Double threshold orders: a new axiomatization

Alexis Tsoukiàs(+), Philippe Vincke(*)

(+) LAMSADE, Université Paris Dauphine
75775 Paris Cedex 16, France
e-mail: tsoukias@lamsade.dauphine.fr
(*) SMG Université Libre de Bruxelles
CP 210/01, Bld. du Triomphe, 1050 Bruxelles, Belgium
e-mail: pvincke@ulb.ac.be

Abstract

The paper presents some new results concerning the axiomatization of double threshold preference structures. Such structures, which have been introduced in order to model a situation of hesitation between the strict preference and the indifference, were not axiomatized through the use of a single characteristic relation. We give two theorems for this purpose, exploiting a four valued logic recently introduced by the authors as a preference modeling language under hesitation.

Introduction

The use of a threshold in a preference structure dates back to Luce (1956) who used this concept in order to face information discrimination. A semiorder is a preference structure with a constant threshold modeling intransitivity of indifference. An interval order extends such a structure when the threshold is a function $\rho : A \mapsto \mathcal{R}^+$ (A being the set of alternatives on which the order applies and $\mathcal{R}^+$ are the positive real numbers). instead of a constant one. Definitions, characterizations and numerical representations of such structures are widely reported in the literature (see Roubens and Vincke, 1985 and Fishburn, 1985). They have been adapted for the purpose of this paper and presented in Annex E.

Since the seventies the concept of "pseudo-order" has been introduced in the literature (see Roy, 1977) in order to represent preference structures with a double threshold. The concept of a pseudo-order is based on the idea that between indifference and preference there might be a zone of ambiguity and/or uncertainty where it is difficult to make the distinction between the two relations. Such a "zone" has been called "weak preference" (see Roy and Vincke, 1984). Different tentatives have been done to axiomatize such preference structures (Vincke, 1980, Vincke, 1988, Roy and Vincke, 1987, Valadares Tavares 1988, Moreno 1992) including also some studies on more general properties of preference structures with thresholds (see Doignon, 1987, Abbas and Vincke, 1993). The results in these papers present however all the same weakness. First of all there is no distinct status between P (the strict preference) and Q (the weak preference) since they are both asymmetric. Secondly it is not possible to define a characteristic relation representing the preference structure and endowed with specific properties. Thus it is impossible to define P, Q and I from a single characteristic relation S as it happens for other conventional preference structures (see Roubens and Vincke, 1985).

Such a problem has no solution in the frame of conventional preference modeling. Recently Tsoukiàs and Vincke (1995) demonstrated that given a language with n (finite) truth values as a basis for modeling, it is possible to define (under a sound axiomatization) at most $n(n+1)/2$ distinct preference relations using a single characteristic relation. For $n = 2$ the three relations are usually called "preference", "indifference" and "incomparability". The "weak preference" relation simply does not exist under the classic two valued logic which is the basic language in conventional preference modeling. Our proposition is to demonstrate that using a four valued logic it is possible to give a "simple" and "elegant" axiomatization of any double threshold based preference structure, including pseudo-orders, generalizing well known concepts of binary relations properties.

The paper is organized as follows. Section 1 presents an outline of the formalism used. Section 2 redefines the concept of a double threshold order and presents the main result of the paper, that is the characterization of such preference structure and its numerical representation. Section 3 presents a brief discussion of the results comparing them with conventional preference modeling. Annex A presents briefly the logic adopted in the paper. Annex B presents the semantics of the preference relations defined in the **PC** preference structure. Annex C presents the demonstrations of the lemmas used in the paper. Annex D contains the demonstrations of the two main results of the paper. Annex E redefines the usual concepts of semi order and interval order under the new formalism.

1 The problem and the formalism

1.1 The problem

Consider a set A of objects and a complete and reflexive binary relation S on A (read $s(x,y)$: x is at least as good as y). We can establish an asymmetric binary relation P such that $\forall x,y \;\; p(x,y) \equiv s(x,y) \wedge \neg s(y,x)$ and a reflexive and symmetric binary relation I such that $\forall x,y \;\; i(x,y) \equiv s(x,y) \wedge s(y,x)$, representing strict preference and indifference respectively. Clearly $S = P \cup I$. Notice that if S is not complete we can introduce a third binary relation R such that $\forall x,y \;\; r(x,y) \equiv \neg s(x,y) \wedge \neg s(y,x)$ named incomparability. The collection $\langle P, I, R \rangle$ is a preference structure whose characteristic relation is S. Such a preference structure fulfills the following axioms:
A1: any preference structure should define a partition of its application domain;
A2: any preference structure should follow the principle of "independence from irrelevant alternatives";
A3: each preference relation of the preference structure should be characterized uniquely by its properties;
which enable a sound preference modeling.

Some well known types of preference structures, concerned with the use of thresholds are the following (they are used in many MCDA methods).

1. The Interval Order. There exist two real valued functions $g: A \mapsto \mathcal{R}$ and $k: A \mapsto \mathcal{R}^+$ such that:
   - $\forall x,y \;\; p(x,y) \Leftrightarrow g(x) > g(y) + k(y)$
   - $\forall x,y \;\; i(x,y) \Leftrightarrow g(y) + k(y) > g(x) > g(y) - k(x)$.

   Considering the characteristic relation S, the necessary and sufficient conditions to have such a structure are that S is complete and:
   - $\forall x,y,z,w \;\; s(x,y) \wedge s(z,w) \rightarrow s(x,w) \vee s(z,y)$

2. The Semi Order. There exist a real valued function $g: A \mapsto \mathcal{R}$ and a constant $k > 0$ such that:
   - $\forall x,y \;\; p(x,y) \Leftrightarrow g(x) > g(y) + k$
   - $\forall x,y \;\; i(x,y) \Leftrightarrow g(y) + k > g(x) > g(y) - k$.

   Considering the characteristic relation S, the necessary and sufficient conditions to have such a structure are that S is complete and:
   - $\forall x,y,z,w \;\; s(x,y) \wedge s(z,w) \rightarrow s(x,w) \vee s(z,y)$
   - $\forall x,y,z,w \;\; s(x,y) \wedge s(y,z) \rightarrow s(x,w) \vee s(w,z)$

3. The Double Threshold Order. There exist three real valued functions $g : A \mapsto \mathcal{R}$, $l : A \mapsto \mathcal{R}^+$ and $k : A \mapsto \mathcal{R}^+$ such that:
   - $\forall x, y \;\; l(x) > k(x)$.
   - $\forall x, y \;\; p(x, y) \Leftrightarrow g(x) > g(y) + l(y)$
   - $\forall x, y \;\; q(x, y) \Leftrightarrow g(y) + l(y) > g(x) > g(y) + k(y)$.
   - $\forall x, y \;\; i(x, y) \Leftrightarrow g(y) + k(y) > g(x) > g(y) - k(x)$.

   In this case a third asymmetric binary relation Q has been introduced in order to represent the concept of "weak preference relation".

4. The Pseudo Order. It is a particular case of a double threshold order such that:
   - $\forall x, y \;\; g(x) > g(y) \Leftrightarrow g(x) + l(x) > g(y) + l(y)$
   - $\forall x, y \;\; g(x) > g(y) \Leftrightarrow g(x) + k(x) > g(y) + k(y)$.

The problem is in the last two structures and consists in the following observations.
- It is not possible to have a single binary relation S whose specific properties may characterize the preference structure. Although $S = P \cup Q \cup I$ it is not possible to define P or Q or I from S since only P, I or R are definable using classic logic (in Tsoukiàs and Vincke 1995 there is a more formal treatment of this problem).
- It is not possible to distinguish P from Q since they have the same properties (they are both asymmetric and that is all).

Actually the use of classic logic as a modeling language is too poor to capture the situation of hesitation introduced by the relation Q and therefore the whole structure is weak. In order to solve such a problem and provide a solid axiomatization to such a preference structure it is necessary to adopt another modeling language (which will always be a logic since we work with mathematical structures) and therefore a new extended preference structure.

1.2 The logic

In the following we briefly present the basic concepts of the logic formalism we use in the paper. The basic property of such a logic is to explicitly represent situations of hesitation due either to lack of information (missing or uncertain) or to excess of information (ambiguous or contradictory). A detailed presentation of the logic can be found in Annex A and Tsoukiàs (1996).

