Complexity guarantees and polling strategies for Riemannian direct-search methods

Clément Royer (Université Paris Dauphine-PSL)

SIAM Conference on Optimization

Edinburgh, June 4 2026

The work

arXiv > math > arXiv:2511.15360

Mathematics > Optimization and Control

[Submitted on 19 Nov 2025 (v1), last revised 29 May 2026 (this version, v2)]

Complexity guarantees and polling strategies for Riemannian direct-search methods

Bastien Cavarretta, Florentin Goyens, Clément W. Royer, Florian Yger

Motivation: Adversarial attacks

- Fancy application.
- Need for black-box algorithms on spheres, matrices, etc.

Motivation: Adversarial attacks

- Fancy application.
- Need for black-box algorithms on spheres, matrices, etc.

Starting point: Direct search on manifolds

- Already done (without complexity).
- Already implemented (how?)

Story behind the paper

Motivation: Adversarial attacks

- Fancy application.
- Need for black-box algorithms on spheres, matrices, etc.

Starting point: Direct search on manifolds

- Already done (without complexity).
- Already implemented (how?)

Results: Geometrical questions

- Directions on manifolds?
- Quality?

Roadmap

1. Riemmanian direct search
2. PSSs and manifolds
3. Comparing strategies

Riemannian optimization setup

$$\underset{\boldsymbol{x} \in \mathcal{M}}{\text{minimize}}\, f(\boldsymbol{x}).$$

- f: Black-box objective, smooth, bounded below.
- $\mathcal{M} \subset \mathbb{R}^n$: Riemannian manifold, $\dim(\mathcal{M}) = m \in \{1, \dots, n\}$.

$$\underset{\boldsymbol{x} \in \mathcal{M}}{\text{minimize}}\, f(\boldsymbol{x}).$$

- f: Black-box objective, smooth, bounded below.
- $\mathcal{M} \subset \mathbb{R}^n$: Riemannian manifold, $\dim(\mathcal{M}) = m \in \{1, \dots, n\}$.

Many people have looked at this

- Zeroth-order methods (and adversarial attacks)
  (Cai et al '20-21, Li et al '22-23,...)

Riemannian optimization setup

$$\underset{\boldsymbol{x} \in \mathcal{M}}{\text{minimize}} \, f(\boldsymbol{x}).$$

- f: Black-box objective, smooth, bounded below.
- $\mathcal{M} \subset \mathbb{R}^n$: Riemannian manifold, $\dim(\mathcal{M}) = m \in \{1, \dots, n\}$.

Many people have looked at this

- Zeroth-order methods (and adversarial attacks)
  (Cai et al '20-21, Li et al '22-23,...)
- Early generalizations of direct search
  (Dreisigmeyer '06 '07 '18 '19,...)

Riemannian optimization setup

$$\underset{\boldsymbol{x} \in \mathcal{M}}{\text{minimize}} \, f(\boldsymbol{x}).$$

- f: Black-box objective, smooth, bounded below.
- $\mathcal{M} \subset \mathbb{R}^n$: Riemannian manifold, $\dim(\mathcal{M}) = m \in \{1, \dots, n\}$.

Many people have looked at this

- Zeroth-order methods (and adversarial attacks)
  (Cai et al '20-21, Li et al '22-23,...)
- Early generalizations of direct search
  (Dreisigmeyer '06 '07 '18 '19,...)
- More recent "DFO" algorithms
  (Chaminiau et al '26, Kungurtsev et al '24,...).

Problem: $\text{minimize}_{\boldsymbol{x} \in \mathcal{M}} f(\boldsymbol{x})$.

Inputs $\boldsymbol{x}_0 \in \mathcal{M}$, $\alpha_0 > 0$, $\eta > 0$.
For $k = 0, 1, \ldots$

- Choose a set $D_k \subset \mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$.
- If $\exists\ \boldsymbol{d}_k \in D_k$ such that

$$f(\mathcal{R}_{\boldsymbol{x}_k}(\alpha_k \boldsymbol{d}_k)) < f(\boldsymbol{x}_k) - \eta \alpha_k^2 \|\boldsymbol{d}_k\|_{\boldsymbol{x}_k}^2$$

  set $\boldsymbol{x}_{k+1} := \mathcal{R}_{\boldsymbol{x}_k}(\alpha_k \boldsymbol{d}_k)$, $\alpha_{k+1} := 2\alpha_k$.
- Otherwise, set $\boldsymbol{x}_{k+1} := \boldsymbol{x}_k$, $\alpha_{k+1} := 0.5\alpha_k$.

