Revisiting linear programming techniques for nonconvex phase retrieval and unbalanced optimal transport

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Collaborators



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- Gaetano Agazzotti (Dauphine)
- Antonin Chambolle (CNRS, Dauphine)

A Friday afternoon topic

Linear programming (LP)

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0.$$

- Most classical optimization problem.
- Numerous applications: OR, Finance, etc.
- Mature general-purpose solvers: Gurobi, CPLEX, HiGHS.

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What is left to do?

- Classical methods challenged by problem sizes.
- Rise of primal-dual algorithms (nonlinear!) since 2020s.

S. J. Wright, Optimization in theory in practice, 2025

Links between linear and nonlinear programming

Interior-point methods

- Newton-type methods with polynomial guarantees.
- Extend to other convex programming setting such as SDPs:

$$\min_{\boldsymbol{X} \in \mathbb{R}^{n \times n}} \langle \boldsymbol{C}, \boldsymbol{X} \rangle \quad \text{s.t.} \quad \mathcal{A}(\boldsymbol{X}) = \boldsymbol{b}, \boldsymbol{X} \succeq 0.$$

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Simplex algorithms

- Classical algorithm for linear programming.
- Numerous variants but no polynomial guarantees.
- Network simplex: Structured variant for transportation problems?

$\rightarrow$ Can we adapt these methods to **nonlinear settings**?

What this talk is about

Nonlinear algorithms

- Goal: Reach high precision.
- Tools: Modern optimization techniques/perspectives.

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Strict saddle optimization

- Special classes of nonconvex problems.
- Good theory/practice of nonlinear algorithms.
- Focus: Phase retrieval.

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- Special classes of nonconvex problems.
- Good theory/practice of nonlinear algorithms.
- Focus: Phase retrieval.

Adapted network simplex

- Extension of network simplex methods to nonlinear settings.
- Focus on unbalanced optimal transport.

Outline

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An easy example

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} \; x_1 - x_2 \qquad \text{s.t.} \qquad x_1 + x_2 = 1, \boldsymbol{x} \geq 0.$$

$\rightarrow$ Inequality constraints: difficult part in LP!

$\rightarrow$ Naive/Old approach: Square variables!

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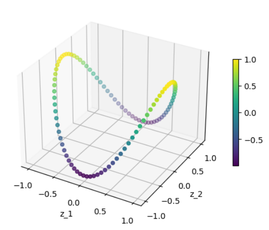
Hadamard parametrization

$$\min_{\boldsymbol{z}\in\mathbb{R}^2} z_1^2 - z_2^2 \qquad \text{s.t.} \qquad z_1^2 + z_2^2 = 1.$$

- Nonconvex objective.
- Sphere constraint.



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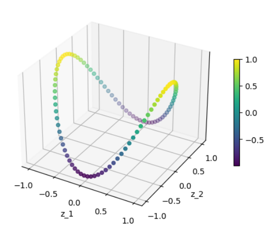
- Nonconvex objective.
  But all minima are global!
- Sphere constraint.
  But tractable optimality conditions!



Benign landscape property

Definition

$f : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{C}^2$ has **benign landscape** if

$$\|\nabla f(\boldsymbol{x})\| = 0 \text{ and } \nabla^2 f(\boldsymbol{x}) \succeq 0 \quad \Leftrightarrow \quad \boldsymbol{x} \in \operatorname*{argmin}_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}).$$

- Every local minimum is global.
- Non-minima with $\|\nabla f(\boldsymbol{x})\| = 0$ are strict saddle points ($\nabla^2 f(\boldsymbol{x}) \not\succeq 0$).

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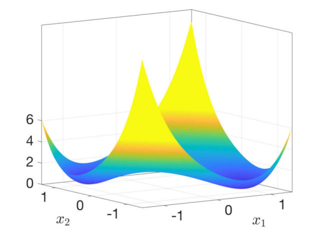
Typical sources of benign nonconvex landscape

- Nonconvex formulations with symmetries.
  $\rightarrow$ Multiple (redundant) solutions, easy to go from one to another.
- Overparameterized models.
  $\rightarrow$ Introduces saddle points but those are strict.

