Revisiting derivative-free optimization algorithms in the context of black-box adversarial attacks

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MILES project in Dauphine

What is MILES?

- Team-project within Dauphine.
- Focus: **Trustworthy AI**.
- Expertise: Machine learning, Game theory, Optimization,...

We have an MCF opening for 2026!

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One of our strengths: Adversarial attacks

Perturb data to fool models

- Computing attacks (Meunier et al '19, Timoz et al '26).
- Game theoretical perspective (Meunier et al '21).
- Robustify models via randomization (Pinot et al. '19 '20 '22, Gnecco Heredia et al '24).

Where this talk fits in

COMPROMIS project (PEPR Cybersécurité)

- Thesis of Bastien Cavarretta.
- Goal: Develop structured adversarial attacks in a black-box setting using derivative-free optimization.

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- Zoo of black-box adversarial attacks.
- Derivative-free methods somewhat disregarded because of scalability issues.

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What we have now

- An algorithmic framework with enhanced scalability (theory).
- Preliminary numerical validation.

Roadmap

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2. Black-box adversarial attacks
3. Adversarial attacks via direct search

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Classification setup

Basic classification paradigm

- Data points $\{\boldsymbol{x}_i, y_i\}_{i=1,\ldots,N}$ with
  - $\boldsymbol{x}_i$ input (e.g. image) viewed as a vector in $\mathcal{X} \subset \mathbb{R}^n$.
  - $y_i \in \{1, \ldots, C\}$ discrete (class) label.
- Goal: Predict y_i given $\boldsymbol{x}_i$ through a model h.
- Training: Compute best h within a model class $\mathcal{H}$.

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Empirical risk minimization

$$\underset{h \in \mathcal{H}}{\text{minimize}} \ \frac{1}{N} \sum_{i=1}^{N} \mathcal{L}(h(\boldsymbol{x}_i), y_i).$$

- Typical model: Logit function $h(\boldsymbol{x}) \in \mathbb{R}^C$ output of a neural network.
- Typical loss: Cross entropy $\mathcal{L}(h(\boldsymbol{x}), y) = -\log\left(\frac{\exp(h(\boldsymbol{x})^y)}{\sum_{c=1}^{C} \exp(h(\boldsymbol{x})^c)}\right)$.

What could go wrong in classification?

Suppose

- Data $\{(\boldsymbol{x}_i, y_i)\}$ (e.g. ImageNet).
- Model h obtained by training on this data?

Can the model be fooled by small perturbations of data points?

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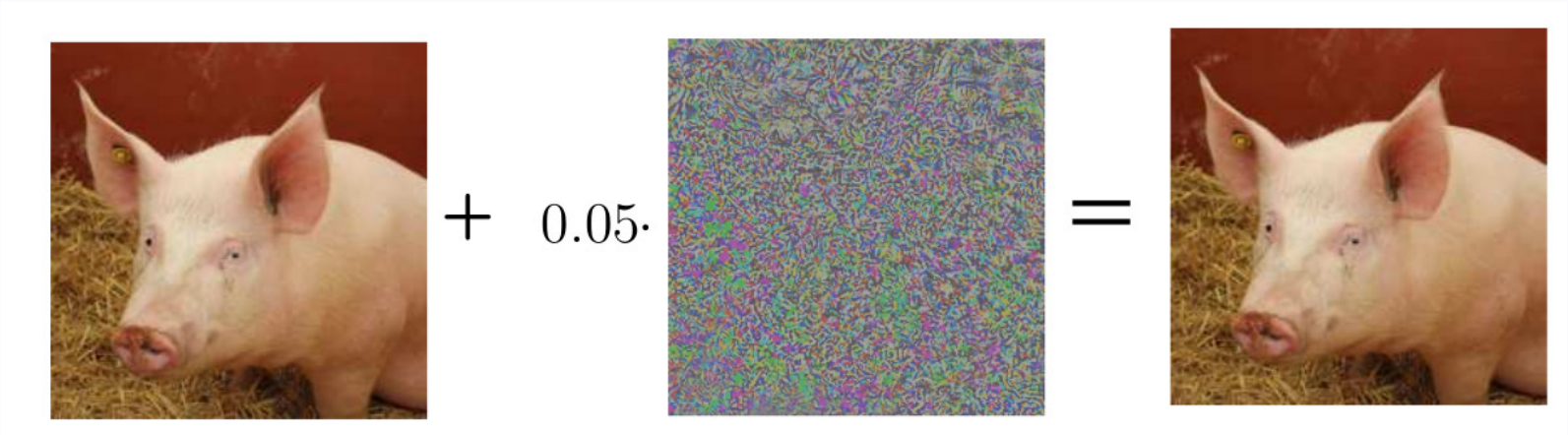

A hog misclassified as an airliner by a ResNet50 with high confidence
`https://adversarial-ml-tutorial.org/introduction/`

Adversarial examples for classification

Adversarial examples/attacks

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- Observed in spam detection (Dalvi et al '04), speech recognition, language processing, etc.

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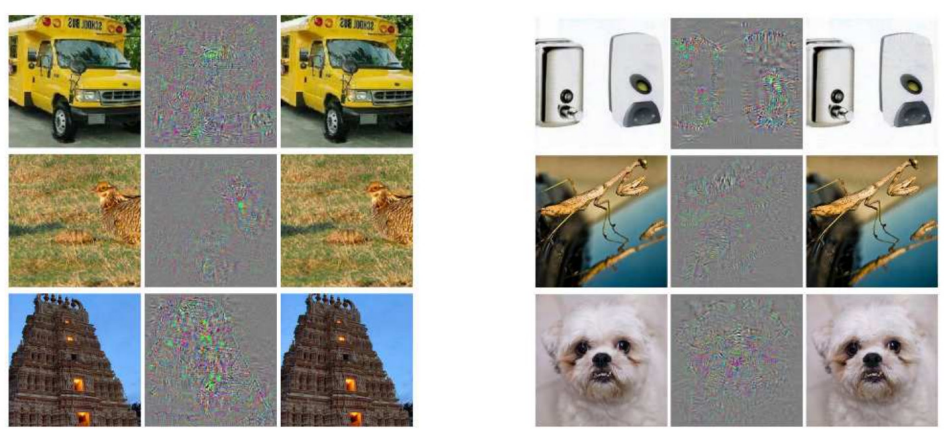

Perturbed data points from ImageNet classified as Ostrich by AlexNet (Szegedy et al '14).

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Adversarial attack from data point $(\boldsymbol{x}, y)$

$$\underset{\boldsymbol{z} \in \mathcal{B}(\boldsymbol{x}, \epsilon) \cap \mathcal{X}}{\text{maximize}} \; \mathcal{L}(h(\boldsymbol{z}), \boldsymbol{y}).$$

- $\mathcal{B}(\boldsymbol{x}, \epsilon) = \{\boldsymbol{z} \mid \|\boldsymbol{z} - \boldsymbol{x}\| \leq \epsilon\}$, $\|\cdot\|$ norm on $\mathbb{R}^n$.
- h either un-trained (data poisoning) or **trained (evasion attack) model**.

$\rightarrow$ Focus for this talk: Evasion attacks.

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Original attack (Szegedy et al '14)

$$\underset{\boldsymbol{z}}{\text{minimize}} \, \|\boldsymbol{z} - \boldsymbol{x}\|_2 \quad \text{s.t.} \quad h(\boldsymbol{z}) = y_{adv}, \boldsymbol{z} \in \mathcal{X}.$$

$\rightarrow$ Targeted attack (label y_{adv})
$\rightarrow$ Nonlinear optimization problem (solved with L-BFGS).

Gradient-based attacks

Idea Solve the attack problem inexactly using gradient-based methods.

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Fast Gradient Method (Goodfellow et al '15)

$$\underset{\boldsymbol{z} \in \mathcal{B}(\boldsymbol{x}, \epsilon)}{\text{maximize}} \quad (\boldsymbol{z} - \boldsymbol{x})^{\mathrm{T}} \nabla_{\boldsymbol{x}} \mathcal{L}(h(\boldsymbol{x}), y).$$

- First-order approximation of $\mathcal{L}(h(\boldsymbol{z}), y) - \mathcal{L}(h(\boldsymbol{x}), y)$.
- Compute $\nabla_{\boldsymbol{x}} \mathcal{L}(h(\boldsymbol{x}), y)$ through automatic differentiation.
- Closed-form expressions when using $\|\cdot\|_2$ and $\|\cdot\|_\infty$.

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Projected Gradient Descent/PGD (Kurakin et al '17, Madry et al '18)

$$\boldsymbol{z}_{t+1} = \Pi_{\mathcal{B}(\boldsymbol{x},\epsilon)} \left(\boldsymbol{z}_t + \alpha \nabla_x \mathcal{L}(h(\boldsymbol{z}_t), y)\right)$$

- $\Pi_{\mathcal{B}(\boldsymbol{x},\epsilon)}$: Projection.
- Multiple steps of gradient ascent for the problem.

