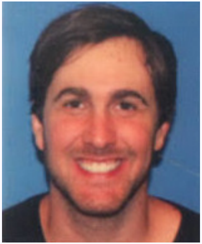
Positive spanning sets and their connections to polyhedra

Clément Royer

LIPN, 19 mars 2026

Collaborators

Denis Cornaz, Sébastien Kerleau, Warren Hare, Gabriel Jarry-Bolduc

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   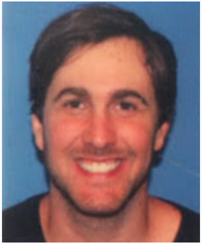

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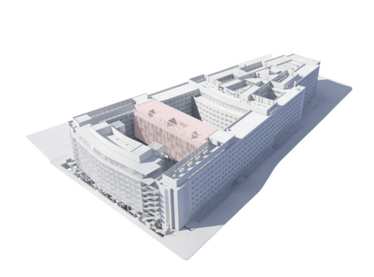
Disclaimer

- Talk/Plots based on Sébastien's thesis (Nov. 2025).
- Less polyhedra than I planned (sorry)
- But graphs, polytopes, and pictures!

Motivation: Dauphine's *Nouveau Campus*



- Building renovation, one wing at a time.
- Faculty/Staff to be moved during renovation.

Our task: Allocate office space during renovation.

Motivation: Dauphine's *Nouveau Campus* ('ed)

Our model for the Dauphine problem

- Huge integer LP, solved via Gurobi.
- ~ 20 hyperparameters defining the model.
- Parallel runs on the department server.

Sub-task: Optimize hyperparameters to get the best LP optimal value.

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Problem challenges

- Cannot differentiate (easily) within Gurobi.
  $\rightarrow$ **Derivative-free algorithms!**
- Cannot solve certain instances (48 hours for a feasible point!)
  $\rightarrow$ **Resiliency to straggler evaluations**.

Blackbox optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

Blackbox/Derivative-free optimization

- **Derivatives unavailable for algorithmic use.**
- Only access to values of f.
- Cost Number of function evaluations.
- Those can take a long time to compute!

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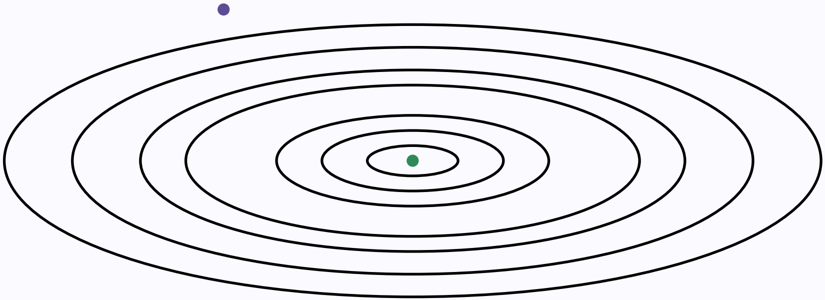
Direct search methods

- Proceed by polling the space along suitable directions.
- Rich convergence theory (Kolda et al '03, Dzahini et al. '25).

How direct search works



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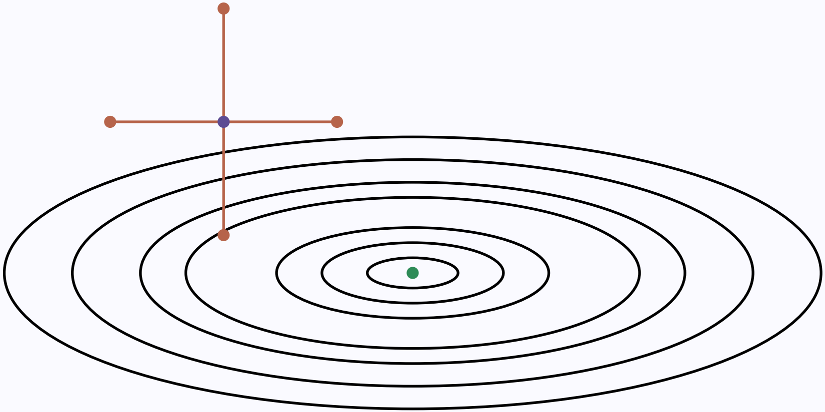