The logic, which is a four-valued, first order language, is based on a net distinction between the "negation" (which represents the part of the

universe verifying the negation of a predicate) and the "complement" (which represents the part of the universe which does not verify a predicate) since the two concepts not necessarily coincide. The four truth values represent four epistemic states of an agent towards a sentence (ϕ) that is:
ϕ is true (t): there is evidence that it is true and there is no evidence that it is false;
ϕ is false (f): there is no evidence that it is true and there is evidence that it is false;
ϕ is unknown (u): there is neither evidence that it is true nor that it is false;
ϕ is contradictory (k): there is both evidence that it is true and that it is false.
The logic is based on a solid algebraic structure being a Boolean algebra on a bilattice of the set of its truth values (k and u are incomparable on one dimension of the bilattice and t and f are incomparable on the other dimension of the bilattice). The logic extends the one introduced by Belnap, 1976 and uses results from Ginsberg, 1988 and Fitting, 1991.

The logic introduced deals with uncertainty. A set $\mathcal{A}$ may be defined, but the membership of an object a to the set may be not sure either because the information is not sufficient or because the information is contradictory.

In order to distinguish these two principal sources of uncertainty the knowledge about the *"membership"* of a in $\mathcal{A}$ and about the *"non-membership"* of a in $\mathcal{A}$ are evaluated independently since they are not necessarily complementary. Under this perspective from a given knowledge we have two possible entailments, one, positive, about membership and one, negative, about non-membership. Therefore any predicate is defined by two sets, its positive and its negative extension in the universe of discourse. Since the negative extension does not necessarily correspond to the complement of the positive extension of the predicate we can expect that the two extensions possibly overlap (due to the independent evaluation) and that there exist parts of the universe of discourse that do not belong to any of the two extensions. The four truth values capture these situations.

Under such a logic, for any predicate $\phi(x_1, \cdots x_n)$ (denoted α), we may have the following sentences:
- $\neg\alpha$ (not α, the negation);
- $\not\sim\alpha$ (perhaps not α, the weak-negation);
- $\sim\alpha$ (the complement of α, $\sim\alpha \equiv \neg\not\sim\neg\not\sim\alpha$);
- $\mathbf{T}\alpha$ (the true extension of α);
- $\mathbf{K}\alpha$ (the contradictory extension of α);

- $\mathbf{U}\alpha$ (the unknown extension of α);
- $\mathbf{F}\alpha$ (the false extension of α);
- $\triangle\alpha$ (presence of truth for α);
- $\triangle\neg\alpha$ (presence of truth for $\neg\alpha$).

1.3 The PC preference structure

Under such a logic a new basic preference structure has been conceived, named **PC**, which is a t-tuple $\langle P, T, K, L, I, J, H, U, V, R\rangle$ of ten preference relations. A characteristic relation S of the preference structure is used which we denote by $s(x, y)$ and read as *"x is at least as good as y"*. The **PC** preference structure has been characterized in Tsoukiàs and Vincke (1995) as being a maximal, well defined fundamental relational system of preferences. In other words it forms a partition of its application domain $A \times A$ (that is a "fundamental relational system of preferences"), uses the maximum of possible relations compatible with the language (that is "maximal") and each basic preference relation is uniquely characterized by its properties (that is "well defined") and/or the characteristic relation. The characterization thus fulfills the three axioms previously introduced.

The formal definitions of the ten basic preference relations are summarized in the following table 1 (notice that as usually $s^{-1}(x, y) \equiv s(y, x)$). The reader may notice that since $s(x, y)$ and $s^{-1}(x, y)$ can each have one of the four truth values, there exist 10 different possible combinations (instead of 3 as in the classic logic case) which are exactly the 10 relations of the **PC** preference structure.

$A \times A$	$\mathbf{T}s^{-1}(x, y)$	$\mathbf{F}s^{-1}(x, y)$	$\mathbf{U}s^{-1}(x, y)$	$\mathbf{K}s^{-1}(x, y)$
$\mathbf{T}s(x, y)$	$\mathbf{T}i(x, y)$	$\mathbf{T}p(x, y)$	$\mathbf{T}k(x, y)$	$\mathbf{T}h(x, y)$
$\mathbf{F}s(x, y)$	$\mathbf{T}p^{-1}(x, y)$	$\mathbf{T}r(x, y)$	$\mathbf{T}v^{-1}(x, y)$	$\mathbf{T}t^{-1}(x, y)$
$\mathbf{U}s(x, y)$	$\mathbf{T}k^{-1}(x, y)$	$\mathbf{T}v(x, y)$	$\mathbf{T}u(x, y)$	$\mathbf{T}l^{-1}(x, y)$
$\mathbf{K}s(x, y)$	$\mathbf{T}h^{-1}(x, y)$	$\mathbf{T}t(x, y)$	$\mathbf{T}l(x, y)$	$\mathbf{T}j(x, y)$

Table 1: the **PC** preference structure

The semantics of the ten basic preference relations are discussed in Annex B (see also Tsoukiàs and Vincke, 1997). We just emphasize that the double threshold order as introduced in section 1.1 presents, between the two unambiguous situations of strict preference and indifference, a situation of hesitation due to the conflicting information carried by the presence of the interval between the two thresholds. In such an interval we consider that the difference $g(x) - g(y)$ is sufficient to discriminate two objects, but not yet sufficient to establish a strict preference. Such a situation is captured by the relation H of the **PC** preference structure in which $s(x, y)$ is true, but $s^{-1}(x, y)$ is contradictory. On the basis of this observation we give in the next section the necessary and sufficient conditions for which a binary relation S is a double threshold order. Clearly the "weak preference" relation Q becomes in our case the relation H.

2 Double threshold orders

From Tsoukiàs and Vincke (1995) we remind the following result:

- strict preference (P) is asymmetric:
  $\forall x \;\; \mathbf{F}p(x, x)$
  $\forall x, y \;\; \mathbf{T}p(x, y) \rightarrow \mathbf{F}p(y, x)$
- semi-indifference (H) is semi-reflexive and weakly symmetric:
  $\forall x \;\; \mathbf{K}h(x, x)$
  $\forall x, y \;\; \mathbf{T}h(x, y) \rightarrow \mathbf{K}h(y, x)$
- strict indifference (I) is reflexive and symmetric:
  $\forall x \;\; \mathbf{T}i(x, x)$
  $\forall x, y \;\; \mathbf{T}i(x, y) \rightarrow \mathbf{T}i(y, x)$

In the following we introduce a definition (cfr. with Vincke, 1988) and two theorems concerning the characterization of a double threshold order. Actually the theorems demonstrate the equivalence of three different ways to define a double threshold order. Therefore any one of the three could be adopted as a definition and the other two as theorems. The demonstrations of the theorems are in annex D. Such demonstrations heavily use a set of lemmas and corrolaries which are all in annex C together with their demonstrations. This is in order to facilitate the reader in understanding the principal result of the paper.

Definition 2.1 *A* **PC** *preference structure is a double-threshold order* **iff**

- $J = K = L = T = U = V = R = \emptyset$
- $\forall x, y, z, w \;\; \mathbf{T}h(x, y) \wedge \mathbf{T}i(y, z) \wedge \mathbf{T}h(z, w) \rightarrow \mathbf{T}p(x, w) \vee \mathbf{T}h(x, w)$
- $\forall x, y, z, w \;\; \mathbf{T}h(x, y) \wedge \mathbf{T}i(y, z) \wedge \mathbf{T}p(z, w) \rightarrow \mathbf{T}p(x, w)$

- $\forall x, y, z, w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$
- $\forall x, y, z, w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}h^{-1}(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$

Before we announce the first theorem we remind that from the definition of the **PC** preference structure we have that:
$\forall x, y \;\; \mathbf{T}s(x,y) \equiv \mathbf{T}p(x,y) \vee \mathbf{T}i(x,y) \vee \mathbf{T}h(x,y) \vee \mathbf{T}k(x,y)$
$\forall x, y \;\; \mathbf{K}s(x,y) \equiv \mathbf{T}h^{-1}(x,y) \vee \mathbf{T}j(x,y) \vee \mathbf{T}l(x,y) \vee \mathbf{T}t(x,y)$
$\forall x, y \;\; \mathbf{U}s(x,y) \equiv \mathbf{T}k^{-1}(x,y) \vee \mathbf{T}l^{-1}(x,y) \vee \mathbf{T}u(x,y) \vee \mathbf{T}v(x,y)$
$\forall x, y \;\; \mathbf{F}s(x,y) \equiv \mathbf{T}p^{-1}(x,y) \vee \mathbf{T}t^{-1}(x,y) \vee \mathbf{T}v^{-1}(x,y) \vee \mathbf{T}r(x,y)$
We can give now the first theorem characterizing double threshold orders.