More attacks, and defenses (sample!)

- **CW Attack** (Carlini & Wagner '17b): Modified version of (Szegedy et al '14) using changes of variables.
  $\rightarrow$ (Carlini & Wagner '17a): Circumvent detection of adversarial examples.

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- **Circumventing existing defenses**
  - (Athalye et al '18) 7/9 defenses from ICLR 2018.
  - (Tramer et al '20) 13 defenses from ICML/ICLR/NeurIPS.
  - (Sitawarin et al '22) breaks (Raff et al '19).

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- **Methodology for evaluating defenses**
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- A never-ending game of cat and mouse...
- Randomized defenses (popular topic in MILES) not necessarily better (Lucas et al '23).

Summing up

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What if the model is not available?

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Black-box adversarial attacks

Recall: Adversarial attack problem

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Black-box setup

- h model (e.g. neural net) only available through queries.
- Different from previous *white-box* models.

First attempts at black-box attacks

Proxy-based attacks (Papernot et al. '17, Tu et al '18)

- Replace h with a proxy network $\tilde{h}$.
- Build white-box attacks for such a network!

$\rightarrow$ Efficient for computing many adversarial examples.

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$\rightarrow$ A derivative-free optimization technique!

Derivative-free optimization

$$\text{maximize}_{\boldsymbol{z} \in \mathbb{R}^n} \quad f(\boldsymbol{z}).$$

Derivative-free algorithms

(Audet & Hare '25, Conn et al '09, Larson et al '21)

- **Derivatives unavailable for algorithmic use**.
- Function calls typically considered expensive.

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- **Direct search** Explore the space through directions.
- **Model based** Use surrogate for f to compute new points (similar to proxy but iterative).

A numerical comparison (from Ughi et al '22)

Goal ℓ_∞ attacks

$$\underset{\boldsymbol{z}\in\mathcal{B}_\infty(\boldsymbol{x},\epsilon_\infty)}{\text{maximize}} \quad \mathcal{L}(h(\boldsymbol{x}), y).$$

- $\mathcal{B}_\infty(\boldsymbol{x}, \epsilon_\infty) = \{\boldsymbol{z} \mid \max_i |z_i - x_i| \leq \epsilon\}$.
- h: ResNet50 with or without adversarial training (Engstrom et al '19).
- Dataset: CIFAR-10, $32 \times 32 \times 3$ images, 10 classes.
- 900 attacks (Assign all other labels to one of 100 images).

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Compared methods

- *Frank-Wolfe* (Chen et al '20): Randomized zeroth-order method.
- *GenAttack* (Alzantot et al '19): Genetic algorithm.
- *Parsimonious* (Moon et al '19), *Square* (Andriushchenko et al '20): Direct search over combinatorial domain.
- BOBYQA (Powell '09) general-purpose DFO solver (in Fortran!)

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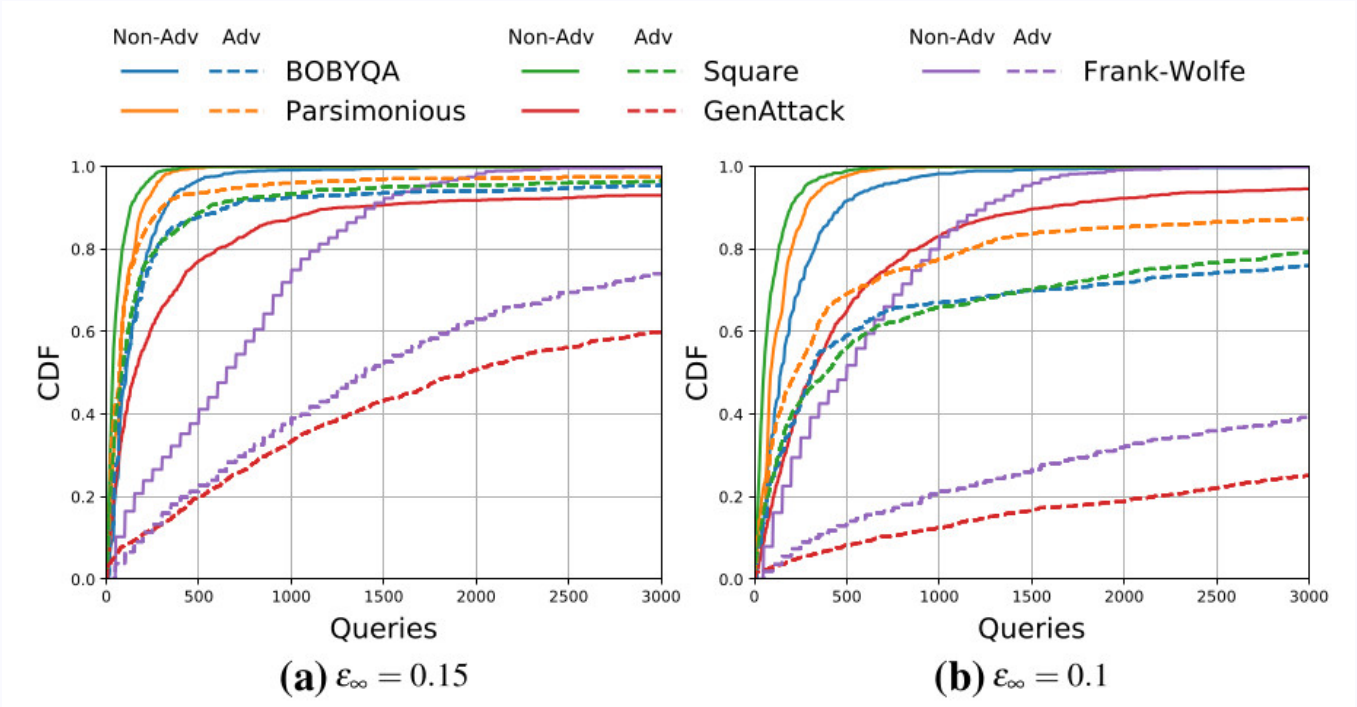

(a) $\varepsilon_\infty = 0.15$ **(b)** $\varepsilon_\infty = 0.1$

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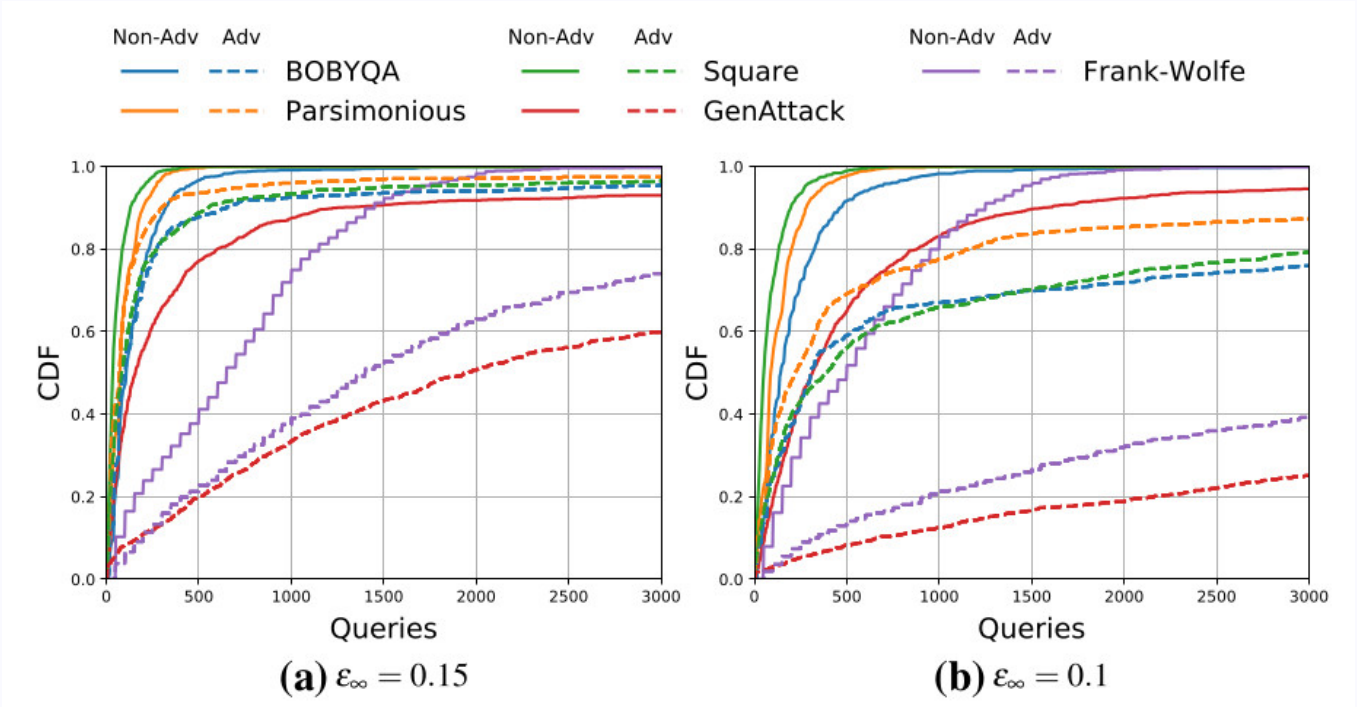

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Good performance for off-the-shelf BOBYQA?

