It's a graph! It's a polytope! No, it's a PSS!

Clément Royer

Derivative-Free Optimization Symposium - June 25, 2026

About this talk

- Derivative-free...

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- Algorithm-free...

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- Optimization-free...

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- But NOT application-free!

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DFO is an application of graph theory!

Outline

1. Positive spanning sets
2. A graphical view of PSS
3. Ear decomposition of PSSs

Outline

Positive spanning sets

Definitions

- $D = [d_1 \ \cdots \ d_p] \in \mathbb{R}^{n \times p}$ is a **positive spanning set (PSS)** if

$$\mathbb{R}^n = \mathrm{cone}(D) = \left\{ \sum_{i=1}^{p} \lambda_i d_i \;\middle|\; \lambda_i \geq 0 \right\}.$$

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Cardinality [Theorems 7.1 and 7.4 in Audet, Hare 2026]

- D PSS $\Rightarrow |D| \geq n+1$.
- D positive basis $\Rightarrow n+1 \leq |D| \leq 2n$.

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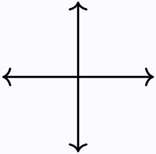
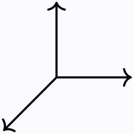
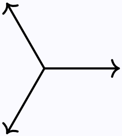
- D PSS $\Rightarrow |D| \geq n+1$.
  Proof: Linear algebra
- D positive basis $\Rightarrow n+1 \leq |D| \leq 2n$.
  Proof: Linear programming/Polytopes

$\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ canonical basis of $\mathbb{R}^n$.

- $D^{\oplus} = [\boldsymbol{e}_1 \ \cdots \ \boldsymbol{e}_n \ \ -\boldsymbol{e}_1 \ \ -\boldsymbol{e}_n]$.
- $D^{\circleddash} = \left[\boldsymbol{e}_1 \ \cdots \ \boldsymbol{e}_n \ \ -\sum_{i=1}^n \boldsymbol{e}_i\right]$.
- $D^{\otimes}$: $n+1$ vectors with uniform angles, including $\boldsymbol{e}_1$.

	$D^{\oplus}$	$D^{\circleddash}$	$D^{\otimes}$
$n = 2$			

"maximal"   "minimal"

Equivalent positive bases (from Price and Coope '03)

Two PSSs D_1 and D_2 are **equivalent** ($D_1 \equiv D_2$) if

$$D_1 = B D_2 \Delta P$$

- B invertible
- Δ diagonal $\succ 0$.
- P permutation matrix.

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What this tells us PSS/PB property independent of

- Choice of basis (Not true for cosine measure).
- Column scaling.
- Column order.

Ex) $D^{\text{◷}} \equiv D^{\text{⊛}}$

Equivalence and positive bases

Equivalence $D_1 \equiv D_2 \;\Leftrightarrow\; D_1 = BD_2\Delta P$.

Equivalence of positive bases

- Any minimal positive basis $D \in \mathbb{R}^{n\times(n+1)}$ is equivalent to $D^{\circleddash} = [I_n \;\; -\sum_i e_i]$.
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- Geometry of PSS/positive bases important for DFO...
- ...but less so to certify PSS/positive basis property?

Outline

Storyline (1/2)

- This is Denis (Cornaz).
- Denis likes graphs.
- Denis thinks PSSs looks like graphs.

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Should we think like Denis?

Some graph theory

$G = (V, A)$ directed graph (digraph) with vertices and arcs.

- G strongly connected if there is a directed path between each pair of vertices.
- G **minimally strongly connected** if no proper subgraph of G is strongly connected.

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Theorem (García-López and Marijuán '12)

If G is **minimally strongly connected** with $|V| = n + 1$, then

$$n + 1 \leq |A| \leq 2n.$$

Network matrix and digraphs

Network matrix

- A function of a graph $G = (V, A)$ and spanning tree $T = (V, \hat{A})$.
- $\{-1, 0, 1\}$ matrix that represents edges in A using edges in $\hat{A}$.