Theorem 2.1 *A binary relation S characterizes a* **PC** *preference structure which is a double threshold order* **iff**
1. $\forall x, y \;\; (\mathbf{T}s(x,y) \vee \mathbf{T}s(y,x)) \wedge \neg \mathbf{U}s(x,y) \wedge \neg \mathbf{U}s(y,x)$.
2. $\forall x, y, z, w \;\; \mathbf{T}s(x,y) \wedge \mathbf{T}s(z,w) \rightarrow \mathbf{T}s(x,w) \vee \mathbf{T}s(z,y)$.
3. $\forall x, y, z, w \;\; \mathbf{T}s(x,y) \wedge \mathbf{K}s(z,w) \rightarrow \mathbf{T}s(x,w) \vee \neg \mathbf{F}s(z,y)$.
4. $\forall x, y, z, w \;\; \mathbf{K}s(x,y) \wedge \mathbf{K}s(z,w) \rightarrow \neg \mathbf{F}s(x,w) \vee \neg \mathbf{F}s(z,y)$.

We give now the theorem about the numerical representation of such preference structure. Such a result updates the ones obtained by Vincke in 1988. Please recall that since the sets A on which the preference relations apply are considered finite and discrete we may use only strict inequalities without any loss of generality of the result.

Theorem 2.2 *A* **PC** *preference structure is a double threshold order* **iff** *there exist three real valued functions, $g : A \mapsto \mathcal{R}$, $k : A \mapsto \mathcal{R}^+$, $l : A \mapsto \mathcal{R}^+$, such that:*
1. $\forall x \;\; k(x) < l(x)$
2. $\forall x, y \;\; \mathbf{T}s(x,y) \Leftrightarrow g(x) > g(y) - k(x)$
3. $\forall x, y \;\; \mathbf{K}s(x,y) \Leftrightarrow g(y) - k(x) > g(x) > g(y) - l(x)$
4. $\forall x, y \;\; \mathbf{F}s(x,y) \Leftrightarrow g(y) - l(x) > g(x)$
5. $\forall x, y \;\; \neg \mathbf{U}s(x,y)$

3 Discussion

The careful reader may notice that a double threshold order could be characterized as a mixture of two interval orders, exactly the s_1 and the s_2 ones. More precisely from the definitions of the two interval orders we have (using classic logic):

$\forall x, y \;\; p(x,y) \equiv s_2(x,y) \wedge \neg s_2^{-1}(x,y)$
$\forall x, y \;\; i(x,y) \equiv s_1(x,y) \wedge s_1^{-1}(x,y)$
$\forall x, y \;\; h(x,y) \equiv s_2^{-1}(x,y) \wedge \neg s_1^{-1}(x,y)$

Moreover the results obtained by our formalism clearly generalize basic results already known in the literature (especially the theorem about the numerical representation). So one may question: why do we need a new characterization which requires a more complicated language to be formulated? We will try to sketch some answers derived from different reasons.

1. *Our approach enables better semantics than the traditional one.* The presence of more than one threshold clearly indicates a zone of uncertainty and/or ambiguity which may be useful to represent.

   In our approach the interval between the two thresholds is identified by a precise epistemic state of the characteristic relation S (to be read as *"at least as good as"*). Moreover such a characterization is not numeric and identifies the precise reasons for which uncertainty / ambiguity arises, that is the presence of contradictory information.

   The different epistemic values of the relation S are explicitly used in the characterization of the double threshold order, thus enabling a clear semantic of the properties associated to such a preference structure.

   Finally no confusion is allowed with preference intensity modeling. The relation h is an uncertain preference (hesitation between preference and indifference) and not a "weak" preference. From this point of view this kind of double threshold order cannot be generalized in multiple threshold orders (as in Doignon, 1987) which are conceived in order to model preference intensity.

2. *Our approach is coherent with the concordance/discordance principle used in decision aid situations.* Moreover it generalizes such a principle in mono-criterion models while traditionally the concordance/discordance concepts have been used in multicriteria decision aid situations.

   It is easy to observe that while the positive reasons, for which the characteristic relation S may be considered to hold (concordance), extend up to the large threshold ($l(x)$), the negative reasons, for which such a relation may be considered not to hold (discordance), start from the small threshold ($k(x)$). In other words the positive and negative evaluations of $s(x,y)$ (and $s^{-1}(x,y)$) are independent and not necessarily complementary. Such a situation may occur when a decision maker is cautious towards his(her) numerical representation and begins to doubt before completely leaving his(her) positive convictions. Such a situation generates an uncertainty and/or ambiguity zone which is explicitly accounted by the contradictory state of S. Such a situation (which normally should entail an inconsistency and therefore the necessity to reconsider the whole model) does not impede neither to model the decision maker's preferences nor to use his(her) numerical representation which may be the only available for the moment or the most confident (the others being even less reliable). We claim that decision aid is possible under inconsistent situations where there are no resources that may further clarify or "improve" the model.

3. *Our approach provides the only axiomatization compatible with the three axioms introduced in section 1.*

   An axiomatic characterization of a preference structure is of course not an absolute necessity and moreover our axioms may be disputed. However when such models are used in decision aid situations, axioms enable one to have a clear idea of the admissible use of the model and of its consequences if it is adopted. We consider that in order to convince analyst (who has to convince on his turn the decision maker) for the appropriateness of a model and of the validity of its results (if applied), an axiomatic characterization is a necessary condition (although it may not be a sufficient one).

Conclusions

A new axiomatization of the double threshold order preference structure is provided in the paper. A four valued paraconsistent first order logic is used as basic language for such an axiomatization. Two main results are reported in the paper, the first concerning the properties of the characteristic relation of the double threshold order (a generalization of the completeness and Ferrers property) and the second concerning its numerical representation.

Such an approach enables a better semantic for the double threshold order, is coherent with the concordance/discordance principle in decision aid situations (besides extending the applicability of such a principle) and is the only consistent with the axioms of "partitioning", "well foundness" and "independence from irrelevant alternatives". The characterization of particular double threshold orders (constant thresholds, consistency conditions etc.) are the next step of the research.

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Annex A

A brief presentation of the DDT logic.

An alphabet of the first order language $\mathcal{L}$, henceforth called DDT, consists of (for a preliminary version, see Doherty et al., 1992):
- a denumerable set of *individual variables* (possibly subscripted):
$x_1, x_2 \cdots y_1, y_2 \cdots z_1, z_2 \cdots$
- the *logical connectives* "$\vee$" (or), "$\wedge$" (and), "$\rightarrow$" (implication), "$\sim$" (complementation), "$\not\sim$" (weak negation) and "$\neg$' (strong negation),
- the *unary operators* "**T**" (true), "**F**" (false), "**U**" (unknown), "**K**" (both), "$\triangle$" (presence of truth),

- the *quantifiers* “$\forall$ (for all) and “$\exists$” (exists),
- the symbols “(” and “)” serving as punctuation marks,
- a countable set of *predicate constants* (i, p, q, r, ...) of positive arity, including “=” for identity.

We use greek letters $\alpha, \beta, \gamma, \cdots$ to represent general formula of the language.

Well-formed formula are defined in the usual way.
If α, β are wff, then $\neg\alpha$, $\nsim\alpha$, $\mathbf{T}\alpha$, $\alpha\wedge\beta$, $\alpha\vee\beta$ etc. are wff.

In the following we give the truth tables of the principal connectives (t,k,u,f being the truth values). In Table 1 are provided the truth tables of the negations and their combinations. In table 2 the truth tables of the three basic binary operations, that is the conjunction, the disjunction and the implication, are provided.

α	$\nsim\alpha$	$\sim\nsim\alpha$	$\sim\alpha$	$\neg\alpha$	$\neg\nsim\alpha$	$\neg\sim\nsim\alpha$	$\neg\sim\alpha$
t	k	u	f	f	k	u	t
k	t	f	u	k	f	t	u
u	f	t	k	u	t	f	k
f	u	k	t	t	u	k	f

Table 1. The truth tables of $\neg$, $\nsim$ and $\sim$ and their combinations.

From an algebraic point of view these eight combinations represent one of the Sylow subgroups of the group of all permutations of four elements and precisely the one preserving complementarity between t and f and between k and u. From this point of view it is easy to observe that the “complementation” can be defined through the other two negations:
$\sim\alpha \equiv \neg\nsim\neg\nsim\alpha \equiv \nsim\neg\nsim\neg\alpha$ ($\equiv$ being the usual connective of identity).

We introduce now the truth tables for the basic binary operators.