Off-the-shelf solver and adversarial

BOBYQA solver (Original, Powell '09)

- Typically used up to a few hundred variables.
- Requires to build n-dimensional models.
- Here $n = 3072$!

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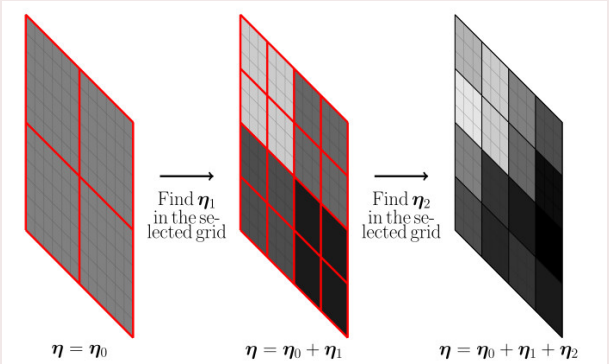
BOBYQA for adversarial (Ughi et al '19)



- Random sub-sampling of variables (also used in ZOO/Frank-Wolfe).
- Hierarchical lifting, built through randomness (also used in GenAttack).

Other derivative-free techniques

True label:	brambling	cannon	pug-dog	fly	armadillo	balloon
Perturbed image **(ZO-PSGD):**						
Prediction label:	goldfinch	plow	bucket	longicorn	croquet ball	parachute
ℓ_2 Distortion:	35.2137	87.7304	72.3397	111.068	172.719	35.9330
# of queries:	7640	150	130	50	210	5830
Perturbed image **(ZO-SMD):**						
Prediction label:	goldfinch	plow	bucket	cicada	croquet ball	parachute
ℓ_2 distortion:	8.9708	26.0126	20.9504	30.0968	45.097	10.9023
# of queries:	29980	350	260	140	570	15830
Perturbed image **(ZO-AdaMM):**						
Prediction label:	goldfinch	plow	bucket	cicada	croquet ball	parachute
ℓ_2 distortion:	8.0502	5.7359	4.5753	4.4456	6.3149	7.7405
# of queries:	53300	1710	790	540	2780	32900
Perturbed image **(ZO-NES):**						
Prediction label:	goldfinch	plow	bucket	cicada	croquet ball	parachute
ℓ_2 distortion:	54.9956	34.5852	28.7035	28.9158	40.1483	51.7116
# of queries:	22110	1080	830	430	2280	15340

Zeroth-order with ImageNet on InceptionV3 (Liu et al '20).

$\rightarrow$ Mostly randomized variants with convergence rates.

Other derivative-free techniques ('ed)

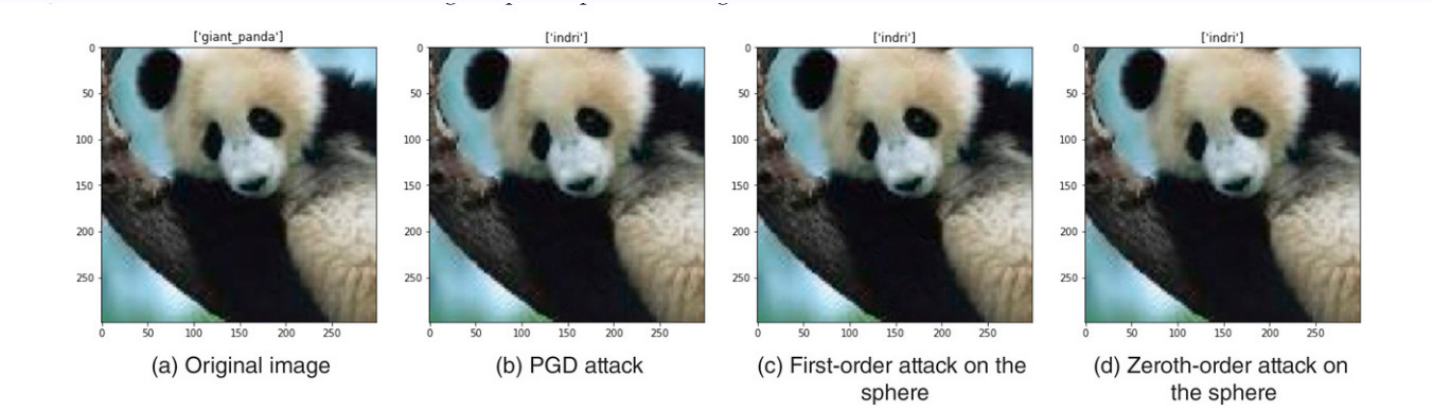
Riemannian Zeroth-Order (Li et al '23)

- Compute attacks on the sphere

$$\underset{\|\boldsymbol{z}-\boldsymbol{x}\|=\epsilon}{\text{maximize}} \ \mathcal{L}(h(\boldsymbol{z}), y).$$

  $\rightarrow$ Riemannian structure!

- Randomized zeroth-order scheme
  $\rightarrow$ Aggregate finite-difference estimates.



(a) Original image (b) PGD attack (c) First-order attack on the sphere (d) Zeroth-order attack on the sphere

Attack on ImageNet, $\|\cdot\| = \|\cdot\|_2$, $\epsilon = 0.05$.

Recap: Black-box adversarial attacks

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- Numerous variants and benchmarking tasks.
- Mostly of zeroth-order type.

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Our objectives

- Adapt general-purpose direct search to adversarial attacks.
- Develop structured attacks with randomness.

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Derivative-free optimization via direct search

$$\underset{z \in \mathbb{R}^n}{\text{maximize}} \quad f(z)$$

Blackbox/Derivative-free optimization

- **Derivatives unavailable for algorithmic use.**
- Only access to values of f.
- **Cost** Number of function evaluations.

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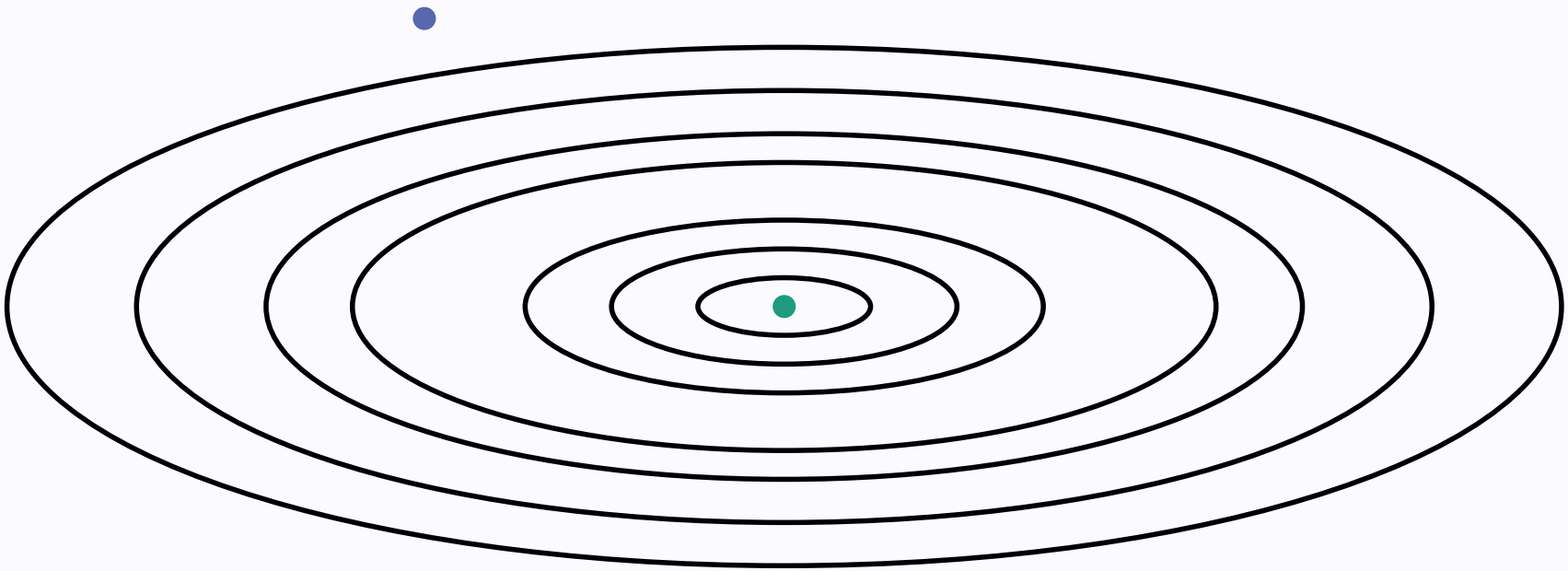
Direct search

- Proceed by polling the space along suitable directions.
- Rich convergence theory (Kolda et al '03, Dzahini et al. '25).