Rank-1 approximation

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \frac{1}{4}\|\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} - \boldsymbol{M}\|_F^2$$



Some examples (pictures from Chi et al '19; Wright and Ma '22)

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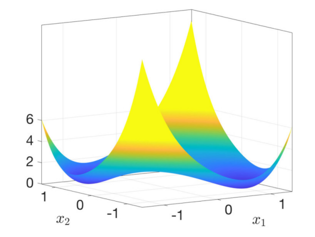
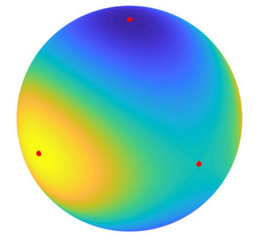


Minimum Eigenvalue problem

$$\min_{\|\boldsymbol{x}\|=1}\ \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$$



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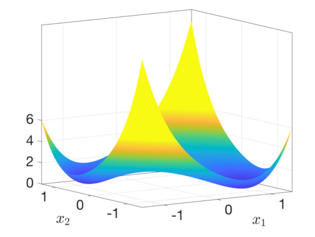

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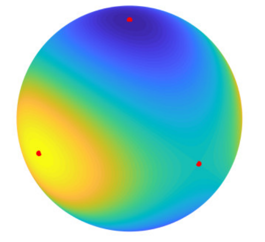

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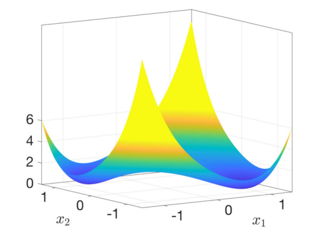

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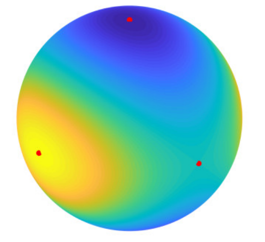

For more: https://sunju.org/research/nonconvex/

Motivation: Nonconvex problems are hard

- Finding points such that $\nabla f(\boldsymbol{x}) = 0$ and $\nabla^2 f(\boldsymbol{x}) \succeq 0$ is NP-hard.
- Modern goal: Find approximate stationary points:

$$\|\nabla f(\boldsymbol{x})\| \le \epsilon_{\mathrm{g}} \quad \text{and} \quad \nabla^2 f(\boldsymbol{x}) \succeq -\epsilon_{\mathrm{H}} \boldsymbol{I}.$$

$\rightarrow$ *When are these close to solutions?*

From benign landscape to strict saddle

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Our tool: Strict saddle property

- Parametric definition.
- Allows for complexity analysis.
- Applies to manifold optimization.

Main problem

$$\operatorname*{minimize}_{\boldsymbol{x} \in \mathcal{M}} f(\boldsymbol{x}),$$

- f $\mathcal{C}^2$ nonconvex
- $\mathcal{M}$ Riemannian manifold ($\mathbb{C}^{n \times m}$, $\mathbb{R}^{n \times m}$, sphere,...).

Setup: Manifold optimization

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Goal: Approximate stationary points

$$\|g(\boldsymbol{x}_k)\| \le \epsilon_{\mathrm{g}} \quad \text{and} \quad \lambda_{\min}\left(\mathcal{H}(\boldsymbol{x}_k)\right) \ge -\epsilon_{\mathrm{H}}.$$

- $g(\cdot)$ Riemannian gradient.
- $\mathcal{H}(\cdot)$ Riemannian Hessian.