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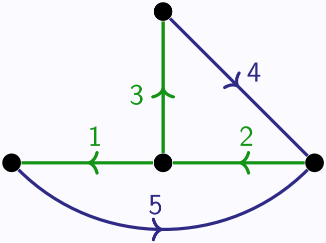
Example



$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

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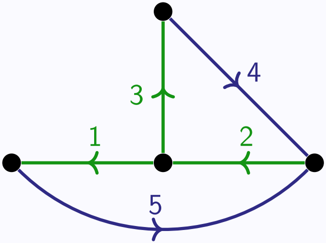

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

This network matrix is a PSS!

Link between graphs and PSSs

Theorem (Cornaz, Kerleau, R. '25)

Let G be a digraph with network matrix M. Then,

- G is strongly connected $\Leftrightarrow$ M is a positive spanning set.
- G is minimally strongly connected $\Leftrightarrow$ M is a positive basis.

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Has this been done before?

Discrete Applied Mathematics 9 (1984) 47–67 47
North-Holland

GALE DIAGRAMS OF CONVEX POLYTOPES AND POSITIVE SPANNING SETS OF VECTORS*

Daniel A. MARCUS
California State Polytechnic University, Pomona, CA 91768, USA

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- "*It can be shown [11]*"

[11] D. Marcus, Circulation polytopes associated with directed graphs, to appear.

- Our (algebraic) proof seems new!

Graphs and PSSs

There is a connection!

- Strongly connected digraphs define PSSs (network matrices).
- Minimally strongly connected digraphs define positive bases.

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How can we exploit this connection?

- New characterization of PSS?
- New examples of PSSs?

Outline

Storyline (2/2)

- This is Sébastien (Kerleau).
- Sébastien got his PhD in 2025.
- Sébastien got into graphs and PSSs.

Ear decomposition of digraphs

Theorem (Bang-Jensen & Gutin 2009)

A digraph is strongly connected iff it has an ear decomposition.

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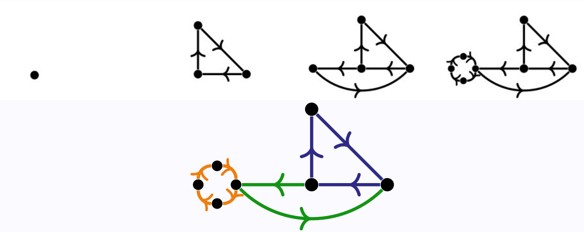

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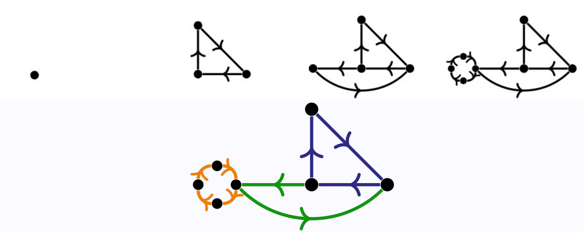

Key points

- Identify circuits in graphs.
- Can be done in linear time (NOT cosine measure).

An ear decomposition for PSS

Theorem (Cornaz, Kerleau, R. 2025)

1. D is a PSS iff
   $D \equiv \begin{bmatrix} I_n & N & X \end{bmatrix}$
   N **negative echelon form**,
   X arbitrary.
2. If D is a positive basis, then
   $D \equiv \begin{bmatrix} I_n & N \end{bmatrix}$
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$$N = \begin{bmatrix} -1 & \times & \times & \times \\ -1 & \times & \times & \times \\ 0 & -1 & \times & \times \\ 0 & -1 & \times & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \times \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

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- Related to a decomposition theorem by (Romanowicz, 1987).
- Part 2 is an iff for orthogonally structured positive bases (Hare, Jarry-Bolduc, Planiden 2023).

New positive bases from graphs (1/2)

Recall We now how to generate positive bases of $\mathbb{R}^n$ with

- $n+1$ vectors: $\equiv [I_n \ \ -\sum_i e_i]$.
- $2n$ vectors: $\equiv [I_n \ \ -I_n]$.