Riemannian direct search

Problem: $\text{minimize}_{\boldsymbol{x}\in\mathcal{M}} f(\boldsymbol{x})$.

Inputs $\boldsymbol{x}_0 \in \mathcal{M}$, $\alpha_0 > 0$, $\eta > 0$.
For $k = 0, 1, \ldots$

- Choose a set $D_k \subset \mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$.
- If $\exists\ \boldsymbol{d}_k \in D_k$ such that

$$f(\mathcal{R}_{\boldsymbol{x}_k}(\alpha_k\,\boldsymbol{d}_k)) < f(\boldsymbol{x}_k) - \eta\alpha_k^2\|\boldsymbol{d}_k\|_{\boldsymbol{x}_k}^2$$

  set $\boldsymbol{x}_{k+1} := \mathcal{R}_{\boldsymbol{x}_k}(\alpha_k\boldsymbol{d}_k)$, $\alpha_{k+1} := 2\alpha_k$.
- Otherwise, set $\boldsymbol{x}_{k+1} := \boldsymbol{x}_k$, $\alpha_{k+1} := 0.5\alpha_k$.

- $\mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$: Tangent space for $\mathcal{M}$ at $\boldsymbol{x}_k$ with norm $\|\cdot\|_{\boldsymbol{x}_k}$
  $\rightarrow$ Respects local geometry.
- $\mathcal{R}_{\boldsymbol{x}_k}(\alpha_k\boldsymbol{D}_k)$: Retraction (brings back onto the manifold)
  $\rightarrow$ Guarantees feasible iterates.

Guarantees (Kungurtsev, Rinaldi, Zeffiro '24)

Asymptotic convergence

Suppose

- f $\mathcal{C}^1$ on $\mathcal{M}$, grad f (Riemannian gradient) Lipschitz.
- Directions D_k are well-chosen.

Then,

$$\liminf_{k \to \infty} \| \operatorname{grad} f(\boldsymbol{x}_k) \|_{\boldsymbol{x}_k} = 0.$$

Guarantees (Kungurtsev, Rinaldi, Zeffiro '24)

Asymptotic convergence

Suppose

- f $\mathcal{C}^1$ on $\mathcal{M}$, grad f (Riemannian gradient) Lipschitz.
- Directions D_k are well-chosen.

Then,

$$\liminf_{k \to \infty} \| \operatorname{grad} f(\boldsymbol{x}_k) \|_{\boldsymbol{x}_k} = 0.$$

From theory to practice

- Well-chosen $\Leftrightarrow$ D_k **positive spanning set** (PSS) of $\mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$.
- In practice: Project PSS of $\mathbb{R}^n$ onto $\mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$.

Guarantees (Kungurtsev, Rinaldi, Zeffiro '24)

Asymptotic convergence

Suppose

- f $\mathcal{C}^1$ on $\mathcal{M}$, grad f (Riemannian gradient) Lipschitz.
- Directions D_k are well-chosen.

Then,

$$\liminf_{k \to \infty} \|\operatorname{grad} f(\boldsymbol{x}_k)\|_{\boldsymbol{x}_k} = 0.$$

From theory to practice

- Well-chosen $\Leftrightarrow$ D_k **positive spanning set** (PSS) of $\mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$.
- In practice: Project PSS of $\mathbb{R}^n$ onto $\mathcal{T}_{\boldsymbol{x}_k}\mathcal{M}$.

This talk: Is projection the best choice?

Definitions

- **Cosine measure of $D \subset \mathbb{R}^m$**

$$\mathrm{cm}(D) = \min_{\boldsymbol{v} \neq 0} \max_{\substack{\boldsymbol{d} \in D \\ \boldsymbol{d} \neq 0}} \frac{\boldsymbol{d}^{\mathrm{T}} \boldsymbol{v}}{\|\boldsymbol{d}\| \|\boldsymbol{v}\|}.$$

Positive Spanning Sets (PSSs) in $\mathbb{R}^m$

Definitions

- **Cosine measure of $D \subset \mathbb{R}^m$**

$$\mathrm{cm}(D) = \min_{\boldsymbol{v} \neq 0} \max_{\substack{\boldsymbol{d} \in D \\ \boldsymbol{d} \neq 0}} \frac{\boldsymbol{d}^{\mathrm{T}} \boldsymbol{v}}{\|\boldsymbol{d}\| \|\boldsymbol{v}\|}.$$

- **Positive spanning set** D such that $\mathrm{cm}(D) > 0$.