Strict saddle property on manifold $\mathcal{M}$

Definition (Ge et al '15 → Goyens, R. '24)

$f : \mathbb{R}^n \to \mathbb{R}$ is $(\alpha, \beta, \gamma, \delta)$-strict saddle if for any $\boldsymbol{x} \in \mathbb{R}^n$, one of these properties holds:

1. $\|g(\boldsymbol{x})\| \geq \alpha$;
2. $\lambda_{\min}\left(\mathcal{H}(\boldsymbol{x})\right) \leq -\beta$;
3. There exists $\boldsymbol{x}^*$ local minimum of f such that

$$\|\boldsymbol{x} - \boldsymbol{x}^*\| \leq \delta \quad \text{and} \quad \lambda_{\min}(\mathcal{H}(\boldsymbol{y})) \geq \gamma \quad \forall \boldsymbol{y},\ \|\boldsymbol{y} - \boldsymbol{x}^*\| \leq 2\delta.$$

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Interpretation: 3 regions in the space

1. Large Riemannian gradient g.
2. Negative curvature for Riemannian Hessian $\mathcal{H}$.
3. Near minimum+geodesic strong convexity.

N.B. Already studied for special problem classes (Pumir et al '18, Sun et al '16-'18).

Simple strict saddle functions

Strict saddle: $\{\|\nabla f(x)\| \geq \alpha\}$ or $\{\nabla^2 f(x) \not\succeq -\beta I\}$ or $\{\|x - x^*\| \leq \delta$ and $\nabla^2 f(y) \succeq \gamma I \ \forall y: \ \|y - x^*\| \leq 2\delta\}$.

Strongly convex⇒Strict saddle!

If f is γ-strongly convex, then it is $\left(\alpha, \beta, \gamma, \frac{2\alpha}{\gamma}\right)$-strict saddle for any positive α and β.

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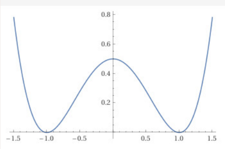
A "truly" strict saddle function

$$f : x \in \mathbb{R} \mapsto \frac{1}{2}(x^2 - 1)^2$$



Example: Matrix completion

$$\underset{\boldsymbol{U}\in\mathbb{R}^{n\times r},\boldsymbol{V}\in\mathbb{R}^{m\times r}}{\text{minimize}} f(\boldsymbol{U},\boldsymbol{V}) := \left\|\mathcal{P}_\Omega(\boldsymbol{U}\boldsymbol{V}^{\mathrm{T}} - \boldsymbol{M})\right\|_F^2, \qquad \Omega \subset [n]\times[m].$$

Assume Nice structure for $\boldsymbol{M}$ (incoherence), probability of sampling $\boldsymbol{M}_{ij}$ large enough.

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Theorem (Ge et al. '17)

Let $(\boldsymbol{U},\boldsymbol{V}) \in \mathbb{R}^{n\times r}\times\mathbb{R}^{m\times r}$. Then, whp, there exists $\alpha > 0$ such that one of these cases occur

1. $\|\nabla f(\boldsymbol{U},\boldsymbol{V})\| \geq \alpha$
2. The Hessian at U, V has negative curvature, i.e.
$$\lambda_{\min}\left(\nabla^2 f(\boldsymbol{U},\boldsymbol{V})\right) < -\mathcal{O}(\sigma_{\min}(M))$$
3. $(\boldsymbol{U},\boldsymbol{V})$ is at distance at most $\mathcal{O}(\frac{\alpha}{\sigma_{\min}(\boldsymbol{M})})$ from a global minimum.

Algorithms for strict saddle problems

- **Gradient descent** Can be shown to escape strict saddle points almost surely (Lee et al '19).
- **Perturbed gradient descent** does so in polynomial time (Jin et al '17, Ma et al '25).
- **Line-search algorithms** can also be analyzed in this way (O'Neill and Wright '23).

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Our choice: Trust-region methods

- Initially developed for nonconvex problems.
- Use Newton-type steps (like interior-point methods in LP).

Inputs: $\boldsymbol{x}_0 \in \mathcal{M}$, $\Delta_0 > 0$, $\eta > 0$.