Example: Coordinate search



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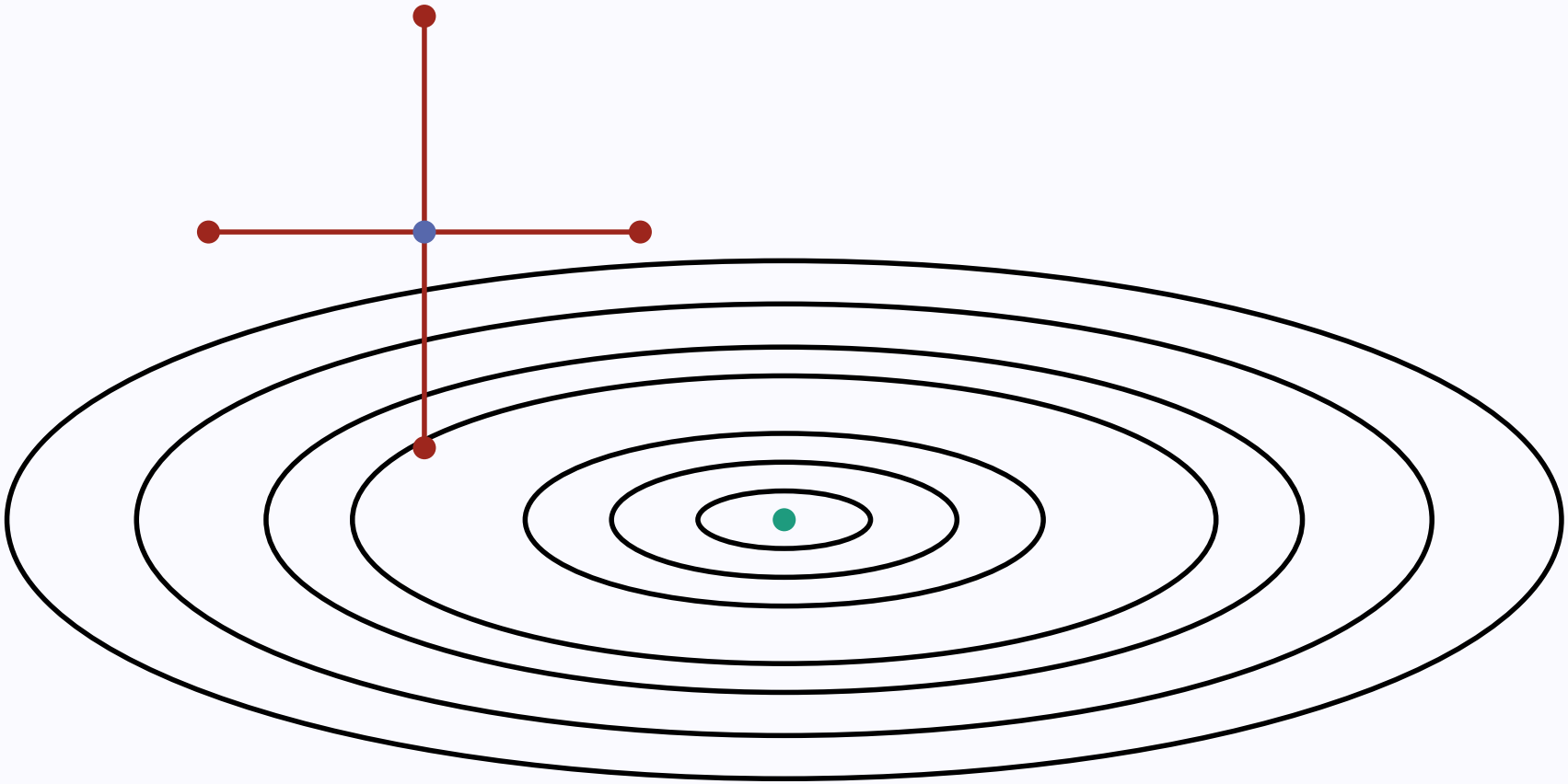

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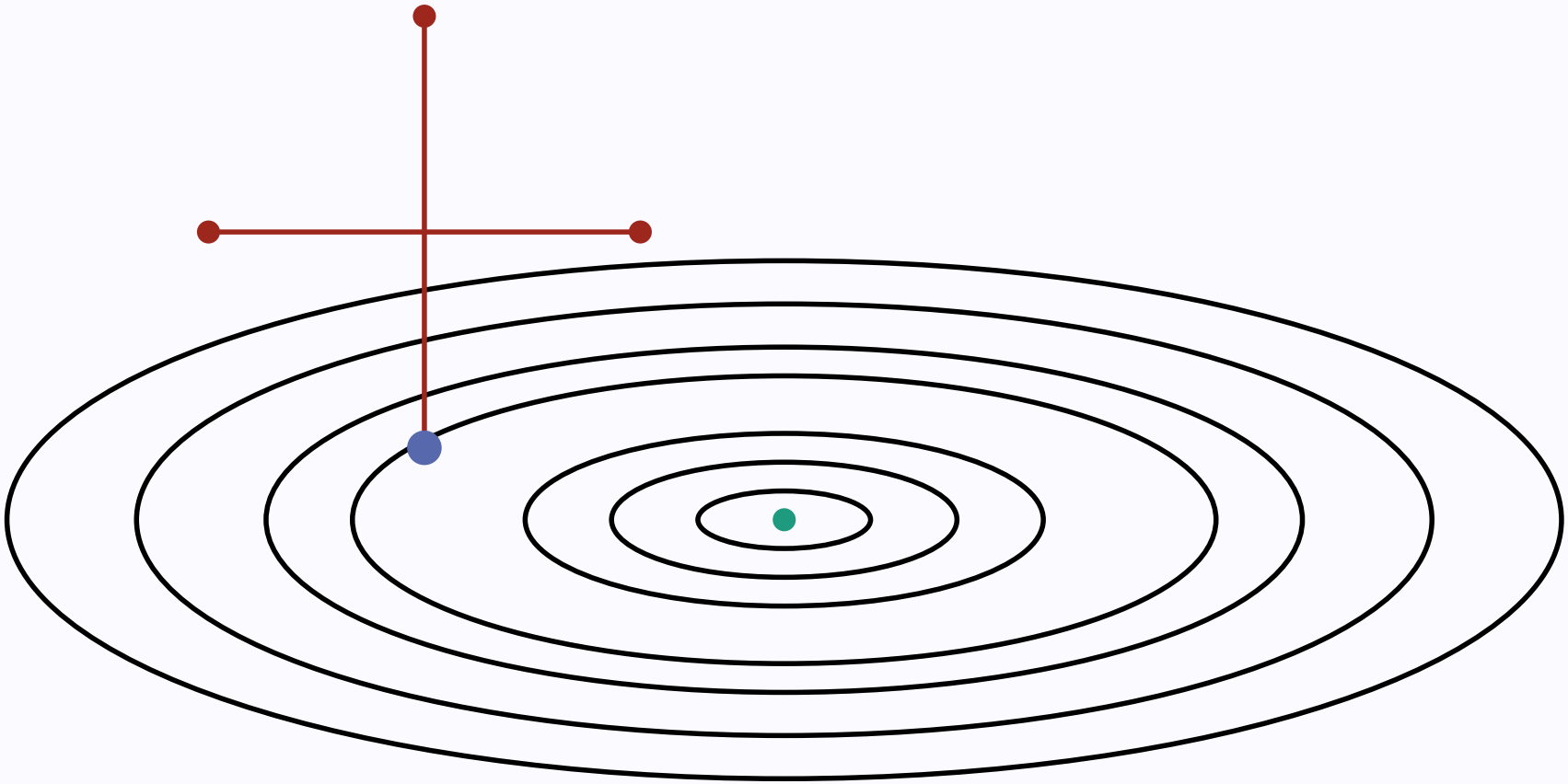