Positive Spanning Sets (PSSs) in $\mathbb{R}^m$

Definitions

- **Cosine measure of $D \subset \mathbb{R}^m$**

$$\mathrm{cm}(D) = \min_{\boldsymbol{v} \neq 0} \max_{\substack{\boldsymbol{d} \in D \\ \boldsymbol{d} \neq 0}} \frac{\boldsymbol{d}^{\mathrm{T}} \boldsymbol{v}}{\|\boldsymbol{d}\| \|\boldsymbol{v}\|}.$$

- **Positive spanning set** D such that $\mathrm{cm}(D) > 0$.

In direct search

- Typical assumption: $\mathrm{cm}(D_k) \geq \kappa \in (0, 1)$.
- For simplicity: Fix $D_k = D$ PSS.

Complexity of direct search (Euclidean case)

Setup Direct search for $\min_{\boldsymbol{x} \in \mathbb{R}^m} f(\boldsymbol{x})$ using $D_k = D$ PSS.

Theorem (Vicente '12)

The method satisfies $\|\nabla f(\boldsymbol{x}_K)\| \leq \epsilon$ after at most

$$\mathcal{O}\left(\, |D| \, \mathrm{cm}(D)^{-2} \, \epsilon^{-2} \, \right) \qquad \text{calls to } f.$$

Complexity of direct search (Euclidean case)

Setup Direct search for $\min_{\boldsymbol{x} \in \mathbb{R}^m} f(\boldsymbol{x})$ using $D_k = D$ PSS.

Theorem (Vicente '12)

The method satisfies $\|\nabla f(\boldsymbol{x}_K)\| \leq \epsilon$ after at most

$$\mathcal{O}\left(\ |D|\ \mathrm{cm}(D)^{-2}\,\epsilon^{-2}\ \right) \qquad \text{calls to } f.$$

Complexity measure of D: $\chi(D) = |D|\,\mathrm{cm}(D)^{-2}$.

Setup Direct search for $\min_{\mathbf{x}\in\mathbb{R}^m} f(\mathbf{x})$ using $D_k = D$ PSS.

Theorem (Vicente '12)

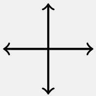
The method satisfies $\|\nabla f(\mathbf{x}_K)\| \le \epsilon$ after at most

$$\mathcal{O}\left(\, |D| \; \mathrm{cm}(D)^{-2} \, \epsilon^{-2} \, \right) \qquad \text{calls to } f.$$

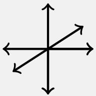
Complexity measure of D: $\chi(D) = |D| \, \mathrm{cm}(D)^{-2}$.

Main example today: $D^{\oplus} = \{\pm \mathbf{e}_i\}$

- $|D^{\oplus}| = 2m$.
- $\mathrm{cm}(D^{\oplus}) = \frac{1}{\sqrt{m}}$.
- $\chi(D^{\oplus}) = 2m^2$.



$m = 2$



$m = 3$

Positive spanning sets for manifold $\mathcal{M}$

For $\boldsymbol{x} \in \mathcal{M}$:

- **Cosine measure of $D_{\boldsymbol{x}} \subset \mathcal{T}_{\boldsymbol{x}}\mathcal{M}$**

$$\mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}}) = \min_{\substack{\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}}\mathcal{M} \\ \boldsymbol{v} \neq 0}} \max_{\substack{\boldsymbol{d} \in D_{\boldsymbol{x}} \\ \boldsymbol{d} \neq 0}} \frac{\langle \boldsymbol{d}, \boldsymbol{v} \rangle_{\boldsymbol{x}}}{\|\boldsymbol{d}\|_{\boldsymbol{x}} \|\boldsymbol{v}\|_{\boldsymbol{x}}}.$$

$\rightarrow$ Special case of cosine measure w.r.t. a subspace (Audet et al '25).