For k=$0, 1, 2, \dots$

1. Define $m_k(\boldsymbol{x}_k + \boldsymbol{s}) := \langle g(\boldsymbol{x}_k), \boldsymbol{s}\rangle + \frac{1}{2}\langle \boldsymbol{s}, \mathcal{H}(\boldsymbol{x}_k)\boldsymbol{s}\rangle$ and compute

$$\boldsymbol{s}_k \in \operatorname*{argmin}_{\substack{\boldsymbol{s}\in\mathcal{T}^{\mathcal{M}}_{\boldsymbol{x}_k} \\ \|\boldsymbol{s}\|\le\Delta_k}} m_k(\boldsymbol{x}_k + \boldsymbol{s}).$$

Trust region for $\min_{\boldsymbol{x} \in \mathcal{M}} f(\boldsymbol{x})$

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4. If $\rho_k \ge \eta$, set $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k^{\mathcal{M}}$ and $\Delta_{k+1} = 2\Delta_k$.

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5. Otherwise, set $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k$ and $\Delta_{k+1} = 0.5\Delta_k$.

Complexity results for trust region

Goal: Compute $\boldsymbol{x}_K$ such that $\|g(\boldsymbol{x}_K)\| \le \epsilon_{\mathrm{g}}$ and $\lambda_{\min}(\mathcal{H}(\boldsymbol{x}_K)) \ge -\epsilon_{\mathrm{H}}$.

For general nonconvex f (Boumal et al '19)

$$K = \mathcal{O}\left(\max\{\epsilon_{\mathrm{g}}^{-2}, \epsilon_{\mathrm{H}}^{-3}\}\right).$$

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For strict saddle f (Goyens and R. '24)

If f is $(\alpha, \beta, \gamma, \delta)$-strict saddle, $K = K_f + K_\epsilon$, with

$$\begin{aligned} K_f &= \mathcal{O}\left(\max\left\{\alpha^{-2}\beta^{-1}, \alpha^{-2}\gamma^{-1}, \beta^{-3}, \gamma^{-3}, \gamma^{-2}\delta^{-1}\right\}\right) \\ K_\epsilon &= \log\log\left[\mathcal{O}\left(\gamma\epsilon_{\mathrm{g}}^{-1}\right)\right]. \end{aligned}$$

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$$K_\epsilon = \log\log\left[\mathcal{O}\left(\gamma\epsilon_{\mathrm{g}}^{-1}\right)\right].$$

- $\log\log$ dependency in ϵ_{g} (none on ϵ_{H})!
- Complexity depends more on landscape parameters!

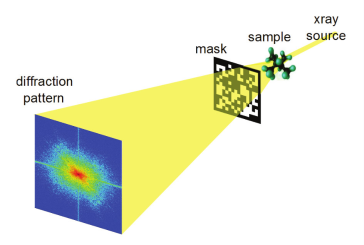
Outline

Phase retrieval

Phase retrieval problem

$$
\begin{array}{ll}
\text{find} & \boldsymbol{x} \in \mathbb{C}^n \\
\text{such that} & |\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle| = b_i, \quad i = 1, \dots, m.
\end{array}
$$

- $b_i = |\langle \boldsymbol{x}^*, \boldsymbol{a}_i \rangle|$: Complex moduli.
- $\boldsymbol{a}_i$: Measurement vectors (Gaussian, wavelets, etc).



(Picture from Candès et al '15)

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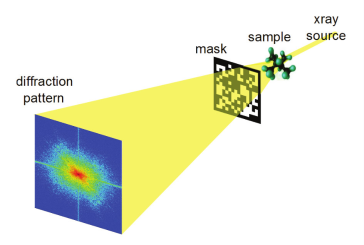

(Picture from Candès et al '15)

Gives rise to convex and nonconvex optimization problems!

Strict saddle phase retrieval ('ed)

Nonconvex vector formulation (Sun et al '18)

$$\underset{\boldsymbol{x}\in\mathbb{C}^n}{\text{minimize}} \;\; f(\boldsymbol{x}) = \frac{1}{2m}\sum_{i=1}^{m}(b_i^2 - |\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle|^2)^2$$

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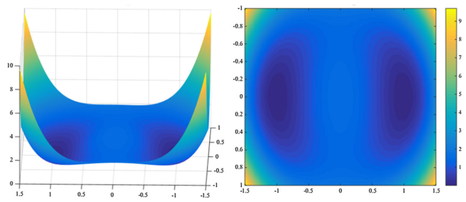
→ For some universal $c > 0$, f is $\left(\frac{c}{n\log(m)}, c, c, \frac{c}{n\log(m)}\right)$-strict saddle with high probability.