$\wedge$	u	f	t	k
u	u	f	u	f
f	f	f	f	f
t	u	f	t	k
k	f	f	k	k

$\vee$	u	f	t	k
u	u	u	t	t
f	u	f	t	k
t	t	t	t	t
k	t	k	t	k

$\rightarrow$	t	k	u	f
t	t	k	u	f
k	t	t	u	u
u	t	k	t	k
f	t	t	t	t

Table 2. The truth tables of $\wedge$, $\vee$ and $\rightarrow$.

The implication introduced here corresponds to the conventional strong monotonic implication. In fact $\alpha\rightarrow\beta$ should be read as *"either the complement of α or β"* (actually we have $\alpha\rightarrow\beta \equiv \sim\alpha\vee\beta$).

A two valued fragment of the language, named DDT^2, can be created introducing some strong unary operators. Their definitions are as follows:

1. $\mathbf{T}\alpha =_{def} \alpha\wedge\sim\neg\alpha$.
2. $\mathbf{F}\alpha =_{def} \sim\alpha\wedge\neg\alpha$.
3. $\mathbf{U}\alpha =_{def} \sim\not\sim\alpha\wedge\neg\not\sim\alpha$.
4. $\mathbf{K}\alpha =_{def} \not\sim\alpha\wedge\not\sim\neg\alpha$.
5. $\triangle\alpha =_{def} \mathbf{T}\alpha\vee\mathbf{K}\alpha$.
6. $\triangle\neg\alpha =_{def} \mathbf{F}\alpha\vee\mathbf{K}\alpha$.

The truth tables for the defined operators are presented in table 3. Actually it is easy to verify that $\triangle\alpha \equiv \mathbf{T}(\alpha\vee\not\sim\alpha)$ and $\triangle\neg\alpha \equiv \mathbf{T}(\neg\alpha\vee\not\sim\neg\alpha)$.

α	$\mathbf{T}\alpha$	$\mathbf{K}\alpha$	$\mathbf{U}\alpha$	$\mathbf{F}\alpha$	$\triangle\alpha$	$\triangle\neg\alpha$
t	t	f	f	f	t	f
k	f	t	f	f	t	t
u	f	f	t	f	f	f
f	f	f	f	t	f	t

Table 3. Truth Tables for the strong unary operators.

Annex B

Semantics of the ten basic preference relations of the **PC** preference structure.

1. $p(x, y)$ (strict preference):
   we can definitely establish that "x is at least as good as y" as there exist sufficient positive reasons supporting it and there are not enough negative reasons against it, while we can surely establish that "y is not as least as good as x" for the opposite reasons; therefore "x is strictly better than y".

2. $t(x, y)$ (semi preference):

   we have doubts concerning "x is at least as good as y" as there exist both sufficient positive reasons supporting it and sufficient negative reasons against it, while we can surely establish that "y is not as least as good as x" because we do not have sufficient positive reasons to assume the opposite and there exist sufficient negative reasons against it; therefore "x could be better than y, but we have some doubts due to the presence of strong negative reasons about $s(x, y)$".

3. $k(x, y)$ (quasi preference):

   we can surely establish that "x is at least as good as y" as there exist sufficient positive reasons supporting it and there are not sufficient negative reasons against it, while we cannot establish that "y is not as least as good as x" or the opposite because we have neither sufficient positive nor negative reasons; therefore "x could be better than y, but we have some doubts because we do not know what happens with $s^{-1}(x, y)$".

4. $l(x, y)$ (maximal hesitation):

   we have doubts concerning "x is at least as good as y" as there exist both sufficient positive reasons supporting it and sufficient negative reasons against it, while we cannot establish that "y is not as least as good as x" or the opposite because we have neither sufficient positive nor negative reasons; therefore "x could be better than y, but we have some doubts both because we have evidence indicating that may hold $\neg s(x, y)$ and we do not know anything about $s^{-1}(x, y)$".

5. $i(x, y)$ (strict indifference):

   we can surely establish that "x is at least as good as y" as there exist sufficient positive reasons supporting it and there are not sufficient negative reasons against it, while we can also surely establish that "y is as least as good as x" for the same reasons; therefore "x and y are strictly indifferent as they could be considered equivalent".

6. $j(x, y)$ (weak indifference):

   we have doubts concerning "x is at least as good as y" as there exist both sufficient positive reasons supporting it and sufficient negative reasons against it, while we have also doubts to establish that "y is at least as good as x" for the same reasons; therefore "x and y could be indifferent", but we doubt due to the presence of strong evidence against it in both directions.

7. $h(x, y)$ (semi indifference):

   we can surely establish that "x is at least as good as y" as there exist sufficient positive reasons supporting it and there are not sufficient negative reasons against it, while we have doubts about "y is as least as good as x" because of the presence of both positive and negative reasons; therefore "x could be indifferent to y (notice in this case that "indifference" is is not symmetric), but we have some doubts to about $s^{-1}(x, y)$".

8. $r(x, y)$ (incomparability):

   we can surely establish that "x is not at least as good as y" as there are not sufficient positive reasons supporting the opposite and there are sufficient negative reasons against it, while we can also surely establish that "y is not as least as good as x" for the same reasons; therefore "x and y are in conflicting position due to strong contrasting information".

9. $u(x, y)$ (weak incomparability):

   we cannot establish that "x is at least as good as y" as there exist neither sufficient positive reasons supporting it nor sufficient negative reasons against it, while we cannot establish that "y is as least as good as x" for the same reasons; therefore "we cannot establish what holds between x and y because of the absence of relevant information".

10. $v(x, y)$ (semi incomparability):

    we can surely establish that "x is not at least as good as y" as there are not sufficient positive reasons supporting the opposite and there are sufficient negative reasons against it, while we cannot establish that "y is as least as good as x" because of the absence of either positive or negative reasons; therefore "x could be in opposition to y, but we have some doubts due to the absence of all the necessary information".

Annex C

In the following we give a sequence of lemmas and corollaries which will be used in the demonstration of the theorems presented in annex D. They all hold for a double threshold preference structure as introduced in definition 2.1.

Lemma 3.1

1. $\forall x, y \;\; \mathbf{T}p(x, y) \wedge \mathbf{T}p(y, z) \rightarrow \mathbf{T}p(x, z)$

2. $\forall x, y \;\; \mathbf{T}h(x, y) \wedge \mathbf{T}p(y, z) \rightarrow \mathbf{T}p(x, z)$

3. $\forall x, y \;\; \mathbf{T}h(x, y) \wedge \mathbf{T}h(y, z) \rightarrow \mathbf{T}p(x, z) \vee \mathbf{T}h(x, z)$

It is easy to verify then the following corollary:

Corollary 3.1

4. $\forall x, y, z, w \;\; \mathbf{T}p(x, y) \wedge \mathbf{T}p(y, z) \wedge \mathbf{T}p(z, w) \rightarrow \mathbf{T}p(x, w)$

5. $\forall x, y, z, w \;\; \mathbf{T}p(x, y) \wedge \mathbf{T}h(y, z) \wedge \mathbf{T}p(z, w) \rightarrow \mathbf{T}p(x, w)$

6. $\forall x, y, z, w \;\; \mathbf{T}h(x, y) \wedge \mathbf{T}p(y, z) \wedge \mathbf{T}p(z, w) \rightarrow \mathbf{T}p(x, w)$

7. $\forall x, y, z, w \;\; \mathbf{T}h(x, y) \wedge \mathbf{T}h(y, z) \wedge \mathbf{T}p(z, w) \rightarrow \mathbf{T}p(x, w)$

8. $\forall x,y,z,w\ \ \mathbf{T}h(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}h(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)$

We continue now with some more results.

Lemma 3.2

9. $\forall x,y,z,w\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}i(y,z)\wedge\mathbf{T}h(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)$

10. $\forall x,y\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}h(y,z)\rightarrow\mathbf{T}p(x,z)\vee\mathbf{T}h(x,z)$

It is easy now to verify the following corollary.