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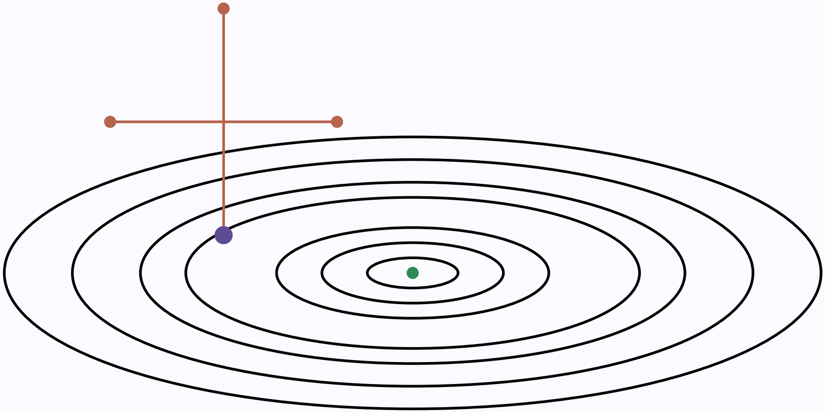

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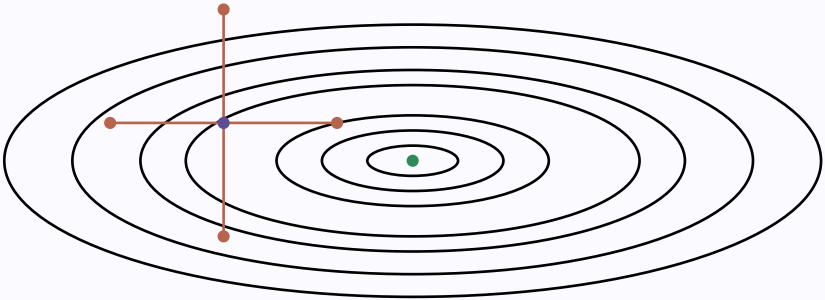

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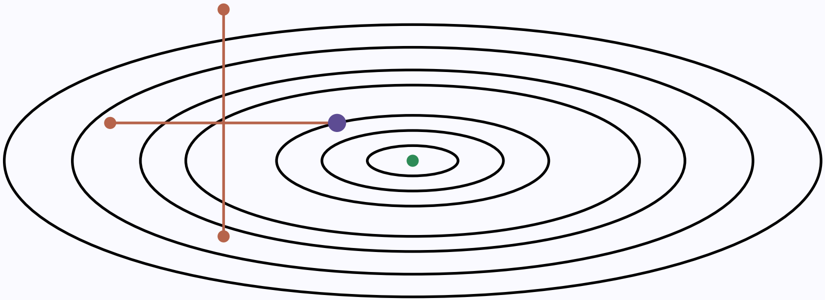

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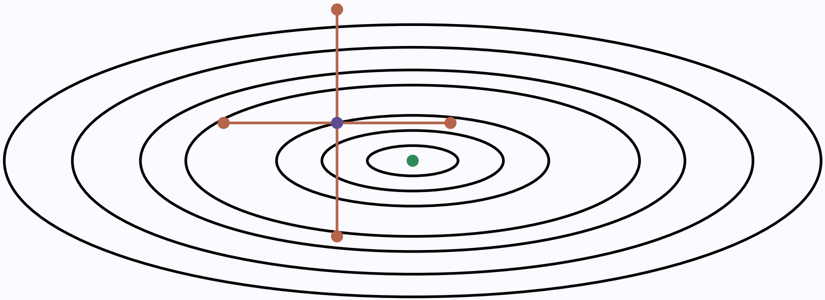

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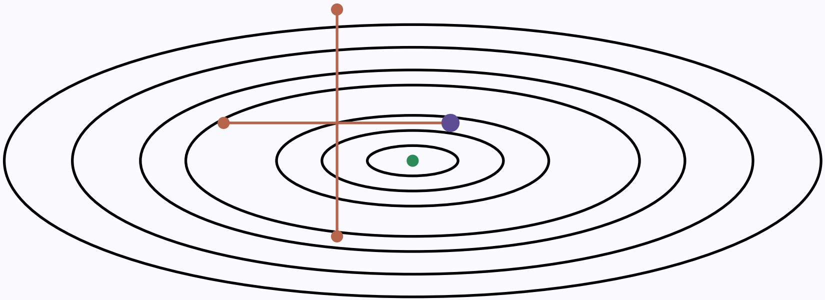