Trust region for phase retrieval

Phase retrieval (Sun et al '18)

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If $\{\boldsymbol{a}_i\}$ are Gaussian and $m = \mathcal{O}(n\log^3(n))$, the objective is $(\frac{c}{n\log(m)}, c, c, \frac{c}{n\log(m)})$-strict saddle for some $c > 0$ w.h.p..

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Complexity results for trust region (Goyens, R. '24)

- Basic trust region:

$$\mathcal{O}(\max\left\{\epsilon_{\mathrm{g}}^{-2}, \epsilon_{\mathrm{H}}^{-3}\right\}) \quad \text{iterations.}$$

- Using strict saddle:

$$\mathcal{O}(n^2) + \log\log(\mathcal{O}(\epsilon_{\mathrm{g}}^{-1})) \quad \text{iterations.}$$

From strict saddle to good nonconvex?

So far

- Good results for trust region on strict saddle functions.
- Example: Strict saddle phase retrieval.

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- Not all nonconvex problems are strict saddle.
- Yet trust region can work well.

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The case of phase retrieval

- Popular formulations: Factored SDPs.
- **Goal:** Study trust region on those!

Phase retrieval SDPs

Recall phase retrieval: Find $\boldsymbol{x} \in \mathbb{C}^n$ such that $|\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle| = b_i \ \forall i$.

PhaseLift SDP relaxation (Candès et al. '11)

$$
\begin{array}{ll}
\min_{\boldsymbol{X} \in \mathbb{C}^{n \times n}} & \operatorname{trace}(\boldsymbol{X}) \\
\text{subject to} & \langle \boldsymbol{X}, \boldsymbol{a}_i \boldsymbol{a}_i^* \rangle = b_i^2 \ \forall i \\
 & \boldsymbol{X} \succeq 0
\end{array}
$$

→ Convex problem, can be solved in polynomial time (like LP!).

→ In practice, cost of matrix linear algebra prohibitive.

→ Expected: **Rank-1 solution**.

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Factored PhaseLift formulation

$$\min_{\boldsymbol{U} \in \mathbb{C}^{n\times r}} f(\boldsymbol{U}) := \frac{1}{2} \|\mathcal{A}(\boldsymbol{U}\,\boldsymbol{U}^{\mathrm{T}}) - \boldsymbol{b}\|_F^2, \qquad \mathcal{A}(\boldsymbol{X}) = [\langle \boldsymbol{X}, \boldsymbol{a}_i\,\boldsymbol{a}_i^* \rangle]_i\,.$$

→ Burer-Monteiro factorization (Burer and Monteiro '03).

→ Not strict saddle for small r!

Simplified PhaseLift instance $\min_{\boldsymbol{U} \in \mathbb{C}^{n \times r}} f(\boldsymbol{U}) := \frac{1}{2}\|\boldsymbol{U}\,\boldsymbol{U}^{\mathrm{T}} - \boldsymbol{b}\|_F^2.$

Trust region on PhaseLift

Simplified PhaseLift instance $\min_{\boldsymbol{U} \in \mathbb{C}^{n \times r}} f(\boldsymbol{U}) := \frac{1}{2} \| \boldsymbol{U} \boldsymbol{U}^{\mathrm{T}} - \boldsymbol{b} \|_F^2$.

Theorem (Goyens, R., Waldspurger '25+)

If $r = 2$, $\boldsymbol{U}_0$ close enough to a solution, then

$$f(\boldsymbol{U}_K) - \min_{\boldsymbol{U}} f(\boldsymbol{U}) \leq \epsilon \left(f(\boldsymbol{U}_0) - \min_{\boldsymbol{U}} f(\boldsymbol{U}) \right)$$

for $K = \mathcal{O}(\log(1/\epsilon))$.