Corollary 3.2

11. $\forall x,y,z,w\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}h(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)$

12. $\forall x,y,z,w\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}h(z,w)\rightarrow$
$\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)$

13. $\forall x,y,z,w\ \ \mathbf{T}h(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}h(z,w)\rightarrow$
$\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)$

We continue with some other useful lemmas

Lemma 3.3

14. $\forall x,y,z,w\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}i(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

15. $\forall x,y,z,w\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}i(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

16. $\forall x,y,z,w\ \ \mathbf{T}h(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}i(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

17. $\forall x,y,z,w\ \ \mathbf{T}h(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}i(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

18. $\forall x,y,z,w\ \ \mathbf{T}i(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}p(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

19. $\forall x,y,z,w\ \ \mathbf{T}i(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}h(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

20. $\forall x,y,z,w\ \ \mathbf{T}i(x,y)\wedge\mathbf{T}p(y,z)\wedge\mathbf{T}i(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

21. $\forall x,y,z,w\ \ \mathbf{T}i(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}p(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

22. $\forall x,y,z,w\ \ \mathbf{T}i(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}h(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

23. $\forall x,y,z,w\ \ \mathbf{T}i(x,y)\wedge\mathbf{T}h(y,z)\wedge\mathbf{T}i(z,w)\rightarrow\mathbf{T}p(x,w)\vee\mathbf{T}h(x,w)\vee\mathbf{T}i(x,w)$

Demonstration of Lemma 2.1.1 From definition 2.1 we have:
$\forall x,y\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}i(y,y)\wedge\mathbf{T}p(y,z)\rightarrow\mathbf{T}p(x,z)$
From reflexivity of relation I we have $\forall x\ \ \mathbf{T}i(x,x)$, therefore
$\forall x,y\ \ \mathbf{T}p(x,y)\wedge\mathbf{T}p(y,z)\rightarrow\mathbf{T}p(x,z)$

Demonstration of Lemma 2.1.2 From definition 2.1 we have:
$\forall x,y\ \ \mathbf{T}h(x,y)\wedge\mathbf{T}i(y,y)\wedge\mathbf{T}p(y,z)\rightarrow\mathbf{T}p(x,z)$

From reflexivity of relation I we have $\forall x \; \mathbf{T}i(x,x)$, therefore
$\forall x,y \; \mathbf{T}h(x,y) \wedge \mathbf{T}p(y,z) \rightarrow \mathbf{T}p(x,z)$

Demonstration of Lemma 2.1.3 From definition 2.1 we have:
$\forall x,y \; \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,y) \wedge \mathbf{T}h(y,z) \rightarrow \mathbf{T}p(x,z) \vee \mathbf{T}h(x,z)$
From reflexivity of relation I we have $\forall x \; \mathbf{T}i(x,x)$, therefore
$\forall x,y \; \mathbf{T}h(x,y) \wedge \mathbf{T}h(y,z) \rightarrow \mathbf{T}p(x,z) \vee \mathbf{T}h(x,z)$

Demonstration of Lemma 2.2.9 Suppose you have:
$\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(w,x)$
From corollary 2.1.6 we have:
$\forall x,y,z,w \; \mathbf{T}h(z,w) \wedge \mathbf{T}p(w,x) \wedge \mathbf{T}p(x,y) \rightarrow \mathbf{T}p(z,y)$
which contradicts with $\mathbf{T}i(y,z)$
Suppose you have:
$\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}h(w,x)$
From corollary 2.1.7 we have:
$\forall x,y,z,w \; \mathbf{T}h(z,w) \wedge \mathbf{T}h(w,x) \wedge \mathbf{T}p(x,y) \rightarrow \mathbf{T}p(z,y)$
which contradicts with $\mathbf{T}i(y,z)$
Suppose you have:
$\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}i(w,x)$
From definition 2.1 we have:
$\forall x,y,z,w \; \mathbf{T}h(z,w) \wedge \mathbf{T}i(w,x) \wedge \mathbf{T}p(x,y) \rightarrow \mathbf{T}p(z,y)$
which contradicts with $\mathbf{T}i(y,z)$
Therefore $\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w)$

Demonstration of Lemma 2.2.10 From the previous lemma we have:
$\forall x,y \; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,y) \wedge \mathbf{T}h(y,z) \rightarrow \mathbf{T}p(x,z) \vee \mathbf{T}h(x,z)$
From reflexivity of relation I we have $\forall x \; \mathbf{T}i(x,x)$, therefore
$\forall x,y \; \mathbf{T}p(x,y) \wedge \mathbf{T}h(y,z) \rightarrow \mathbf{T}p(x,z) \vee \mathbf{T}h(x,z)$

Demonstration of Lemma 2.3.14 Suppose you have:
$\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}p(w,x)$
From corollary 2.1.4 we have:
$\forall x,y,z,w \; \mathbf{T}p(w,x) \wedge \mathbf{T}p(x,y) \wedge \mathbf{T}p(y,z) \rightarrow \mathbf{T}p(w,z)$
which contradicts with $\mathbf{T}i(z,w)$
Suppose you have:
$\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}h(w,x)$
From corollary 2.1.6 we have:
$\forall x,y,z,w \; \mathbf{T}h(w,x) \wedge \mathbf{T}p(x,y) \wedge \mathbf{T}p(y,z) \rightarrow \mathbf{T}p(w,z)$
which contradicts with $\mathbf{T}i(z,w)$
Therefore
$\forall x,y,z,w \; \mathbf{T}p(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w) \vee \mathbf{T}i(x,w)$

Demonstration of Lemma 2.3.15 Suppose you have:
$\forall x, y, z, w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}h(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}p(w,x)$
From corollary 2.2.11 we have:
$\forall x, y, z, w \;\; \mathbf{T}p(w,x) \wedge \mathbf{T}p(x,y) \wedge \mathbf{T}h(y,z) \rightarrow \mathbf{T}p(w,z) \vee \mathbf{T}h(w,z)$
which contradicts with $\mathbf{T}i(z,w)$
Suppose you have:
$\forall x, y, z, w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}h(w,x)$
From corollary 2.2.13 we have:
$\forall x, y, z, w \;\; \mathbf{T}h(w,x) \wedge \mathbf{T}p(x,y) \wedge \mathbf{T}h(y,z) \rightarrow \mathbf{T}p(w,z) \vee \mathbf{T}h(w,z)$
which contradicts with $\mathbf{T}i(z,w)$
Therefore
$\forall x, y, z, w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}h(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w) \vee \mathbf{T}i(x,w)$

Demonstration of Lemma 2.3.16 Suppose you have:
$\forall x, y, z, w \;\; \mathbf{T}h(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}p(w,x)$
From corollary 2.1.5 we have:
$\forall x, y, z, w \;\; \mathbf{T}p(w,x) \wedge \mathbf{T}h(x,y) \wedge \mathbf{T}p(y,z) \rightarrow \mathbf{T}p(w,z)$
which contradicts with $\mathbf{T}i(z,w)$
Suppose you have:
$\forall x, y, z, w \;\; \mathbf{T}h(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}h(w,x)$
From corollary 2.1.7 we have:
$\forall x, y, z, w \;\; \mathbf{T}h(w,x) \wedge \mathbf{T}h(x,y) \wedge \mathbf{T}p(y,z) \rightarrow \mathbf{T}p(w,z)$
which contradicts with $\mathbf{T}i(z,w)$
Therefore
$\forall x, y, z, w \;\; \mathbf{T}h(x,y) \wedge \mathbf{T}p(y,z) \wedge \mathbf{T}i(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w) \vee \mathbf{T}i(x,w)$

The demonstrations of Lemmas 2.3.17 to 2.3.23 are similar to the previous ones and are omitted.

Annex D

Proof of Theorem 2.1

1. We first demonstrate the necessity part.
1.1 $\forall x, y \;\; (\mathbf{T}s(x,y) \vee \mathbf{T}s(y,x)) \wedge \neg \mathbf{U}s(x,y) \wedge \neg \mathbf{U}s(y,x)$.
It is easy to demonstrate that:
$\forall x, y \;\; \mathbf{T}p(x,y) \equiv \mathbf{T}s(x,y) \wedge \mathbf{F}s(y,x)$
$\forall x, y \;\; \mathbf{T}i(x,y) \equiv \mathbf{T}s(x,y) \wedge \mathbf{T}s(y,x)$
$\forall x, y \;\; \mathbf{T}h(x,y) \equiv \mathbf{T}s(x,y) \wedge \mathbf{K}s(y,x)$
The relation S being double threshold order, it holds that:
$J = K = L = T = U = V = R = \emptyset$ therefore
$\forall x, y \;\; \mathbf{T}p(x,y) \vee \mathbf{T}h(x,y) \vee \mathbf{T}i(x,y) \vee \mathbf{T}h(y,x) \vee \mathbf{T}p(y,x)$, therefore
$\forall x, y \;\; (\mathbf{T}s(x,y) \wedge \mathbf{F}s(y,x)) \vee (\mathbf{T}s(x,y) \wedge \mathbf{K}s(y,x)) \vee$
$(\mathbf{T}s(x,y) \wedge \mathbf{T}s(y,x)) \vee (\mathbf{T}s(x,y) \wedge \mathbf{T}s(y,x)) \vee$