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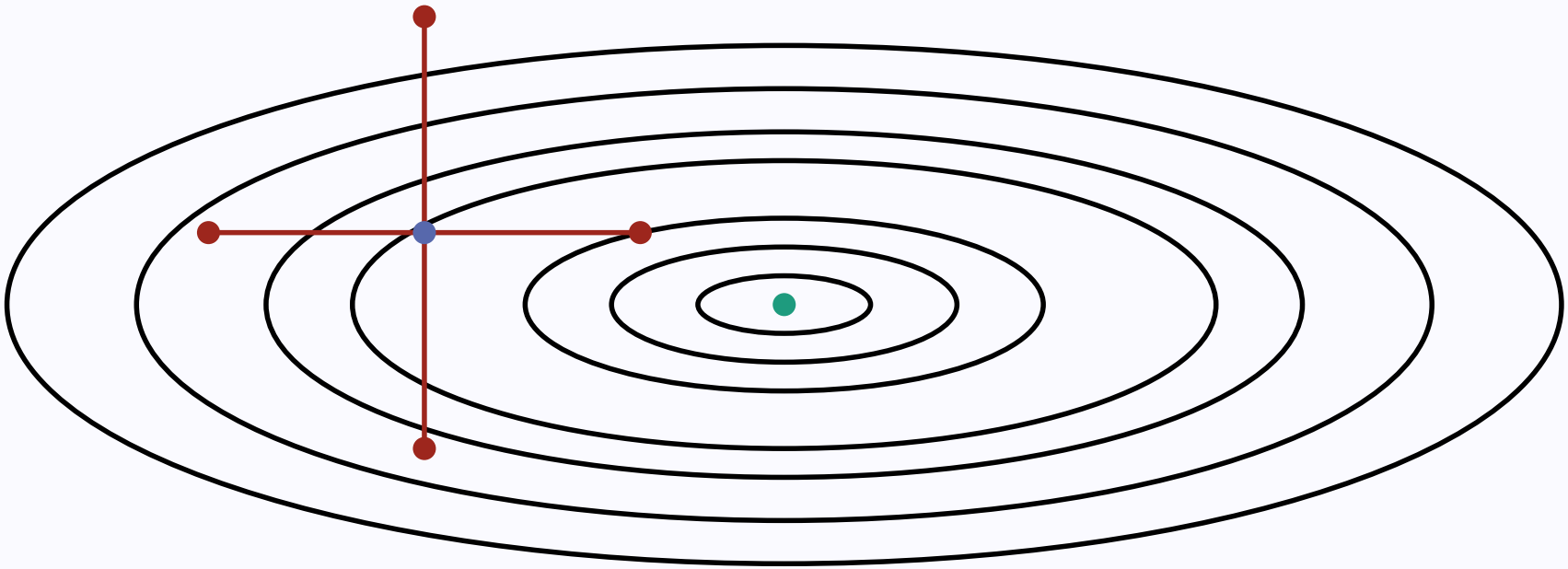

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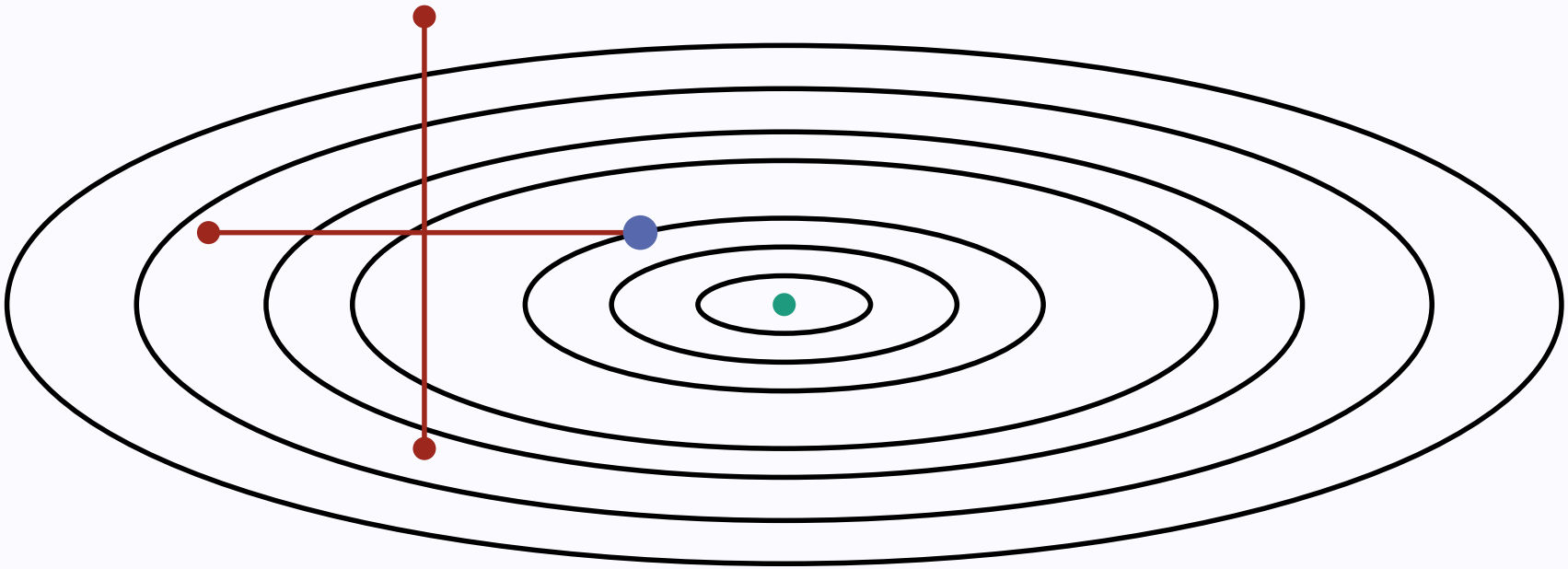

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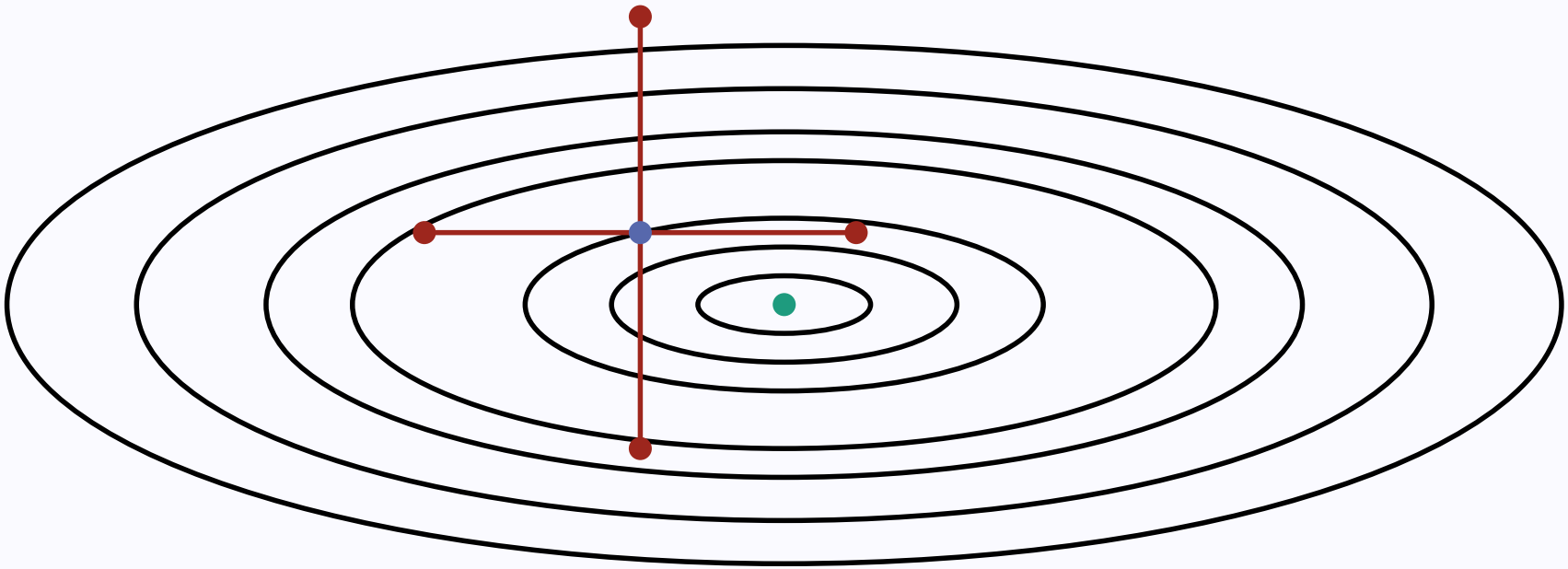