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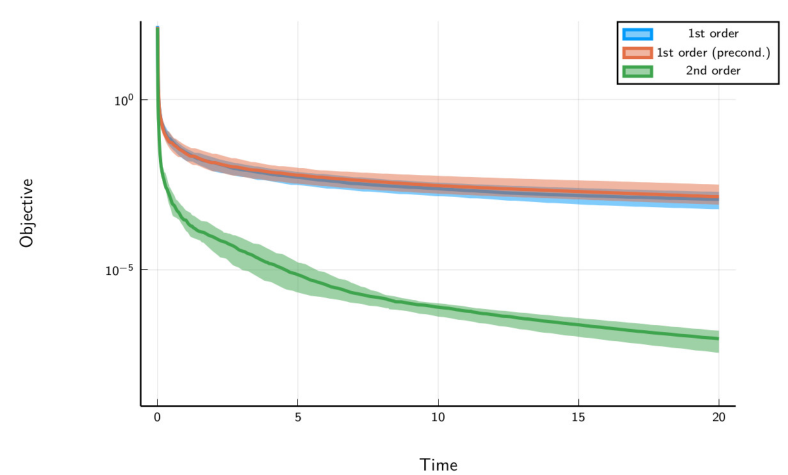
for $K = \mathcal{O}(\log(1/\epsilon))$.

- $\rightarrow$ Worse than the $\log(\log(1/\epsilon))$ strict saddle rate.
- $\rightarrow$ Local result but no need for PŁ/ strong convexity near solution.
- $\rightarrow$ On par with other analyses (Rebjock and Boumal '24-'25)

Ongoing: Extension to more PhaseLift problems, in line with numerics.

Setup:

- $n = 128$, $m = 768$, wavelet measurements, $r = 2$.
- Gradient descent, Preconditioned (Zhang,Fattahi,Zhang '21), Trust region.



Recap: Phase retrieval

Strict saddle phase retrieval

- Amenable to analysis.
- Fast convergence guaranteed for trust region.

SDP-based phase retrieval

- Not always strict saddle.
- But trust region fast anyways!

Outline

Back to linear programming

My main line of research

- Structured nonconvex formulations.
- Can revisit convex algorithms in that space.

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What about convex problems?

- Primal-dual methods for solving LPs are booming.
- Rely on convex optimization algorithms!

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Our approach (w/ G. Agazzotti & A. Chambolle)

- Explore structured LPs from optimal transport.
- Start with classical (simplex) framework.

Optimal transport

$$\min_{\boldsymbol{P} \in \mathcal{U}(\boldsymbol{a},\boldsymbol{b})} \langle \boldsymbol{P}, \boldsymbol{C} \rangle$$

- $\boldsymbol{a}$,$\boldsymbol{b}$ measures on $\mathbb{R}^n$, $\|\boldsymbol{a}\|_1 = \|\boldsymbol{b}\|_1$.
- $\mathcal{U}(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{P} \geq 0 \,|\boldsymbol{P} 1_n = \boldsymbol{a}, \boldsymbol{P}^{\mathrm{T}} 1_n = \boldsymbol{b}\}$.

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Linear program!

- Transportation problem on bipartite graph.
- Challenging to tackle through interior-point methods.
- State of the art: Network simplex (Flamary et al '21).

Optimality conditions

If $\boldsymbol{P}$ solution, $\exists(\boldsymbol{u}, \boldsymbol{v})$ such that

$$\boldsymbol{P} \in \mathcal{U}(\boldsymbol{a}, \boldsymbol{b}), \quad \boldsymbol{C} \geq \boldsymbol{u} \oplus \boldsymbol{v}, \quad \boldsymbol{P} \odot (\boldsymbol{C} - \boldsymbol{u} \oplus \boldsymbol{v}) = 0.$$

Network simplex

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Network simplex idea

- Maintain $(\boldsymbol{P}, \boldsymbol{u}, \boldsymbol{v})$ such that $\boldsymbol{P} \in \mathcal{U}(\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{P} \odot (\boldsymbol{C} - \boldsymbol{u} \oplus \boldsymbol{v}) = 0$.
- If $\boldsymbol{C} \geq \boldsymbol{u} \oplus \boldsymbol{v}$, optimal. Otherwise, find $u_i + v_j > C_{ij}$ and update $(\boldsymbol{P}, \boldsymbol{u}, \boldsymbol{v})$ to satisfy $C_{ij} \geq u_i + v_j$ + other two conditions.