$(\mathbf{K}s(x,y)\wedge\mathbf{T}s(y,x))\vee(\mathbf{F}s(x,y)\wedge\mathbf{T}s(y,x))$, that is
$\forall x,y\ \ (\mathbf{T}s(x,y)\wedge(\mathbf{F}s(y,x)\vee\mathbf{T}s(y,x)\vee\mathbf{K}s(y,x))\vee$
$(\mathbf{T}s(y,x)\wedge(\mathbf{F}s(x,y))\vee\mathbf{T}s(x,y)\vee\mathbf{K}s(x,y))$, that is
$\forall x,y\ \ (\mathbf{T}s(x,y)\wedge\neg\mathbf{U}s(y,x))\vee(\mathbf{T}s(y,x)\wedge\neg\mathbf{U}s(x,y))$
It is easy to verify the identity $\mathbf{T}\alpha\equiv\mathbf{T}\alpha\wedge\neg\mathbf{U}\alpha$ obtaining
$\forall x,y\ \ (\mathbf{T}s(x,y)\wedge\neg\mathbf{U}s(y,x)\wedge\neg\mathbf{U}s(x,y))\vee(\mathbf{T}s(y,x)\wedge\neg\mathbf{U}s(x,y)\wedge\neg\mathbf{U}s(y,x))$, that is
$\forall x,y\ \ (\mathbf{T}s(x,y)\vee\mathbf{T}s(y,x))\wedge\neg\mathbf{U}s(x,y)\wedge\neg\mathbf{U}s(y,x)$.
1.2 $\forall x,y,z,w\ \ \mathbf{T}s(x,y)\wedge\mathbf{T}s(z,w)\rightarrow\mathbf{T}s(x,w)\vee\mathbf{T}s(z,y)$.
Suppose that $\neg\mathbf{T}s(x,w)\wedge\neg\mathbf{T}s(z,y)$.
Due to the completeness property previously demonstrated we obtain:
$\forall x,y,z,w\ \ \mathbf{T}s(w,x)\wedge\mathbf{T}s(y,z)$. Therefore we have:
$\forall x,y,z,w\ \ \mathbf{T}s(z,w)\wedge\mathbf{T}s(w,x)\wedge\mathbf{T}s(x,y)$.
That gives the following combinations:

$$\begin{bmatrix}\mathbf{T}p(z,w)\\ \mathbf{T}h(z,w)\\ \mathbf{T}i(z,w)\end{bmatrix}\wedge\begin{bmatrix}\mathbf{T}p(w,x)\\ \mathbf{T}h(w,x)\end{bmatrix}\wedge\begin{bmatrix}\mathbf{T}p(x,y)\\ \mathbf{T}h(x,y)\\ \mathbf{T}i(x,y)\end{bmatrix}$$

Please remark that due to $\neg\mathbf{T}s(w,x)$ we cannot have $\mathbf{T}i(w,x)$.
It is easy now to verify from the lemmas and corollaries that all the combinations (there are 18) give as result $\mathbf{T}p(z,y)\vee\mathbf{T}h(z,y)\vee\mathbf{T}i(z,y)$ which implies $\mathbf{T}s(z,y)$ and which contradicts $\neg\mathbf{T}s(z,y)$.
1.3 $\forall x,y,z,w\ \ \mathbf{T}s(x,y)\wedge\mathbf{K}s(z,w)\rightarrow\mathbf{T}s(x,w)\vee\neg\mathbf{F}s(z,y)$.
Suppose that $\neg\mathbf{T}s(x,w)\wedge\mathbf{F}s(z,y)$.
Due to the completeness property previously demonstrated we obtain:
$\forall x,y,z,w\ \ \mathbf{T}s(w,x)\wedge\mathbf{T}s(y,z)$. Therefore we have:
$\forall x,y,z,w\ \ \mathbf{T}s(w,x)\wedge\mathbf{T}s(x,y)\wedge\mathbf{T}s(y,z)$.
That gives the following combinations:

$$\begin{bmatrix}\mathbf{T}p(w,x)\\ \mathbf{T}h(w,x)\end{bmatrix}\wedge\begin{bmatrix}\mathbf{T}p(x,y)\\ \mathbf{T}h(x,y)\\ \mathbf{T}i(x,y)\end{bmatrix}\wedge\begin{bmatrix}\mathbf{T}p(y,z)\end{bmatrix}$$

Please remark that due to $\neg\mathbf{T}s(w,x)$ we cannot have $\mathbf{T}i(w,x)$ and that due to $\mathbf{F}s(z,y)$ we have exactly $\mathbf{T}p(y,z)$.
It is easy now to verify from the lemmas and corollaries that all the combinations (there are 6) give as result $\mathbf{T}p(w,z)$ which contradicts $\mathbf{K}s(z,w)$.
1.4 $\forall x,y,z,w\ \ \mathbf{K}s(x,y)\wedge\mathbf{K}s(z,w)\rightarrow\neg\mathbf{F}s(x,w)\vee\neg\mathbf{F}s(z,y)$.
Suppose that $\mathbf{F}s(x,w)\wedge\mathbf{F}s(z,y)$.
Due to the completeness property previously demonstrated we obtain:
$\forall x,y,z,w\ \ \mathbf{T}s(w,x)\wedge\mathbf{T}s(y,z)$. Therefore we have:
$\forall x,y,z,w\ \ \mathbf{T}s(w,x)\wedge\mathbf{T}s(x,y)\wedge\mathbf{T}s(y,z)$.
That gives the following combinations:

$$\begin{bmatrix}\mathbf{T}p(w,x)\end{bmatrix}\wedge\begin{bmatrix}\mathbf{T}p(x,y)\\ \mathbf{T}h(x,y)\\ \mathbf{T}i(x,y)\end{bmatrix}\wedge\begin{bmatrix}\mathbf{T}p(y,z)\end{bmatrix}$$

Please remark that due to $\mathbf{F}s(x,w)$ we have exactly $\mathbf{T}p(w,x)$ and that due to $\mathbf{F}s(z,y)$ we have exactly $\mathbf{T}p(y,z)$.
It is easy now to verify from the lemmas and corollaries that all the combinations (there are 3) give as result $\mathbf{T}p(w,z)$ which contradicts $\mathbf{K}s(z,w)$.

2. We now demonstrate the sufficiency. We have to demonstrate the following:

1. $J = K = L = T = U = V = R = \emptyset$
2. strict preference P is asymmetric and irreflexive:
   $\forall x \ \ \mathbf{F}p(x,x)$
   $\forall x,y \ \ \mathbf{T}p(x,y) \rightarrow \mathbf{F}p(y,x)$
3. semi-indifference H is semi-reflexive and weakly symmetric:
   $\forall x \ \ \mathbf{K}h(x,x)$
   $\forall x,y \ \ \mathbf{T}h(x,y) \rightarrow \mathbf{K}h(y,x)$
4. indifference I is reflexive and symmetric:
   $\forall x \ \ \mathbf{T}i(x,x)$
   $\forall x,y \ \ \mathbf{T}i(x,y) \rightarrow \mathbf{T}i(y,x)$
5. $\forall x,y,z,w \ \ \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w)$
6. $\forall x,y,z,w \ \ \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$
7. $\forall x,y,z,w \ \ \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$
8. $\forall x,y,z,w \ \ \mathbf{T}p(x,y) \wedge \mathbf{T}h^{-1}(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(x,w)$