Positive spanning sets for manifold $\mathcal{M}$

For $\boldsymbol{x} \in \mathcal{M}$:

- **Cosine measure of $D_{\boldsymbol{x}} \subset \mathcal{T}_{\boldsymbol{x}}\mathcal{M}$**

$$\mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}}) = \min_{\substack{\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}}\mathcal{M} \\ \boldsymbol{v} \neq 0}} \max_{\substack{\boldsymbol{d} \in D_{\boldsymbol{x}} \\ \boldsymbol{d} \neq 0}} \frac{\langle \boldsymbol{d}, \boldsymbol{v} \rangle_{\boldsymbol{x}}}{\|\boldsymbol{d}\|_{\boldsymbol{x}} \|\boldsymbol{v}\|_{\boldsymbol{x}}}.$$

$\rightarrow$ Special case of cosine measure w.r.t. a subspace (Audet et al '25).

- **Positive spanning set for $\mathcal{T}_{\boldsymbol{x}}\mathcal{M}$** $\mathrm{cm}(D_{\boldsymbol{x}}) > 0$.

Positive spanning sets for manifold $\mathcal{M}$

For $\boldsymbol{x} \in \mathcal{M}$:

- **Cosine measure of $D_{\boldsymbol{x}} \subset \mathcal{T}_{\boldsymbol{x}}\mathcal{M}$**

$$\mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}}) = \min_{\substack{\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}}\mathcal{M} \\ \boldsymbol{v} \neq 0}} \max_{\substack{\boldsymbol{d} \in D_{\boldsymbol{x}} \\ \boldsymbol{d} \neq 0}} \frac{\langle \boldsymbol{d}, \boldsymbol{v} \rangle_{\boldsymbol{x}}}{\|\boldsymbol{d}\|_{\boldsymbol{x}} \|\boldsymbol{v}\|_{\boldsymbol{x}}}.$$

  $\rightarrow$ Special case of cosine measure w.r.t. a subspace (Audet et al '25).

- **Positive spanning set for $\mathcal{T}_{\boldsymbol{x}}\mathcal{M}$** $\mathrm{cm}(D_{\boldsymbol{x}}) > 0$.

Let $\mathcal{D} = (D_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{M}}, D_{\boldsymbol{x}} \subset \mathcal{T}_{\boldsymbol{x}}\mathcal{M}$.

- **Cosine measure of $\mathcal{D}$**

$$\mathrm{cm}_{\mathcal{M}} = \inf_{\boldsymbol{x} \in \mathcal{M}} \mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}})$$

Positive spanning sets for manifold $\mathcal{M}$

For $\boldsymbol{x} \in \mathcal{M}$:

- **Cosine measure of $D_{\boldsymbol{x}} \subset \mathcal{T}_{\boldsymbol{x}}\mathcal{M}$**

$$\mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}}) = \min_{\substack{\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}}\mathcal{M} \\ \boldsymbol{v} \neq 0}} \max_{\substack{\boldsymbol{d} \in D_{\boldsymbol{x}} \\ \boldsymbol{d} \neq 0}} \frac{\langle \boldsymbol{d}, \boldsymbol{v} \rangle_{\boldsymbol{x}}}{\|\boldsymbol{d}\|_{\boldsymbol{x}} \|\boldsymbol{v}\|_{\boldsymbol{x}}}.$$

$\rightarrow$ Special case of cosine measure w.r.t. a subspace (Audet et al '25).

- **Positive spanning set for $\mathcal{T}_{\boldsymbol{x}}\mathcal{M}$** $\mathrm{cm}(D_{\boldsymbol{x}}) > 0$.

Let $\mathcal{D} = (D_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{M}}, D_{\boldsymbol{x}} \subset \mathcal{T}_{\boldsymbol{x}}\mathcal{M}$.

- **Cosine measure of $\mathcal{D}$**

$$\mathrm{cm}_{\mathcal{M}} = \inf_{\boldsymbol{x} \in \mathcal{M}} \mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}})$$

- **Positive spanning set for $\mathcal{M}$** $\mathrm{cm}_{\mathcal{M}}(\mathcal{D}) > 0$.

Complexity bounds for manifold setting

Assume

- $\mathcal{D} = (D_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{M}}$ PSS of $\mathcal{M}$.
- $D_k = D_{\boldsymbol{x}_k}$ for every k.

Complexity bounds for manifold setting

Assume

- $\mathcal{D} = (D_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{M}}$ PSS of $\mathcal{M}$.
- $D_k = D_{\boldsymbol{x}_k}$ for every k.