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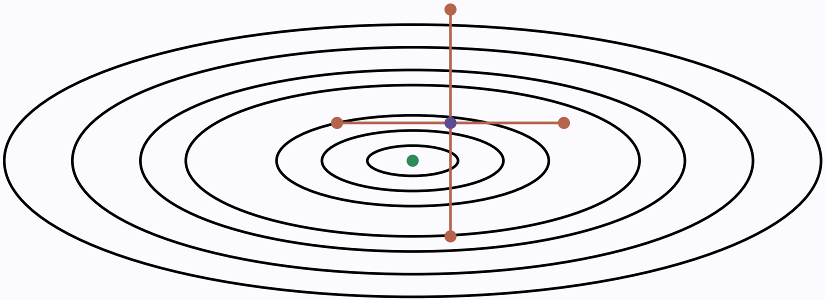

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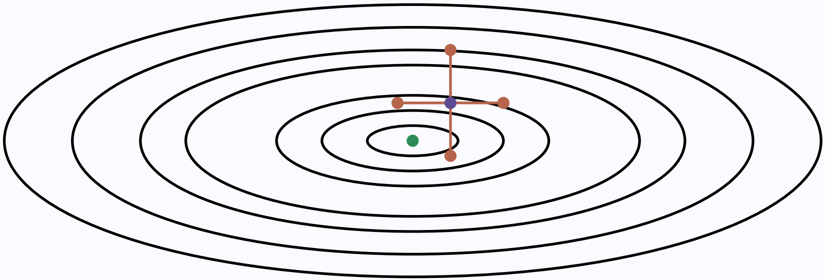

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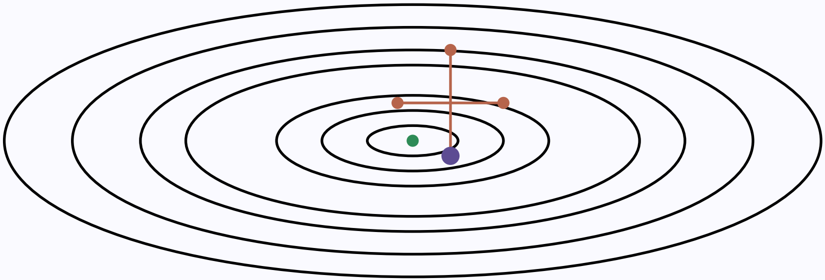

A direct-search method

Inputs: $x_0 \in \mathbb{R}^n$, $\alpha_0 > 0$, D nonzero vectors in $\mathbb{R}^n$.

Iteration i**:** Given (x_i, α_i),

- If $\exists\ d_i \in D$ such that

$$f(x_i + \alpha_i\, d_i) < f(x_i) - \alpha_i^2$$

  set $x_{i+1} := x_i + \alpha_i d_i$, $\alpha_{i+1} := 2\alpha_i$.
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Choosing directions

- Need good geometry within $\mathbb{R}^n$.
- Cost: Want as few directions as possible.

The rest of the talk

Choosing directions

- Positive spanning sets are a good choice!
- Some are better than others.

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Surprising discrete connection

- Can generate directions from graphs...
- ...and improve structural understanding of positive spanning sets.

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Surprising discrete connection

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- ...and improve structural understanding of positive spanning sets.

Resilient positive spanning sets

- Polytope theory more informative than graphs.
- There are open questions!

Outline

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Positive spanning sets (Davis, 1954)

Definitions

- $D = [d_1 \ \cdots \ d_p] \in \mathbb{R}^{n \times p}$ is a **positive spanning set (PSS)** if

$$\mathbb{R}^n = \text{cone}(D) = \left\{ \sum_{i=1}^{p} \lambda_i d_i \;\middle|\; \lambda_i \geq 0 \right\}.$$

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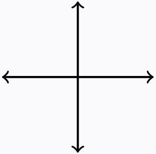
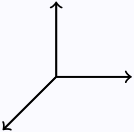
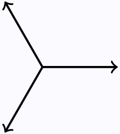
Cardinality

- D PSS $\Rightarrow |D| \geq n+1$.
- D positive basis $\Rightarrow n+1 \leq |D| \leq 2n$.
  $\rightarrow$ Can be proven via linear programming (Audet '11).