2.1 Doing the demonstration of 1.2 backwards it is easy to verify the condition.
2.2 It follows from the definition of P. A demonstration is also available in Tsoukiàs and Vincke (1995).
2.3 It follows from the definition of H. A demonstration is also available in Tsoukiàs and Vincke (1995).
2.4 It follows from the definition of I. A demonstration is also available in Tsoukiàs and Vincke (1995).
2.5 $\forall x,y,z,w \ \ \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w)$, that is:
$\forall x,y,z,w \ \ \mathbf{T}s(x,y) \wedge \mathbf{K}s(y,x) \wedge \mathbf{T}s(y,z) \wedge \mathbf{T}s(z,y) \wedge \mathbf{T}s(z,w) \wedge \mathbf{K}s(w,z) \rightarrow$
$\mathbf{T}s(x,w) \wedge (\mathbf{K}s(w,x) \vee \mathbf{F}s(w,x))$
Suppose that is $\mathbf{T}s(w,x)$, then we have from 1.2:
$\forall x,y,z,w \ \ \mathbf{T}s(w,x) \wedge \mathbf{T}s(y,z) \rightarrow \mathbf{T}s(w,z) \vee \mathbf{T}s(y,x)$
which contradicts $\mathbf{K}s(z,w) \wedge \mathbf{K}(x,y)$,
that is $\neg \mathbf{T}s(w,x)$ and from the completeness result we have:
$\mathbf{T}s(x,w) \wedge (\mathbf{K}s(w,x) \vee \mathbf{F}s(w,x))$.
2.6 $\forall x,y,z,w \ \ \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$, that is:
$\forall x,y,z,w \ \ \mathbf{T}s(x,y) \wedge \mathbf{K}s(y,x) \wedge \mathbf{T}s(y,z) \wedge \mathbf{T}s(z,y) \wedge \mathbf{T}s(z,w) \wedge \mathbf{F}s(w,z) \rightarrow$
$\mathbf{T}s(x,w) \wedge \mathbf{F}s(w,x)$
As previously it is possible to demonstrate that $\mathbf{T}s(x,w)$

and that $\mathbf{K}s(w,x) \vee \mathbf{F}s(w,x)$.
Suppose now that $\mathbf{K}s(w,x)$ then we have from 1.3
$\forall x,y,z,w \;\; \mathbf{K}s(w,x) \wedge \mathbf{T}s(y,z) \rightarrow \mathbf{T}s(y,x) \vee \neg \mathbf{F}s(w,z)$
which contradicts $\mathbf{F}s(w,z) \wedge \mathbf{K}(y,x)$,
that is $\neg \mathbf{K}s(w,x)$ and from the completeness result we have:
$\mathbf{T}s(x,w) \wedge \mathbf{F}s(w,x)$.
2.7 $\forall x,y,z,w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$
The demonstration is exactly as the previous one.
2.8 $\forall x,y,z,w \;\; \mathbf{T}p(x,y) \wedge \mathbf{T}h^{-1}(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(x,w)$, that is:
$\forall x,y,z,w \;\; \mathbf{T}s(x,y) \wedge \mathbf{F}s(y,x) \wedge \mathbf{K}s(y,z) \wedge \mathbf{T}s(z,y) \wedge \mathbf{T}s(z,w) \wedge \mathbf{F}s(w,z) \rightarrow$
$\mathbf{T}s(x,w) \wedge \mathbf{F}s(w,x)$
As previously it is possible to demonstrate that $\mathbf{T}s(x,w)$
and that $\mathbf{K}s(w,x) \vee \mathbf{F}s(w,x)$.
Suppose now that $\mathbf{K}s(w,x)$ then we have from 1.4
$\forall x,y,z,w \;\; \mathbf{K}s(w,x) \wedge \mathbf{K}s(y,z) \rightarrow \neg \mathbf{F}s(y,x) \vee \neg \mathbf{F}s(w,z)$
which contradicts $\mathbf{F}s(w,z) \wedge \mathbf{F}(y,x)$,
that is $\neg \mathbf{K}s(w,x)$ and from the completeness result we have:
$\mathbf{T}s(x,w) \wedge \mathbf{F}s(w,x)$.

And this completes the demonstration of Theorem 2.1.

■

Proof of Theorem 2.2

1. We first demonstrate sufficiency. We assume that the three real valued functions exist and we have to demonstrate that the corresponding preference **PC** preference structure is a double threshold order, that is the eight conditions of definition 2.1.

1.1 It is easy to observe that since $\forall x,y \;\; \neg \mathbf{U}s(x,y)$ there are only nine combinations between $s(x,y)$ and $s^{-1}(x,y)$. However from the definitions of $s(x,y)$ it is also possible to observe that $\forall x,y \;\; \mathbf{K}s(x,y) \rightarrow \mathbf{T}s^{-1}(x,y)$ and that also $\forall x,y \;\; \mathbf{F}s(x,y) \rightarrow \mathbf{T}s^{-1}(x,y)$ thus remaining only five possibilities and precisely:
- $\mathbf{T}s(x,y) \wedge \mathbf{T}s^{-1}(x,y) \equiv \mathbf{T}i(x,y)$
- $\mathbf{T}s(x,y) \wedge \mathbf{K}s^{-1}(x,y) \equiv \mathbf{T}h(x,y)$
- $\mathbf{T}s(x,y) \wedge \mathbf{F}s^{-1}(x,y) \equiv \mathbf{T}p(x,y)$
- $\mathbf{K}s(x,y) \wedge \mathbf{T}s^{-1}(x,y) \equiv \mathbf{T}h^{-1}(x,y)$
- $\mathbf{F}s(x,y) \wedge \mathbf{T}s^{-1}(x,y) \equiv \mathbf{T}p^{-1}(x,y)$

Therefore all relations except p,h,i are empty.

From the above definitions it is possible to obtain also:
- $\forall x,y \;\; \mathbf{T}p(x,y) \Leftrightarrow g(x) > g(y) + l(y)$
- $\forall x,y \;\; \mathbf{T}h(x,y) \Leftrightarrow g(y) + l(y) > g(x) > g(y) + k(y)$
- $\forall x,y \;\; \mathbf{T}i(x,y) \Leftrightarrow g(y) + k(y) > g(x) > g(y) - k(x)$

1.2, 1.3 and 1.4 can be immediately derived from the definitions and the properties of the **PC** preference structure.

1.5 We have to demonstrate that:

$\forall x,y,z,w \ \ \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}h(z,w) \rightarrow \mathbf{T}p(x,w) \vee \mathbf{T}h(x,w)$

From $\mathbf{T}h(x,y)$ we have: $g(x) > g(y) + k(y)$

From $\mathbf{T}i(y,z)$ we have: $g(y) > g(z) - k(y)$

From $\mathbf{T}h(z,w)$ we have: $g(z) > g(w) + k(w)$

And summing we obtain: $g(x) > g(w) + k(w)$

which is equivalent to: $\mathbf{T}p(x,w) \vee \mathbf{T}h(x,w)$

1.6 We have to demonstrate that:

$\forall x,y,z,w \ \ \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$

From $\mathbf{T}h(x,y)$ we have: $g(x) > g(y) + k(y)$

From $\mathbf{T}i(y,z)$ we have: $g(y) > g(z) - k(y)$

From $\mathbf{T}p(z,w)$ we have: $g(z) > g(w) + l(w)$

And summing we obtain: $g(x) > g(w) + l(w)$

which is equivalent to: $\mathbf{T}p(x,w)$

1.7 We have to demonstrate that:

$\forall x,y,z,w \ \ \mathbf{T}p(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$

From $\mathbf{T}p(x,y)$ we have: $g(x) > g(y) + l(y)$

From $\mathbf{T}i(y,z)$ we have: $g(y) > g(z) - k(y)$

From $\mathbf{T}p(z,w)$ we have: $g(z) > g(w) + l(w)$

And summing we obtain: $g(x) > g(w) + k(w) + l(y) - k(y)$

But since $l(y) - k(y) > 0$

we obtain: $g(x) > g(w) + l(w)$

which is equivalent to: $\mathbf{T}p(x,w)$

1.8 We have to demonstrate that:

$\forall x,y,z,w \ \ \mathbf{T}p(x,y) \wedge \mathbf{T}h^{-1}(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$

From $\mathbf{T}p(x,y)$ we have: $g(x) > g(y) + l(y)$

From $\mathbf{T}h^{-1}(y,z)$ we have: $g(y) > g(z) - l(y)$

From $\mathbf{T}p(z,w)$ we have: $g(z) > g(w) + l(w)$

And summing we obtain: $g(x) > g(w) + l(w)$

which is equivalent to: $\mathbf{T}p(x,w)$

We give now the demonstration of the necessity part. Given a double threshold order (as defined by definition 2.1 and theorem 2.1) we have to demonstrate the existence of the three real valued functions.