Theorem (Cavarretta, Goyens, R., Yger '25+)

The method satisfies $\|\operatorname{grad} f(\boldsymbol{x}_k)\|_{\boldsymbol{x}_k} \le \epsilon$ after at most

$$\mathcal{O}\left(\chi_{\mathcal{M}}(\mathcal{D})\, \epsilon^{-2} \right) \quad \text{calls to } f$$

where

$$\chi_{\mathcal{M}}(\mathcal{D}) = \sup_{\boldsymbol{x} \in \mathcal{M}} |D_{\boldsymbol{x}}| \underbrace{\left[\inf_{\boldsymbol{x} \in \mathcal{M}} \operatorname{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}}) \right]^{-2}}_{\operatorname{cm}_{\mathcal{M}}(\mathcal{D})^{-2}}.$$

Assume

- $\mathcal{D} = (D_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{M}}$ PSS of $\mathcal{M}$.
- $D_k = D_{\boldsymbol{x}_k}$ for every k.

Theorem (Cavarretta, Goyens, R., Yger '25+)

The method satisfies $\|\operatorname{grad} f(\boldsymbol{x}_k)\|_{\boldsymbol{x}_k} \leq \epsilon$ after at most

$$\mathcal{O}\left(\ \chi_{\mathcal{M}}(\mathcal{D})\,\epsilon^{-2}\ \right) \quad \text{calls to } f$$

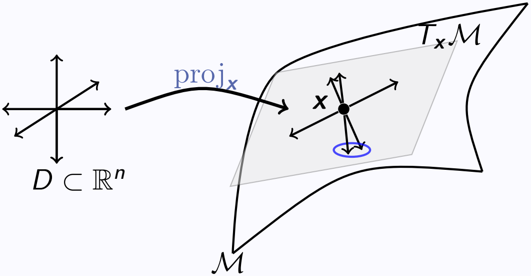
where

$$\chi_{\mathcal{M}}(\mathcal{D}) = \sup_{\boldsymbol{x}\in\mathcal{M}} |D_{\boldsymbol{x}}| \underbrace{\left[\inf_{\boldsymbol{x}\in\mathcal{M}} \mathrm{cm}_{\boldsymbol{x}}(D_{\boldsymbol{x}})\right]^{-2}}_{\mathrm{cm}_{\mathcal{M}}(\mathcal{D})^{-2}}.$$

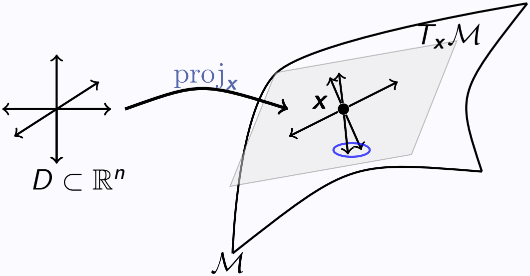
How do we generate $\mathcal{D}$?

Building manifold PSSs: Projected way

Idea Use PSSs from ambient space and project.



Idea Use PSSs from ambient space and project.

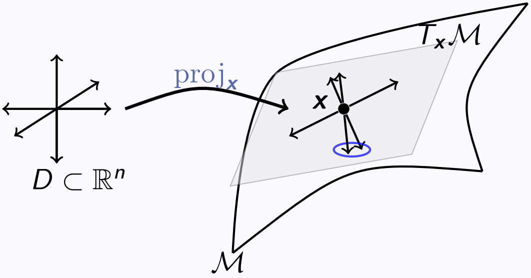


Projected PSS From D PSS of $\mathbb{R}^n$, build

$$\mathcal{P}(D) = \{P_{\boldsymbol{x}}(D)\}_{\boldsymbol{x}\in\mathcal{M}}, \qquad P_{\boldsymbol{x}}(D) \text{ projection of } D \text{ onto } \mathcal{T}_{\boldsymbol{x}}\mathcal{M}.$$

Building manifold PSSs: Projected way

Idea Use PSSs from ambient space and project.