PSS	$D^{\oplus}$	$D^{\oslash}$	$D^{\otimes}$
$n = 2$			

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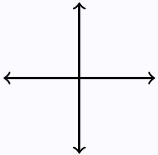
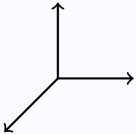
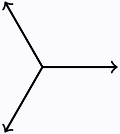
$n = 2$			

$\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ canonical basis of $\mathbb{R}^n$.

- $D^{\oplus}$: Coordinate vectors and their negatives in $\mathbb{R}^n$.
- $D^{\oslash}$: $n+1$ vectors with uniform angles, including $\boldsymbol{e}_1$.
- $D^{\otimes}$: Coordinate vectors and $\boldsymbol{e}_{n+1} = -\sum_{i=1}^{n} \boldsymbol{e}_i$.

Back to direct search

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- What is the best possible D?
- In theory, expressed via *complexity measure*.

Complexity measure

- Cosine measure: $\mathrm{cm}(D) = \max_{d \in D} \min_{\|\boldsymbol{v}\|=1} \frac{d^{\mathrm{T}} \boldsymbol{v}}{\|d\|}$.
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Examples

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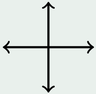
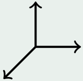
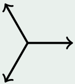
$n=2$			
$\|D\|$	$2n$	$n+1$	$n+1$
$\mathrm{cm}(D)$	$\frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{n^2+2(n-1)\sqrt{n}}}$	$\frac{1}{n}$
$\chi(D)$	$2n^2$	$n^3 + O(n^{3/2})$	$n^3 + O(n^2)$

Why consider multiple PSS?

- Best set in theory conjectured to be $D^{\oplus}$ (Dodangeh et al '16)
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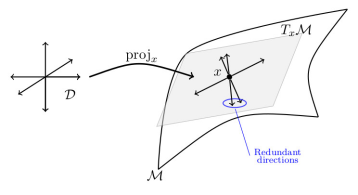

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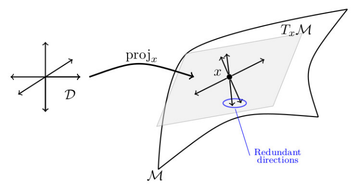

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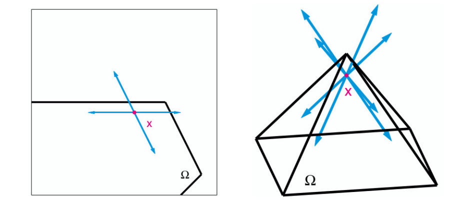

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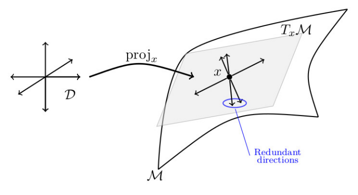

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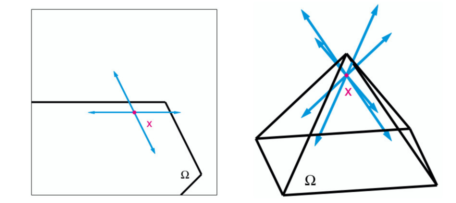

In short

Positive spanning sets

- Span the space by conic combinations ($\geq n+1$ vectors).
- If minimal: Positive bases ($\leq 2n$ vectors).

Key aspects

- **Cardinality**.
- Structure/geometry of a PSS.

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Bringing PSSs to Dauphine

- *"These PSSs look like graphs"* (D. Cornaz, 2020).
- *"What are you talking about?"* (C. Royer, 2020).

Outline

Some (unrelated?) graph theory

$G = (V, E)$ directed graph (digraph).

- G strongly connected if there is a directed path between each pair $(i, j) \in V^2$.
- G **minimally strongly connected** if no proper subgraph of G is strongly connected.