We introduce two binary relations s_1 and s_2 as follows:

- $\forall x,y \ \ \mathbf{T}s_1(x,y) \equiv \mathbf{T}p(x,y) \vee \mathbf{T}h(x,y) \vee \mathbf{T}i(x,y)$
- $\forall x,y \ \ \mathbf{F}s_1(x,y) \equiv \mathbf{T}p^{-1}(x,y) \vee \mathbf{T}h^{-1}(x,y)$
- $\forall x,y \ \ \neg\mathbf{K}s_1(x,y) \wedge \neg\mathbf{U}s_1(x,y)$
- $\forall x,y \ \ \mathbf{T}s_2(x,y) \equiv \mathbf{T}p(x,y) \vee \mathbf{T}h(x,y) \vee \mathbf{T}i(x,y) \vee \mathbf{T}h^{-1}(x,y)$
- $\forall x,y \ \ \mathbf{F}s_2(x,y) \equiv \mathbf{T}p^{-1}(x,y)$
- $\forall x,y \ \ \neg\mathbf{K}s_2(x,y) \wedge \neg\mathbf{U}s_2(x,y)$

We therefore have:

- $\forall x, y \;\; \mathbf{T}p_1(x,y) \equiv \mathbf{T}p(x,y) \vee \mathbf{T}h(x,y)$
- $\forall x, y \;\; \mathbf{T}i_1(x,y) \equiv \mathbf{T}i(x,y)$
- $\forall x, y \;\; \mathbf{T}p_2(x,y) \equiv \mathbf{T}p(x,y)$
- $\forall x, y \;\; \mathbf{T}i_2(x,y) \equiv \mathbf{T}i(x,y) \vee \mathbf{T}h(x,y) \vee \mathbf{T}h^{-1}(x,y)$

any other relation of **PC** being empty (demonstration obvious).
We claim that s_1 and s_2 are interval orders (for the extension of the concept of interval order and semi order under the DDT language see Annex D).

1. We have to demonstrate that:
   1. (strong completeness) $\forall x, y \;\; (\mathbf{T}s_1(x,y) \vee \mathbf{T}s_1^{-1}(x,y))$
   $\wedge \neg \mathbf{K}s_1(x,y) \wedge \neg \mathbf{K}s_1^{-1}(x,y) \wedge \neg \mathbf{U}s_1(x,y) \wedge \neg \mathbf{U}s_1^{-1}(x,y)$
   2. (strong Ferrers property).
   $\forall x, y, z, w \;\; \mathbf{T}p_1(x,y) \wedge \mathbf{T}i_1(y,z) \wedge \mathbf{T}p_1(z,w) \rightarrow \mathbf{T}p_1(x,w)$
   The first property is obvious by definition of s_1.
   The second property is equivalent to:
   $\forall x, y, z, w \;\; (\mathbf{T}p(x,y) \vee \mathbf{T}h(x,y)) \wedge \mathbf{T}i(y,z) \wedge (\mathbf{T}p(z,w) \vee \mathbf{T}h(z,w)) \rightarrow$
   $(\mathbf{T}p(x,w) \vee \mathbf{T}h(x,w)$
   which is verified since $\langle p, h, i \rangle$ is a double threshold order.

2. We have to demonstrate that:
   1. (strong completeness) $\forall x, y \;\; (\mathbf{T}s_2(x,y) \vee \mathbf{T}s_2^{-1}(x,y))$
   $\wedge \neg \mathbf{K}s_2(x,y) \wedge \neg \mathbf{K}s_2^{-1}(x,y) \wedge \neg \mathbf{U}s_2(x,y) \wedge \neg \mathbf{U}s_2^{-1}(x,y)$
   2. (strong Ferrers property).
   $\forall x, y, z, w \;\; \mathbf{T}p_2(x,y) \wedge \mathbf{T}i_2(y,z) \wedge \mathbf{T}p_2(z,w) \rightarrow \mathbf{T}p_1(x,w)$
   The first property is obvious by definition of s_2.
   The second property is equivalent to:
   $\forall x, y, z, w \;\; \mathbf{T}p(x,y) \wedge (\mathbf{T}i(y,z) \vee \mathbf{T}h(y,z) \vee \mathbf{T}h^{-1}s(y,z)) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$
   which is verified since $\langle p, h, i \rangle$ is a double threshold order.

Therefore there exist real valued functions $g_1(x)$, $g_2(x)$, $k(x)$, $l(x)$, such that:
$\forall x, y \;\; \mathbf{T}s_1(x,y) \Leftrightarrow g_1(x) > g_1(y) - k(y)$
$\forall x, y \;\; \mathbf{T}s_2(x,y) \Leftrightarrow g_2(x) > g_2(y) - l(y)$
Therefore:
$\forall x, y \;\; \mathbf{T}p_1(x,y) \Leftrightarrow g_1(x) > g_1(y) + k(y)$
$\forall x, y \;\; \mathbf{T}i_1(x,y) \Leftrightarrow g_1(y) + k(y) > g_1(x) > g_1(y) - k(x)$
$\forall x, y \;\; \mathbf{T}p_2(x,y) \Leftrightarrow g_2(x) > g_2(y) + l(y)$
$\forall x, y \;\; \mathbf{T}i_2(x,y) \Leftrightarrow g_2(y) + l(y) > g_2(x) > g_2(y) - l(x)$
We claim that it is always possible to choose g_1 and g_2 so that
$\forall x \;\; g_1(x) = g_2(x)$ and $l(x) > k(x)$ and fulfill the conditions of the double threshold order.
Since $\langle p, h, i \rangle$ form a double threshold order the following holds:
$\forall x, y, z, w \;\; \mathbf{T}h(x,y) \wedge \mathbf{T}i(y,z) \wedge \mathbf{T}p(z,w) \rightarrow \mathbf{T}p(x,w)$

Using the numerical representations of s_1 and s_2 we have respectively:

$$\begin{array}{ll} s_1 & s_2 \\ g_1(x) > g_1(y) + k(y) & g_2(y) + l(y) > g_2(x) > g_2(y) - l(x) \\ g_1(z) + k(z) > g_1(y) > g_1(z) - k(y) & g_2(z) + l(z) > g_2(y) > g_2(z) - l(y) \\ g_1(z) > g_1(w) + k(w) & g_2(z) > g_2(w) + l(w) \end{array}$$

Therefore we have that the following hold simultaneously:
$g_1(x) > g_1(y) + k(y)$
$g_1(y) > g_1(z) - k(y)$
$g_2(z) > g_2(w) + l(w)$
and summing we have:
$g_1(x) + g_2(z) > g_1(z) + g_2(w) + l(w)$
Since $\langle p, h, i\rangle$ is a double threshold order, in the above case $p(x, w)$ holds and therefore $p_2(x, w)$ holds. So we have $g_2(x) > g_2(w) + l(w)$ and this is always possible when $\forall x \;\; g_1(x) = g_2(x)$
Moreover from the definition of p_1 and i_2 it is possible to observe that the two relations have a common extension, so they can commonly hold. In order to keep the extension non empty we may have $p_1(x, y) \wedge i_2(x, y)$ which translates in (remind that $g_1(x) = g_2(x) = g(x)$) :
$g(x) > g(y) + k(y)$ and $g(y) + l(y) > g(x)$
and therefore $l(x) > k(x)$
And this completes demonstration of Theorem 2.2.

■

Annex E

1. Interval orders

   Given a set A and a **PC** preference structure where P and I represent strict preference and indifference respectively, define S as the characteristic relation of such **PC** structure. Then S is an interval order **iff**
   1. $\forall x, y \;\; (\mathbf{T}s(x,y) \vee \mathbf{T}s(y,x)) \wedge \neg\mathbf{K}s(x,y) \wedge \neg\mathbf{K}s(y,x) \wedge \neg\mathbf{U}s(x,y) \wedge \neg\mathbf{U}s(x,y)$ (strong completeness)
   2. $\forall x, y, z, w \;\; \mathbf{T}s(x,y) \wedge \mathbf{T}s(z,w) \rightarrow \mathbf{T}s(x,w) \vee \mathbf{T}s(z,y)$ (strong Ferrers property).

2. Semi-orders

   Given a set A and a **PC** preference structure where P and I represent strict preference and indifference respectively, define S as the characteristic relation of such **PC** structure. Then S is a semi-order **iff**
   1. $\forall x, y \;\; (\mathbf{T}s(x,y) \vee \mathbf{T}s(y,x)) \wedge \neg\mathbf{K}s(x,y) \wedge \neg\mathbf{K}s(y,x) \wedge \neg\mathbf{U}s(x,y) \wedge \neg\mathbf{U}s(x,y)$ (strong completeness)
   2. $\forall x, y, z, w \;\; \mathbf{T}s(x,y) \wedge \mathbf{T}s(z,w) \rightarrow \mathbf{T}s(x,w) \vee \mathbf{T}s(z,y)$ (strong Ferrers property).
   3. $\forall x, y, z, w \;\; \mathbf{T}s(x,y) \wedge \mathbf{T}s(y,z) \rightarrow \mathbf{T}s(x,w) \vee \mathbf{T}s(w,z)$ (strong semi-transitivity).