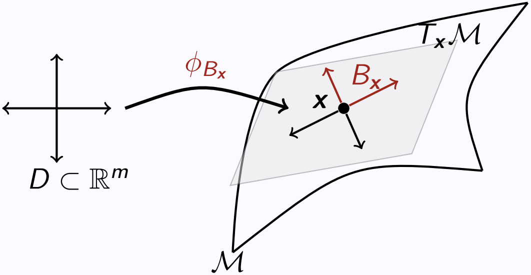


Projected PSS From D PSS of $\mathbb{R}^n$, build

$$\mathcal{P}(D) = \{P_{\boldsymbol{x}}(D)\}_{\boldsymbol{x}\in\mathcal{M}}, \qquad P_{\boldsymbol{x}}(D) \text{ projection of } D \text{ onto } \mathcal{T}_{\boldsymbol{x}}\mathcal{M}.$$

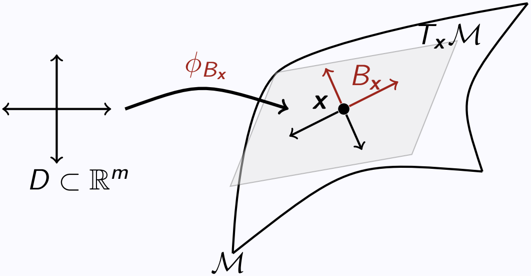
- $\mathrm{cm}_{\mathcal{M}}(\mathcal{P}(D)) \geq \mathrm{cm}(D)$.
- $\chi_{\mathcal{M}}(\mathcal{P}(D)) \leq |D|\,\mathrm{cm}(D)^{-2} = \chi(D)$.

Idea Use isomorphism induced by orthogonal bases between $\mathbb{R}^m$ and $\mathcal{T}_{\boldsymbol{x}}\mathcal{M}$.



Building manifold PSSs: Intrinsic way

Idea Use isomorphism induced by orthogonal bases between $\mathbb{R}^m$ and $\mathcal{T}_{\mathbf{x}}\mathcal{M}$.



Intrinsic PSS From D PSS of $\mathbb{R}^m$ and $\mathcal{B} = (B_{\mathbf{x}})_{\mathbf{x}\in\mathcal{M}}$ bases, build

$$D(\mathcal{B}) = (D(B_{\mathbf{x}}))_{\mathbf{x}\in\mathcal{M}}$$

- $\mathrm{cm}_{\mathcal{M}}(D(\mathcal{B})) = \mathrm{cm}(D)$, $|D(B_{\mathbf{x}})| = |D|$..
- $\chi_{\mathcal{M}}(D(\mathcal{B})) = \chi(D)$.

A simple (?) case for comparison

Manifold $\mathcal{M} = \mathbb{S}^{n-1} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = 1\}$, $m = n - 1$.

Two constructions in $\mathcal{T}_{\boldsymbol{x}}\mathcal{M} = \{\boldsymbol{u} \mid \boldsymbol{u}^{\mathrm{T}}\boldsymbol{x} = 0\}$

	Projected $P_{\boldsymbol{x}}(D^{\oplus})$	Intrinsic $D^{\oplus}(B_{\boldsymbol{x}})$
$\lvert \cdot \rvert$	$\leq 2n$	$2m = 2(n-1)$
$\mathrm{cm}_{\boldsymbol{x}}$	$\geq \frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{m}} = \frac{1}{\sqrt{n-1}}$

A simple (?) case for comparison

Manifold $\mathcal{M} = \mathbb{S}^{n-1} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = 1\}$, $m = n - 1$.

Two constructions in $\mathcal{T}_{\boldsymbol{x}}\mathcal{M} = \{\boldsymbol{u} \mid \boldsymbol{u}^\mathrm{T}\boldsymbol{x} = 0\}$

	Projected $P_{\boldsymbol{x}}(D^{\oplus})$	Intrinsic $D^{\oplus}(B_{\boldsymbol{x}})$
$\lvert \cdot \rvert$	$\leq 2n$	$2m = 2(n-1)$
$\mathrm{cm}_{\boldsymbol{x}}$	$\geq \frac{1}{\sqrt{n-1}} \geq \frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{m}} = \frac{1}{\sqrt{n-1}}$

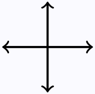
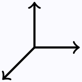
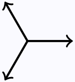
Theorem (Cavarretta, Goyens, R., Yger '25+)

$$2n(n-1) = \chi_{\mathbb{S}^{n-1}}(\mathcal{P}(D^{\oplus})) \geq \chi_{\mathbb{S}^{n-1}}(D^{\oplus}(\mathcal{B})) = 2(n-1)^2$$

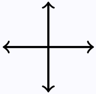
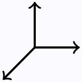
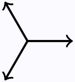
- Improvement especially for small n.
- Technical proof tailored to $D^{\oplus}$.