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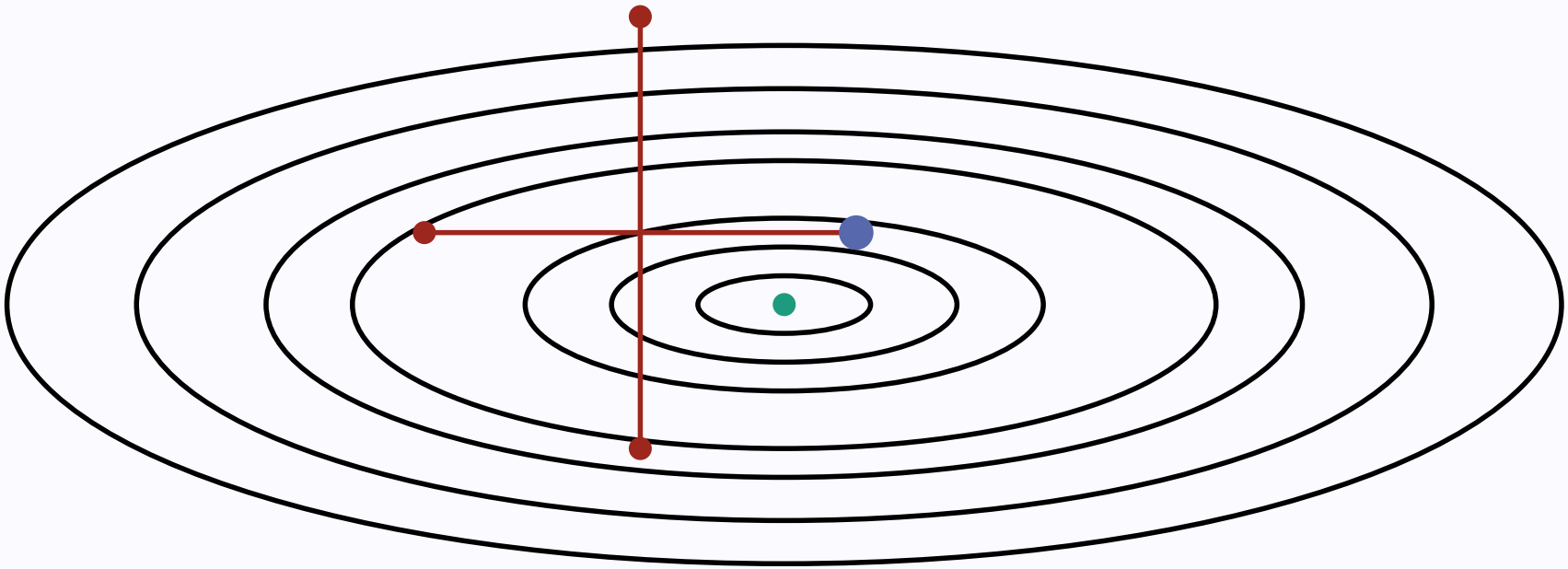

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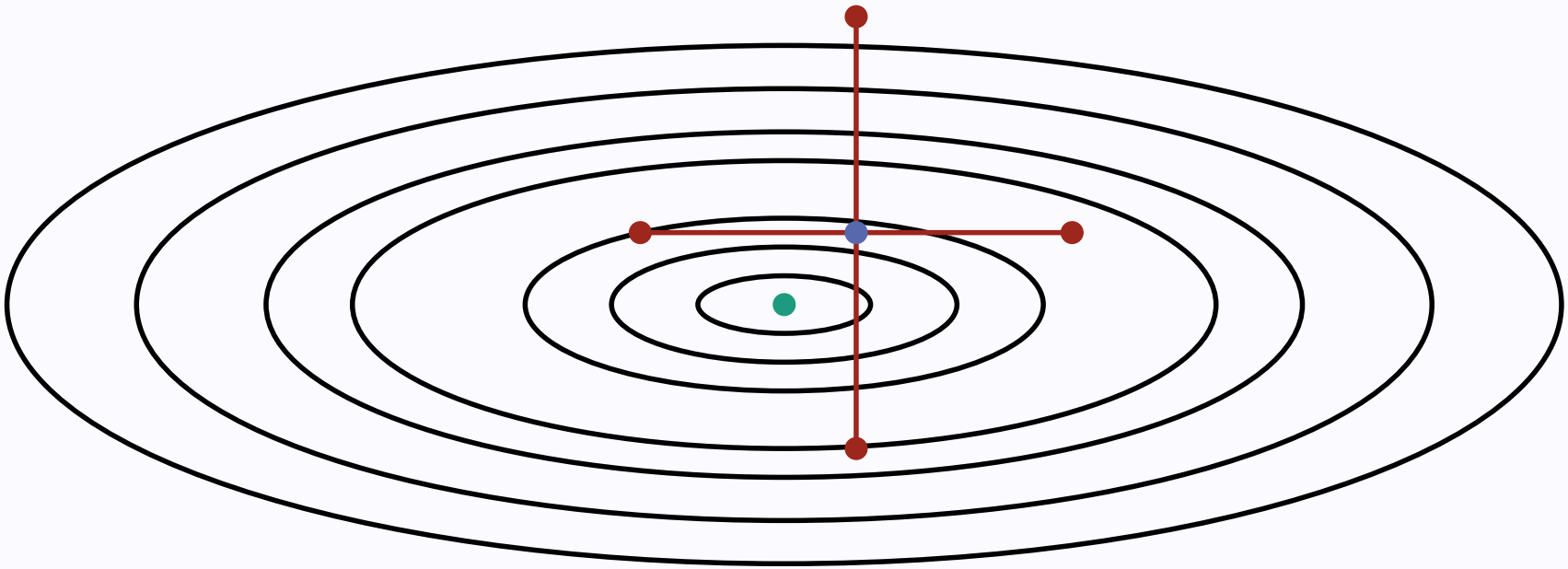

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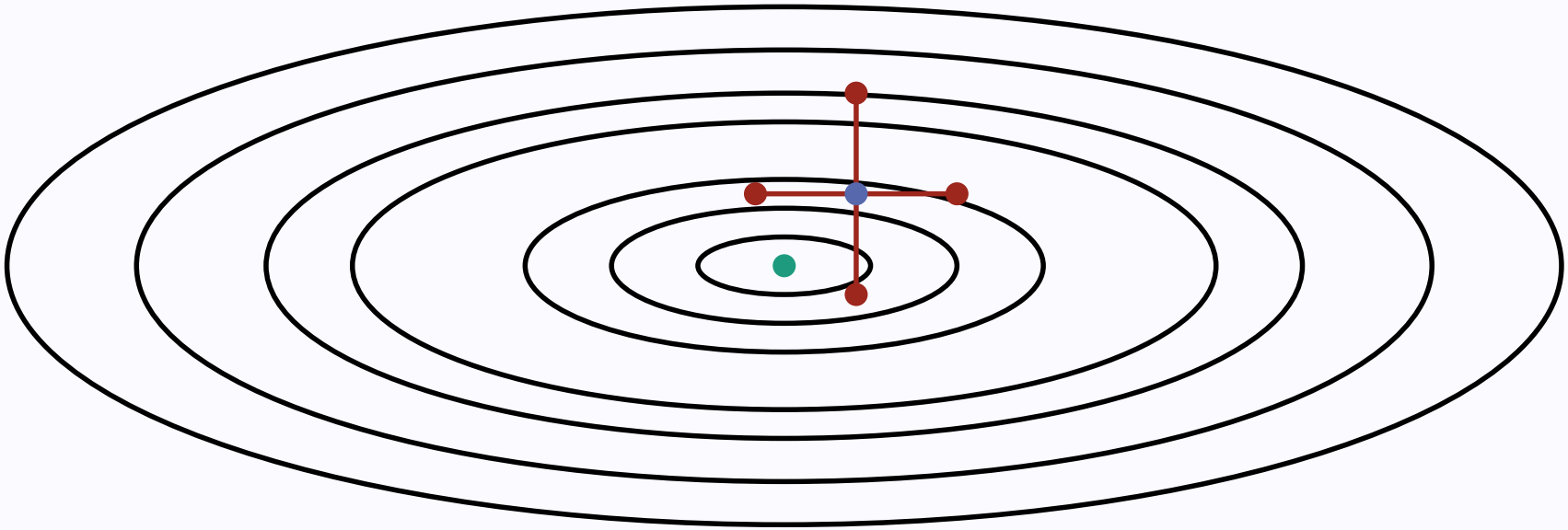