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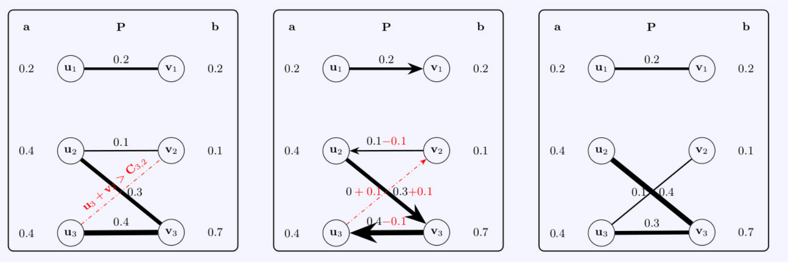

Unbalanced (L2) optimal transport

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Unbalanced optimal transport (UOT)

$$\min_{\boldsymbol{P}\geq 0}\langle \boldsymbol{P},\boldsymbol{C}\rangle+\frac{1}{2\tau}\|\boldsymbol{P}1_n-\boldsymbol{a}\|^2+\frac{1}{2\tau}\|\boldsymbol{P}^{\mathrm{T}}1_n-\boldsymbol{b}\|^2.$$

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- $\tau > 0$: Relaxes marginal constraints.
- Quadratic (nonlinear) optimization problem.

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- $\tau > 0$: Relaxes marginal constraints.
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Goals

- Adapt network simplex to UOT.
- Get efficient implementation.

Network simplex for UOT

Network simplex for OT

- Uses optimality conditions

$$\boldsymbol{P} \in \mathcal{U}(\boldsymbol{a},\boldsymbol{b}), \quad \boldsymbol{C} \geq \boldsymbol{u} \oplus \boldsymbol{v}, \quad \boldsymbol{P} \odot (\boldsymbol{C} - \boldsymbol{u} \oplus \boldsymbol{v}) = 0.$$

- Exploits bipartite graph structure.
- Complexity guarantees depend on the variant.

Network simplex for UOT

- Optimality conditions still easy to check:

$$\boldsymbol{P} \in \mathcal{U}(\boldsymbol{a} - \tau \boldsymbol{u}, \boldsymbol{b} - \tau \boldsymbol{v}), \quad \boldsymbol{C} \geq \boldsymbol{u} \oplus \boldsymbol{v}, \quad \boldsymbol{P} \odot (\boldsymbol{C} - \boldsymbol{u} \oplus \boldsymbol{v}) = 0.$$

- Key: Explicit formulas for updating transport graph!

$\rightarrow$ Can prove complexity results!

Extensions

- Optimal transport:

$$\min_{\boldsymbol{P} \in \mathcal{U}(\boldsymbol{a},\boldsymbol{b})} \langle \boldsymbol{P}, \boldsymbol{C} \rangle$$

- Unbalanced optimal transport:

$$\min_{\boldsymbol{P} \geq 0} \langle \boldsymbol{P}, \boldsymbol{C} \rangle + \frac{1}{2\tau} \|\boldsymbol{P} 1_n - \boldsymbol{a}\|^2 + \frac{1}{2\tau} \|\boldsymbol{P}^{\mathrm{T}} 1_n - \boldsymbol{b}\|^2$$

- Semi-unbalanced optimal transport (SUOT)

$$\min_{\substack{\boldsymbol{P} 1_n = \boldsymbol{a} \\ \boldsymbol{P} \geq 0}} \langle \boldsymbol{P}, \boldsymbol{C} \rangle + \frac{1}{2\tau} \left\| \boldsymbol{P}^{\mathrm{T}} 1_n - \boldsymbol{b} \right\|^2 .$$

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- Can use strictly convex functions/Bregman divergences:

$$\min_{\boldsymbol{P} \geq 0} \langle \boldsymbol{P}, \boldsymbol{C} \rangle + \frac{1}{2\tau} \psi(\boldsymbol{P} 1_n, \boldsymbol{a}) + \frac{1}{2\tau} \psi(\boldsymbol{P}^{\mathrm{T}} 1_n, \boldsymbol{b}).$$

Ex) Kullback-Leibler $\psi(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} x_i \log\left(\frac{x_i}{y_i}\right) - x_i + y_i$.