Numerical study

Three standard PSSs in $\mathbb{R}^m$

PSS	$D^{\oplus}$	$D^{\oslash}$	$D^{\circledast}$
In $\mathbb{R}^2$			
$\lvert \cdot \rvert$	$2m$	$m+1$	$m+1$
$\mathrm{cm}(\cdot)$	$\frac{1}{\sqrt{m}}$	$\frac{1}{\sqrt{m^2+2(m-1)\sqrt{m}}}$	$\frac{1}{m}$
$\chi(\cdot)$	$2m^2$	$m^3 + O(m^{5/2})$	$m^3 + O(m^2)$

Three standard PSSs in $\mathbb{R}^m$

PSS	$D^{\oplus}$	$D^{\obslash}$	$D^{\ovee}$
In $\mathbb{R}^2$			
$\lvert \cdot \rvert$	$2m$	$m+1$	$m+1$
$\mathrm{cm}(\cdot)$	$\frac{1}{\sqrt{m}}$	$\frac{1}{\sqrt{m^2+2(m-1)\sqrt{m}}}$	$\frac{1}{m}$
$\chi(\cdot)$	$2m^2$	$m^3 + O(m^{5/2})$	$m^3 + O(m^2)$

Goal Compare projected VS intrinsic PSSs built from $D^{\oplus}$, $D^{\obslash}$, $D^{\ovee}$.

Toy manifold problems (1/2)

Four problem classes

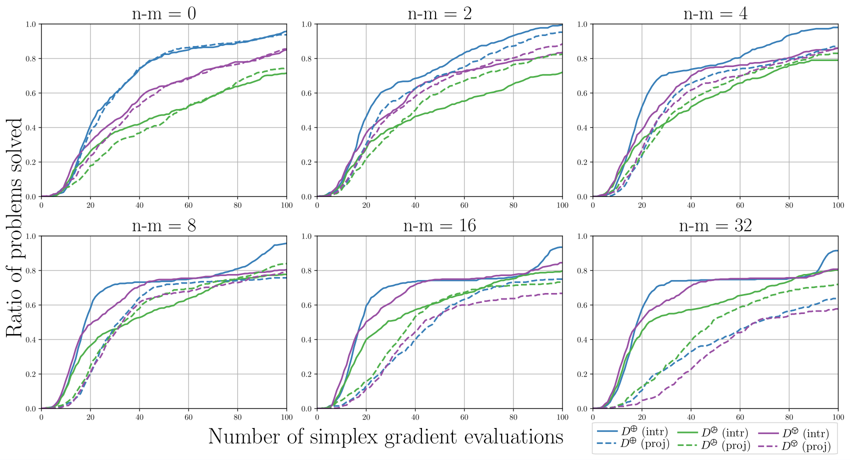
$\mathcal{M}$	$f(x)$	Parameters
$F := \mathrm{span}(z_1, \ldots, z_m)$	$\sum_{i=1}^{10} \|x_i - x\|_2^2$	$(x_i) \subset \mathbb{R}^n$
	$\sum_{i=1}^{10} \|x_i - x\|_2^2$	$(x_i) \subset \mathcal{M}$
	$\frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$	$A \succcurlyeq \frac{1}{10} I_n,\ b \in \mathbb{R}^n$
$\mathbb{S}^{(m+1)-1} \times \{0_{\mathbb{R}^{n-m}}\}$	$\langle x, Ax\rangle$	$A = A^{\mathrm{T}} \in \mathbb{R}^{n \times n}$

Toy manifold problems (1/2)