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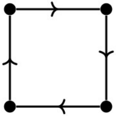

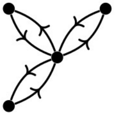

Network matrix

Given $G=(V,A)$ and a spanning tree $T=(V,\hat{A})$, the **network matrix** $M \in \{-1,0,1\}^{|\hat{A}|\times|A|}$ expresses how the path from each edge in A uses edges from $\hat{A}$.

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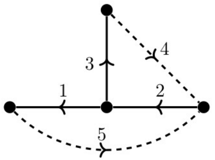
Example



$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Spanning tree (1,2,3), columns of M are arcs in order.

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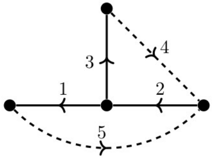

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This network matrix is a PSS!

Link between graphs and PSSs

Theorem (New algebraic proof - Cornaz, Kerleau, R. '25)

Let G be a digraph with network matrix M. Then,

- G is strongly connected $\Leftrightarrow$ M is a positive spanning set.
- G is minimally strongly connected $\Leftrightarrow$ M is a positive basis.

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Corollary: Cardinality of positive bases

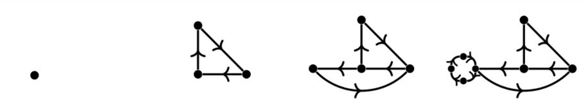
- Cardinality of minimally strongly connected digraphs (García-López and Marijuán '12)

$$G = (V, A) \text{ strongly connected} \quad \Leftrightarrow \quad |V| \leq |A| \leq 2(|V| - 1)$$

- Yields the bound for positive bases in $\mathbb{R}^{|V|-1}$!
  $\rightarrow$ Key: Invariance of positive basis property through transformations (scaling, permutation).

Other consequences

- **Ear decomposition** Characterizes strongly connected digraphs



$\rightarrow$ Applies to PSSs as well!
$\rightarrow$ More explicit than existing decompositions (Romanowicz '87).

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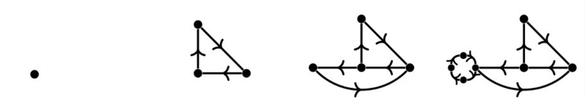

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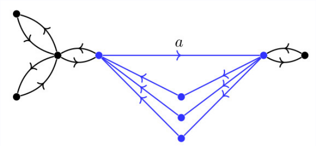
- **New PSS/PB generation** for non-extreme sizes



→ Minimally strongly connected digraph with 9 vertices and 15 edges.
→ Defines a positive basis of size 15 in $\mathbb{R}^8$!

In short: Graphs and PSS

- Strongly connected digraphs $\rightarrow$ PSS.
- Minimality $\rightarrow$ Positive bases.

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What this enables

- Generates PSSs from graphs!
- Characterize positive bases of certain cardinality.

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Main problem

- Sometimes PSSs are not enough.
- Can graphs help there?

Outline

Back to blackbox optimization

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

Blackbox/Derivative-free optimization

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- **Cost** Number of function evaluations.
- **Those can take a long time to compute**.

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Straggler and direct search

- Some evaluations (stragglers) must be ignored at each iteration.
- Motivation: Long solving times on a cluster.

Back to direct search

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- Otherwise, set $x_{i+1} := x_i$, $\alpha_{i+1} := \alpha_i/2$.

Back to direct search

Inputs: $x_0 \in \mathbb{R}^n$, $\alpha_0 > 0$, D nonzero vectors.
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- $\mathcal{S}_i$: Straggler evaluations (not known ahead of time!)
- Need to choose D so as to be robust to missing evaluations.

Positive k-spanning sets (PkSS)

Definition (Marcus '81-'84)

$D \subset \mathbb{R}^n$ is a PkSS with $k \geq 1$ if

- Any $\mathcal{N} \subset D$ with $|\mathcal{N}| = |D| - k + 1$ is a PSS.
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The k-cosine measure (Hare, Jarry-Bolduc, Kerleau, R. '24)

- For any $D \subset \mathbb{R}^n$, the k-cosine measure of D is

$$\mathrm{cm}_k(D) = \min_{\substack{\mathcal{N} \subset D \\ |\mathcal{N}| = |D| - k + 1}} \mathrm{cm}(\mathcal{N}).$$

- D PkSS $\iff \mathrm{cm}_k(D) > 0$.

