cup P \quad x \approx_I y \iff f(x) = f(y) \).

This is once again a Constraint Satisfaction Problem.
Basic Claim

- Any unsupervised decision problem is an optimisation problem.
- Any supervised decision problem is a constraint satisfaction problem.

Since any constraint satisfaction problem can be seen as an optimisation problem, we can definitely focus only to the later ones.
Why $x \succeq y$?

When in reality we just know that:

\[
\begin{align*}
    w & \succeq_1 z \succeq_1 x \succeq_1 y \succeq_1 t \\
    w & \succeq_2 y \succeq_2 x \succeq_2 t \succeq_2 z \\
    w & \succeq_3 t \succeq_3 x \succeq_3 y \succeq_3 z \\
    z & \succeq_4 y \succeq_4 x \succeq_4 t \succeq_4 w \\
    \vdots
\end{align*}
\]

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The Problem

Suppose we have $n$ ordering relations $\succeq_1 \cdots \succeq_n$ on the set $A$. We are looking for an overall ordering relation $\succeq$ on $A$ “representing” the different orders.

[Diagram]

\[ \succeq_i (x, y) \quad \text{f}_i(x), f_i(y) \]

\[ \succeq (x, y) \quad F(x, y) \]
Two fundamental questions

1. How do we consider preferences and differences of preferences along a single criterion/dimension?

2. How do we consider preferences and differences of preferences among several different criteria/dimensions?
Two fundamental questions

1. How do we consider preferences and differences of preferences along a single criterion/dimension?
2. How do we consider preferences and differences of preferences among several different criteria/dimensions?