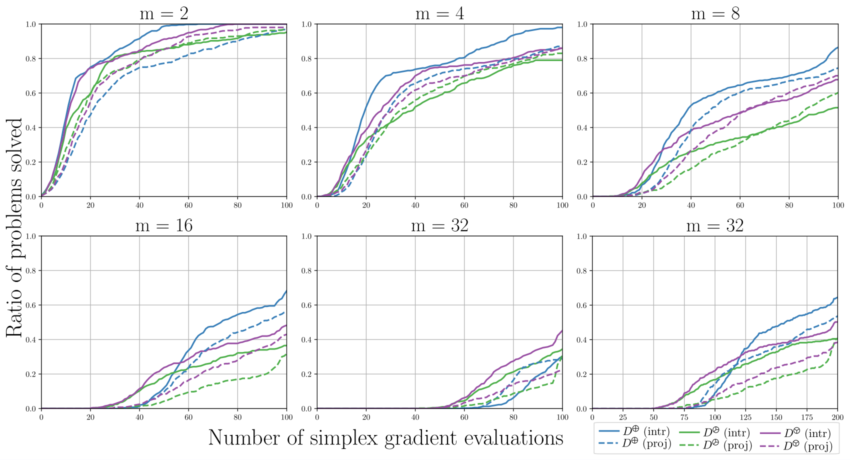
Four problem classes

$\mathcal{M}$	$f(x)$	Parameters
$F := \mathrm{span}(z_1, \ldots, z_m)$	$\sum_{i=1}^{10} \|x_i - x\|_2^2$	$(x_i) \subset \mathbb{R}^n$
	$\sum_{i=1}^{10} \|x_i - x\|_2^2$	$(x_i) \subset \mathcal{M}$
	$\frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$	$A \succcurlyeq \frac{1}{10} I_n,\ b \in \mathbb{R}^n$
$\mathbb{S}^{(m+1)-1} \times \{0_{\mathbb{R}^{n-m}}\}$	$\langle x, Ax\rangle$	$A = A^{\mathrm{T}} \in \mathbb{R}^{n \times n}$

- 100 instances for each class with varying $m/n-m$.
- Data profiles with $100(m+1)$ evaluation budget.



- Varying: $n - m$ with $m = 4$ or m with $n - m = 4$.
- Intrinsic variants outperform projected ones (esp. for large $n - m$).



- Varying: $n - m$ with $m = 4$ or m with $n - m = 4$.
- Intrinsic variants outperform projected ones (esp. for large $n - m$).

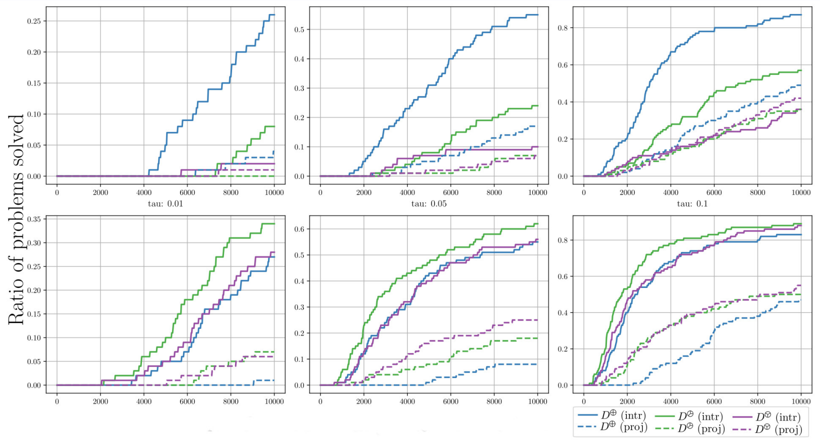
$$\min_{(R_1,R_2)\in SO(5)\times SO(5)} \|R_1 - HR_2\|_F^2$$

where $SO(5) = \{R \in \mathbb{R}^{5\times 5}, R^{\mathrm{T}}R = I_5, \det(R) = 1\}$ and $H \in \mathbb{R}^{5\times 5}$

- $m = 20,\ n - m = 30.$
- Known tangent space.

Setup

- 100 instances with different H.
- Budget of 10000 evaluations.



- Intrinsic better than projected (again).
- Rotations affect quality of $\mathcal{P}(D^{\oplus})$.

Conclusion

Our setup

- Direct search on Riemannian manifolds.
- Complexity guarantees.

Our findings

- Proper definition of PSSs with guarantees.
- Intrinsic version $>$ Projected version?

Up next

- Going probabilistic...
- ...for larger-scale applications.

That's it!

arXiv > math > arXiv:2511.15360

Mathematics > Optimization and Control

[Submitted on 19 Nov 2025 (v1), last revised 29 May 2026 (this version, v2)]

Complexity guarantees and polling strategies for Riemannian direct-search methods

Bastien Cavarretta, Florentin Goyens, Clément W. Royer, Florian Yger arXiv > math > arXiv:2511.15360

Mathematics > Optimization and Control

[Submitted on 19 Nov 2025 (v1), last revised 29 May 2026 (this version, v2)]

Complexity guarantees and polling strategies for Riemannian direct-search methods

Bastien Cavarretta, Florentin Goyens, Clément W. Royer, Florian Yger

Thank you!

`clement.royer@lamsade.dauphine.fr`