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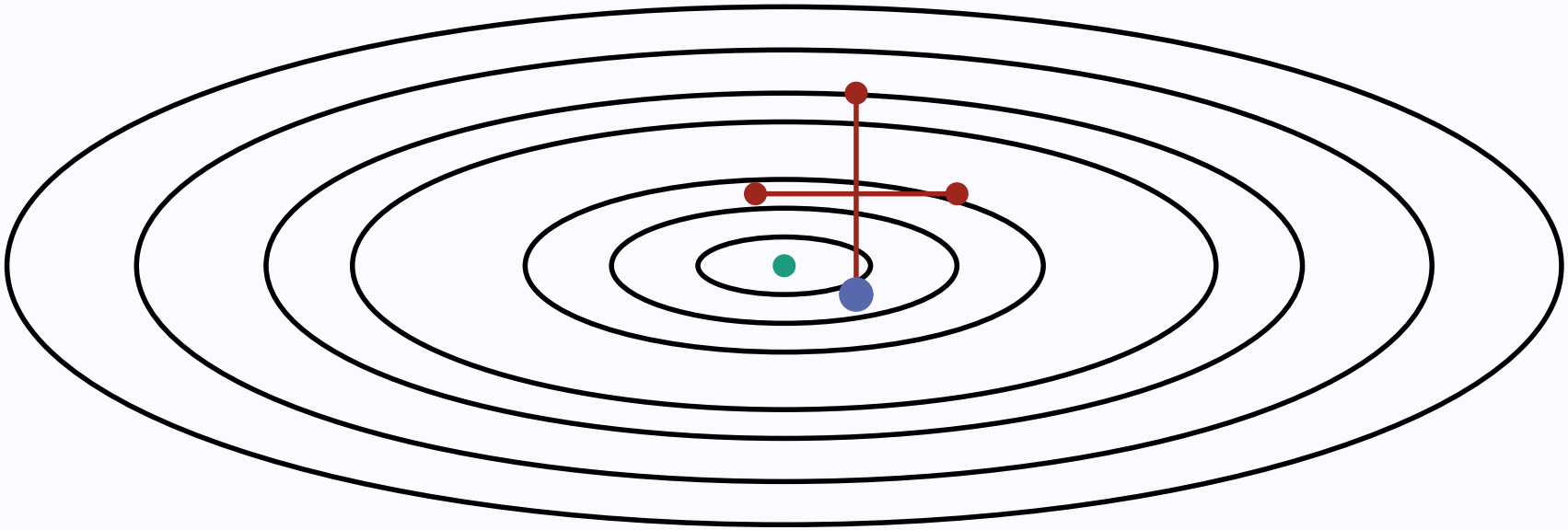

Basic direct-search framework

Problem $\text{maximize}_{z \in \mathbb{R}^n} \quad f(z)$.

Inputs $z_0 \in \mathbb{R}^n$, $\alpha_0 > 0$, $\eta > 0$.
For $k = 0, 1, \ldots$

- Choose $\mathcal{D}_k \subset \mathbb{R}^n$.

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Positive Spanning Set (PSS)

$$\mathcal{D} \subset \mathbb{R}^n \text{ PSS} \qquad \Longleftrightarrow \qquad \text{cm}(\mathcal{D}) = \max_{\boldsymbol{d} \in \mathcal{D}} \min_{\|\boldsymbol{v}\|=1} \frac{\boldsymbol{d}^{\mathrm{T}} \boldsymbol{v}}{\|\boldsymbol{d}\|} > 0.$$

$\text{cm}(\mathcal{D})$: Cosine measure.

Complexity of direct search (smooth setting)

Goal $\|\nabla f(\boldsymbol{z}_K)\| \leq \epsilon_g$.

Theorem (Vicente '12)

If $\mathcal{D}_k = \mathcal{D}$ PSS $\forall k$, the method takes at most

$$\mathcal{O}\left(|\mathcal{D}| \, \mathrm{cm}(\mathcal{D})^{-2} \, \epsilon_g^{-2} \right)$$

calls to f.

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- Best known bound:

$$|\mathcal{D}| = \mathcal{O}(n), \mathrm{cm}(\mathcal{D}) = \mathcal{O}(n^{-1/2}) \quad \Rightarrow \quad \mathcal{O}(n^2 \, \epsilon_g^{-2}).$$

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Can we improve using randomness?

Using randomness

Our approach (w/ Lindon Roberts)

- Consider a random subspace of dimension $r \leq n$;
- Use a PSS to approximate the projected gradient in the subspace;
- Guarantee sufficient gradient information **in probability**.

What it brings us

- Use random directions.
- Much less than ambient dimension n!

Not the only game in town (1/2)

Probabilistic descent (Gratton et al '15)

- Use directions $[\boldsymbol{d} \ -\boldsymbol{d}]$ with $\boldsymbol{d} \sim \mathcal{U}(\mathbb{S}^{n-1})$.
- Complexity improves from $\mathcal{O}(n^2\epsilon^{-2})$ to $\mathcal{O}(n\epsilon^{-2})$ $(m = 2)$.
- Limited to one distribution.

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Gaussian smoothing approach: Draw $\boldsymbol{d} \sim \mathcal{N}(0, \boldsymbol{I})$ and use

$$\frac{f(\boldsymbol{x} + \delta\boldsymbol{d}) - f(\boldsymbol{x})}{\delta}\boldsymbol{d} \quad \text{or} \quad \frac{f(\boldsymbol{x} + \delta\boldsymbol{d}) - f(\boldsymbol{x} - \delta\boldsymbol{d})}{\delta}\boldsymbol{d}.$$

Random gradient-free method (Nesterov and Spokoiny 2017),
Stochastic three-point method (Bergou et al, 2020).

- Also achieve $\mathcal{O}(n\epsilon^{-2})$ bound.
- Use one-dimensional subspace based on Gaussian vectors.
- Use fixed or decreasing stepsizes.

Not the only game in town (2/2)

Zeroth-order (Kozak et al '21, '22)

- Estimate directional derivatives directly.
- Use orthogonal random directions $\boldsymbol{Q} \in \mathbb{R}^{n \times r}$, $\boldsymbol{Q}^{\mathrm{T}} \boldsymbol{Q} = \boldsymbol{I}$.
- Complexity results for convex/PL functions.

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Our approach

- General, subspace-based framework.
- Inspiration: Model-based methods (Cartis and Roberts '23, Dzahini and Wild '22a).

Direct-search algorithm (in subspaces)

Inputs: $\boldsymbol{z}_0 \in \mathbb{R}^n$, $\alpha_0 > 0$, $\eta > 0$.
Iteration k**:** Given $(\boldsymbol{z}_k, \alpha_k)$,

- Choose $\boldsymbol{P}_k \in \mathbb{R}^{r \times n}$ **at random**.
- Choose $\mathcal{D}_k \subset \mathbb{R}^r$ having m vectors.
- If $\exists\ \boldsymbol{d}_k \in \mathcal{D}_k$ such that

$$f(\boldsymbol{z}_k + \alpha_k \boldsymbol{P}_k^{\mathrm{T}} \boldsymbol{d}_k) > f(\boldsymbol{z}_k) + \eta \alpha_k^2 \|\boldsymbol{P}_k^{\mathrm{T}} \boldsymbol{d}_k\|^2,$$

  set $\boldsymbol{z}_{k+1} := \boldsymbol{z}_k + \alpha_k \boldsymbol{P}_k^{\mathrm{T}} \boldsymbol{d}_k$, $\alpha_{k+1} := 2\alpha_k$.
- Otherwise, set $\boldsymbol{z}_{k+1} := \boldsymbol{z}_k$, $\alpha_{k+1} := \alpha_k/2$.

Complexity analysis (smooth case)

Goal $\|\nabla f(\boldsymbol{z}_K)\| \leq \epsilon_g$.

Theorem (Roberts, R. '23)

Assume:

- $\{\boldsymbol{P}_k\}$ $(p, \sigma, P_{\max})$-well aligned with $p > 1/2$.
- $\mathcal{D}_k$ best PSS in r-dimensional subspace.

Let N_{ϵ_g} the number of function evaluations needed to have $\|\nabla f(\boldsymbol{z}_K)\| \leq \epsilon_g$. Then,

$$\mathbb{P}\left(N_\epsilon \leq \mathcal{O}\left(r^2 \phi \epsilon_g^{-2}\right)\right) \geq 1 - \exp\left(-\mathcal{O}\left(\phi \epsilon_g^{-2}\right)\right).$$

where $\phi = \sigma^{-2} P_{\max}^4$.

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How does $r^2\phi$ depend on n?

Randomized subspace techniques

Setup

- Use random $\boldsymbol{P}_k \in \mathbb{R}^{r \times n}$.
- Complexity scales as $r^2 \sigma^{-2} P_{\max}^{-4}$.