Model (Blondel et al '18)

$$\min_{\boldsymbol{P} 1_n = 1_n/K} \langle \boldsymbol{P}, \boldsymbol{C} \rangle + \frac{1}{2\tau} \left\| \boldsymbol{P}^{\mathrm{T}} 1_n - \frac{1_n}{K} \right\|^2$$

- Uniform measure on K uniform points for each image.
- $\boldsymbol{C}_{ij} = \|\boldsymbol{x}_i - \boldsymbol{y}_j\|^2$, $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ sampled points.

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(a) *Femme au béret et à la robe quadrillée*

(b) *La nuit étoilée*

(c) τ = 1e-4

(d) τ = 1e-1

(e) τ = 1e1

(f) τ = 1e4

Benchmark Random, uniform instances.

Competitors High-accuracy solvers ($<10^{-12}$ accuracy).

- UOT: `celer` (Massias et al '18).
- SUOT: Regularization path (Lee et al '19).
- UOT+KL: LBFGSB from Scipy.

Comparison with other solvers

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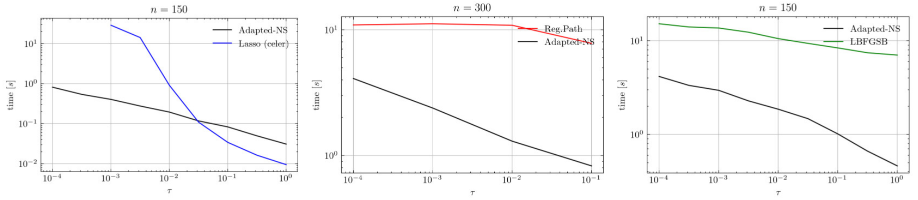


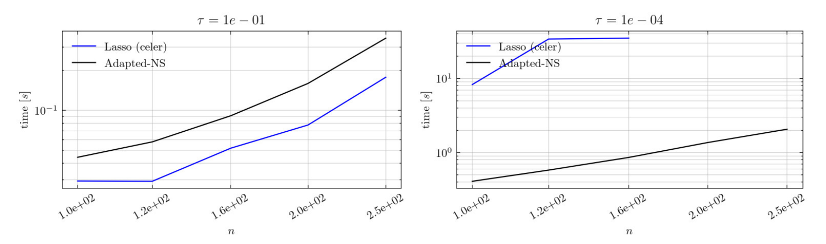
(a) UOT-L^2 (b) SUOT-L^2 (c) UOT-KL

Comparison with other solvers ('ed)

Focus: UOT

$$\min_{\boldsymbol{P}\geq 0} \langle \boldsymbol{P}, \boldsymbol{C} \rangle + \frac{1}{2\tau}\|\boldsymbol{P}1_n - \boldsymbol{a}\|^2 + \frac{1}{2\tau}\|\boldsymbol{P}^{\mathrm{T}}1_n - \boldsymbol{b}\|^2$$

- celer/LASSO approach more efficient with large τ.
- Network simplex better when $\tau \to 0$.



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Unbalanced optimal transport

- Convex (nonlinear) problems with graph structure.
- Can adapt network simplex algorithms!

Perspectives

Next steps

- Get those papers out (+code).
- Study nonconvex formulations of unbalanced optimal transport?

- G. Agazzotti, A. Chambolle and C. W. Royer, *Adapted network simplex algorithms for unbalanced optimal transport*, working paper.
- P. Caucheteux, F. Goyens, C. W. Royer and I. Waldspurger, *Trust-region methods for Burer-Monteiro factorizations of phase retrieval problems*, working paper.
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Merci !

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