Building a PkSS from a PSS: First approach

- $D^{\oplus} := \{e_1, \dots, e_n, -e_1, \dots, -e_n\}$, with $\{e_\ell\}$ coordinate basis vectors.
- $\beta D^{\oplus}$: multiply all vectors by real β.

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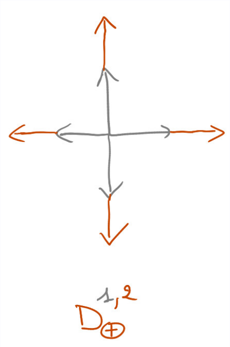
Duplicate vectors

If $\beta_1, \dots, \beta_k > 0$ are distinct,

$$D^{\oplus}_{\beta_{1:k}} := \bigcup_{j=1}^{k} \beta_j D^{\oplus}$$

is a PkSS with

$$\mathrm{cm}_k(D^{\oplus}_{\beta_{1:k}}) = \mathrm{cm}(D^{\oplus}) = \frac{1}{\sqrt{n}}.$$



Building a PkSSfrom a PSS: Second approach

- $D^{\otimes} := \{e_1, \dots, e_n, -\sum_{l=1}^{n} e_l\}$, with $\{e_\ell\}$ coordinate basis vectors.
- $RD^{\otimes}$: Apply **Rotation matrix** $R \in \mathbb{R}^{n \times n}$ to all vectors.

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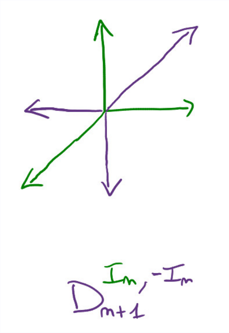
Rotate vectors

Let $R_1, \dots, R_k$ be k distinct positive real numbers. The set

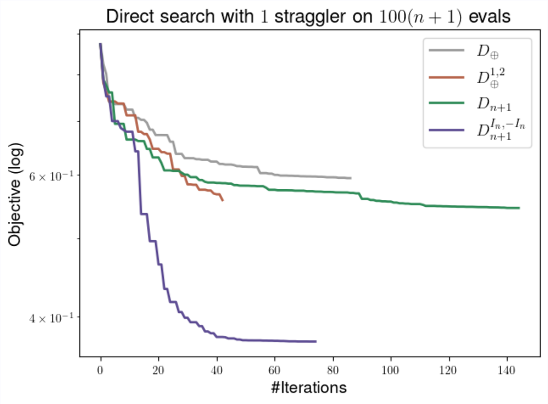
$$D^{\otimes}_{R_{1:k}} := \bigcup_{j=1}^{k} R_j D^{\otimes}$$

is a PkSS with

$$\mathrm{cm}_k(D^{\otimes}_{R_{1:k}}) \geq \mathrm{cm}(D^{\otimes}) = \frac{1}{\sqrt{n^2+2(n-1)\sqrt{n}}}.$$



Numerical illustration



- Nonconvex regression problem, $n = 10$ (Carmon et al '17).
- P2SSs obtained by duplication or rotation.
- Outperform PSSs based on $D^{\oplus}$ or $D^{\oslash}$.

Positive k-basis (PkB)

A PkSS D is a positive k-basis if no proper subset of D is a PkSS.

Intricate definition

- Want minimality (positive basis) and resiliency (PkSS).
- Previous examples: May not be PkBs, definitely not of smallest cardinality.

Generating positive k-bases

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Can we use graphs to generate PkBs?

Outline

The right graph notion

G is k-strongly connected if every pair of vertices is connected by at least k paths.

Minimally 2-strongly connected digraphs.

Generating PkSSs from graphs

The right graph notion

G is k-strongly connected if every pair of vertices is connected by at least k paths.

Minimally 2*-strongly connected digraphs.*

Properties (Dalmazzo '77, Marcus '84)

- G is k-strongly connected if and only if its network matrix is a PkSS.
- If $G = (V, E)$ has $n+1$ vertices and is **minimally** k-strongly connected, then

$$k(n+1) \leq |E| \leq 2kn.$$

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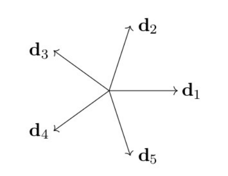

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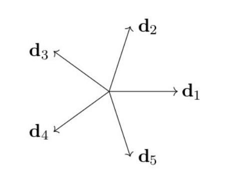

Can tools from polytope theory help?