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Corollary

$D \in \mathbb{R}^{n\times(2n-1)}$ is a positive basis iff

$$D \equiv \begin{bmatrix} I_n & N \end{bmatrix}, \qquad N = \begin{bmatrix} -1 & 0_{n-2}^{\mathrm{T}} \\ -1 & v^{\mathrm{T}} \\ 0_{n-2} & -I_{n-2} \end{bmatrix}, \quad v \in \{-1,0\}^{n-2}.$$

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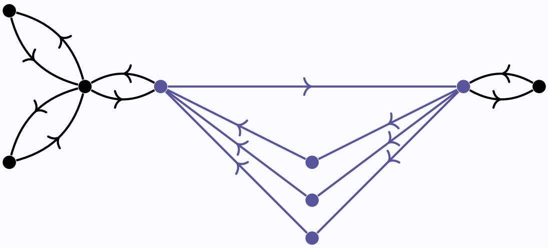
- $v = 0_{n-2}$: Get OSPB (Hare/Jarry-Bolduc/Planiden 2023).
- Can build graphs for which $v \neq 0_{n-2}$!

Positive basis with $2n-1$ elements



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→ Defines a positive basis of size 15 in $\mathbb{R}^8$!

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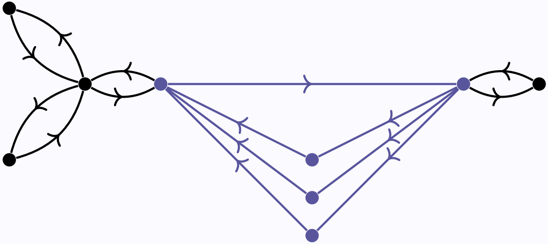

$\rightarrow$ Minimally strongly connected digraph with 9 vertices and 15 edges.
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- Easy visualization of a positive basis.
- Possibly new graph characterizations
  *Ex) Bi-directed forest + **circuits of size 3 with a common arc.***
- Combinatorial structure to analyze case by case
  (done for $2n-2$, $n+2$).

Summary

Positive spanning sets and bases

- Spanning property stable by transformations.
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- Strongly connected digraphs (and minimality).
- Ear-like decomposition.
- Nice visualization of PSS structure!

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Open questions

- Cosine measures of graphical PSSs (of course).
- Use in DFO algorithms (maybe).

References

- D. Cornaz, S. Kerleau and C. W. Royer,
  A characterization of positive spanning sets with ties to strongly connected digraphs,
  Discrete Applied Mathematics, 2025.
- S. Kerleau,
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Thank you!

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Positive k-spanning sets (PkSS)

Definition (Marcus '81-'84)

$D \subset \mathbb{R}^n$ is a PkSS with $k \geq 1$ if i

- Any $\mathcal{N} \subset D$ with $|\mathcal{N}| = |D| - k + 1$ is a PSS.
- Removing $k-1$ vectors from D does not change its PSS nature.

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$\rightarrow k = 1$: Regular definition of PSS.

Generating positive k-bases

Positive k-basis (PkB)

A PkSS D is a positive k-basis if no proper subset of D is a PkSS.

Intricate definition

- Want minimality (positive basis) and resiliency (PkSS).
- Previous examples: May not be PkBs, definitely not of smallest cardinality.

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Can we use graphs to generate PkBs?

The right graph notion

G is k-strongly connected if every pair of vertices is connected by at least k paths.

Minimally 2-strongly connected digraphs.

Generating PkSSs from graphs

The right graph notion

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Minimally 2-strongly connected digraphs.

Properties (Dalmazzo '77, Marcus '84)

- G is k-strongly connected if and only if its network matrix is a PkSS.
- If $G = (V, E)$ has $n+1$ vertices and is **minimally** k-strongly connected, then

$$k(n+1) \leq |E| \leq 2kn.$$

PkSSs are not just graphs

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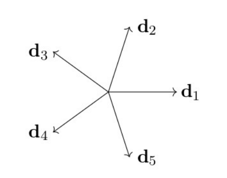

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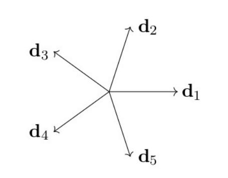

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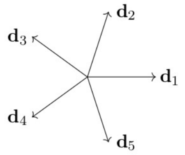

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What graphs bring here

- Representation.
- Easy construction.