Gale diagram

X_D is a Gale diagram of $D \in \mathbb{R}^{n \times p}$ if the columns of X_D^{T} form a basis for the null space of D.

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Theorem (Shephard '71)

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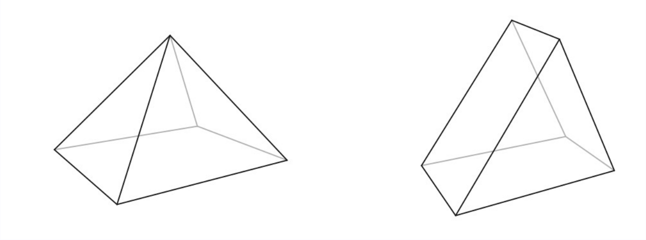
For polytopes

Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^m$ and $P \in \mathbb{R}^{p \times m}$ the matrix of its vertices coordinates. A Gale diagram for P is called a Gale diagram for $\mathcal{P}$.

Classes of polytopes $\mathcal{P}$ convex polytope, P matrix of vertices.

- $\mathcal{P}$ is lifted if last row of P is all ones.
- $\mathcal{P}$ is k-neighborly if each set of $\leq k$ vertices generates a face.
- $\mathcal{P}$ is k-unneighborly if each vertex can be included in a set of $k+1$ vertices that do not generate a face.



4-neighborly and 1-unneighborly polytopes in $\mathbb{R}^3$ (Woltzlaw '09).

Generating PkSS from polytopes

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Theorem (Marcus '84)

Let $\mathcal{P}$ be a lifted polytope in $\mathbb{R}^p$ with $q > p$ vertices, and P its vertex matrix. Let X_P be a Gale diagram for $\mathcal{P}$.

- X_P is a PkSS for $\mathbb{R}^{q-p} \Leftrightarrow \mathcal{P}$ $k-1$-neighborly.
- If $\mathcal{P}$ is also $k-1$-unneighborly, X_P positive k-basis!

Cardinality of positive k-bases

Corollary Minimal cardinality of a PkB is $2k + n - 1$.

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Construction for fixed (k,n)

- Set $q = 2k+n-1$, $p = q-n$.
- Pick q different values $t_1,\dots,t_q$ and let

$$\mathcal{P} = \mathrm{conv}\left\{ \begin{bmatrix} 1 \\ t_i \\ \vdots \\ t_i^{q-1} \end{bmatrix} \ i=1,\dots,q \right\}.$$

- X_P (Gale diagram) gives a positive k-basis of cardinality q.

Example $(k=n=2,\ t_i=i)$

$$X_P = \begin{bmatrix} -1 & 3 & -3 & 1 & 0 \\ -3 & 8 & -6 & 0 & 1 \end{bmatrix}.$$

Cardinality of positive k-bases

Marcus' counterexample ('84)

- Original conjecture: maximum size of a positive k-basis is $2kn$.
- Claimed to have found a counterexample

> **Note added in proof**
> As this goes to press, the author has discovered an unneighborly polytope of dimension 36 having only 49 vertices.

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Theorem (Woltzlaw '09)

- If $n \leq 12$ or $k = 1$, largest cardinality of positive k-bases is $2kn$.
- Otherwise, it is at most $kn(n+1)^{k-1}$.

$\rightarrow$ Largest examples found have cardinality $\mathcal{O}(kn^2)$.

$\rightarrow$ **Exact value still unknown!**

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Positive spanning sets and bases

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Open questions

- Cardinality of positive k-bases.
- Polytopes give explicit constructions.

References

- D. Cornaz, S. Kerleau and C. W. Royer, *A characterization of positive spanning sets with ties to strongly connected digraphs*, Discrete Applied Mathematics, 2025.
- W. Hare, G. Jarry-Bolduc, S. Kerleau and C. W. Royer, *Using orthogonally structured positive bases for constructing positive k-spanning sets with cosine measure guarantees*, Linear Algebra and Applications, 2024.
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Thank you!

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