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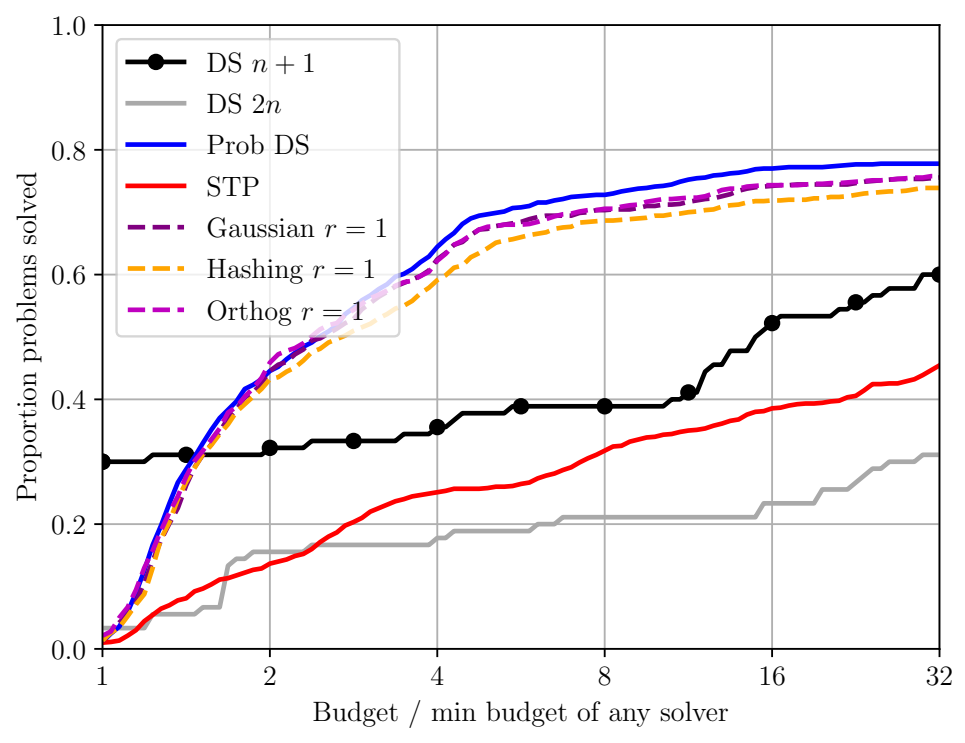
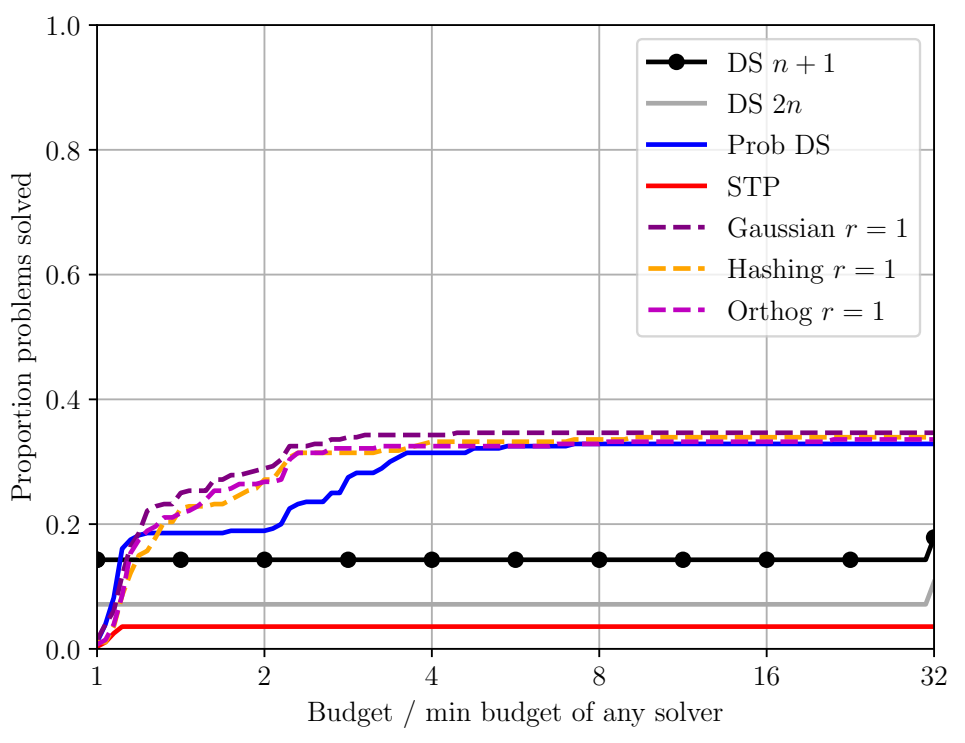
$\boldsymbol{P}_k$	σ	$P_{\max}$
Gaussian	$\Theta(\sqrt{n/r})$	$\Theta(\sqrt{n/r})$
Hashing	$\Theta(\sqrt{n/r})$ (Dzahini & Wild '22b)	$\sqrt{n}$
Orthogonal	$\sqrt{n/r}$	$\sqrt{n/r}$.

Best bound

$$r = \mathcal{O}(1) \quad \Rightarrow \quad \mathcal{O}(n\epsilon_g^{-2}).$$

Numerical illustration





Left: Medium scale; Right: Large scale.

- Operating in random subspaces works!
- But always a (hidden) dependency on n!

$\rightarrow$ `pip install directsearch`

Takeways

Direct search

- Simple to implement.
- Theoretical guarantees$\rightarrow$ Dependency on n.
- Competitor: Zeroth-order schemes!

Random subspaces

- Help scale things up (on standard DFO benchmarks).
- Still dependent on problem dimension n (not enough for adversarial attacks).

Riemannian optimization setup

$$\underset{\boldsymbol{Z} \in \mathcal{M}}{\text{maximize}}\, f(\boldsymbol{Z}).$$

- f: Black-box objective.
- $\boldsymbol{Z}$: Matrix/Tensor object (e.g. $32 \times 32 \times 3$ images).
- $\mathcal{M}$: Known Riemannian manifold.

Motivating examples

- Spheres in Matrix/Tensor spaces: $\mathcal{M} = \{\boldsymbol{Z}, \|\boldsymbol{Z} - \boldsymbol{X}\| = \epsilon\}$.
- Symmetric Positive Definite Matrices (SPDNet).
- Low-rank matrix/tensors.

Riemannian direct search

Problem: maximize$_{\boldsymbol{Z} \in \mathcal{M}} f(\boldsymbol{Z})$.

Inputs $\boldsymbol{Z}_0 \in \mathcal{M}$, $\alpha_0 > 0$, $\eta > 0$.
For $k = 0, 1, \ldots$

- Choose a PSS $\mathcal{D}(\boldsymbol{Z}_k)$ for $\mathcal{T}_{\boldsymbol{Z}_k}\mathcal{M}$.
- If $\exists\ \boldsymbol{D}_k \in \mathcal{D}_k$ such that

$$f(\mathcal{R}_{\boldsymbol{Z}_k}(\alpha_k \boldsymbol{D}_k)) > f(\boldsymbol{Z}_k) + \eta \alpha_k^2 \|\boldsymbol{D}_k\|_{\boldsymbol{Z}_k}^2$$

  set $\boldsymbol{Z}_{k+1} := \mathcal{R}_{\boldsymbol{Z}_k}(\alpha_k \boldsymbol{D}_k)$, $\alpha_{k+1} := 2\alpha_k$.
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- $\mathcal{T}_{\boldsymbol{Z}_k}\mathcal{M}$: Tangent space for $\mathcal{M}$ at $\boldsymbol{Z}_k$ with norm $\|\cdot\|_{\boldsymbol{Z}_k}$ $\rightarrow$ Rigorous extension of PSS possible!
- $\mathcal{R}_{\boldsymbol{Z}_k}(\alpha_k \boldsymbol{D}_k)$: Retraction (brings back onto the manifold) $\rightarrow$ Guarantees feasible iterates!

A theoretically sound approach

Goal $\|\text{grad} f(\boldsymbol{Z}_K)\| \leq \epsilon_g$ (Riemannian gradient).

Theorem (Cavarretta, Goyens, R., Yger '25)

The method takes at most

$$\mathcal{O}\left(\max_{\boldsymbol{Z} \in \mathcal{M}} |\mathcal{D}(\boldsymbol{Z})| \, \text{cm}(\mathcal{D}(\boldsymbol{Z}))^{-2} \, \epsilon_g^{-2} \right)$$

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Extension: probabilistic variant

- Sample tangent subspaces (Gutman & Ho-Nguyen '22).
- Generalization of Euclidean case (Roberts, R. '23).

Test case: Low-rank attacks

Recall: Adversarial attack problem (matrix version)

$$\underset{\boldsymbol{Z} \in \mathcal{B}(\boldsymbol{X}, \epsilon)}{\text{maximize}} \ \mathcal{L}(h(\boldsymbol{Z}), y)$$

$\rightarrow \mathcal{B}(\cdot)$ ℓ_2 or nuclear norm ball.

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- Observation: Projected Gradient Descent computes approximately low-rank attacks ($\boldsymbol{Z} - \boldsymbol{X}$ has many small singular values).

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- Approach: Enforce low-rank perturbation:

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and solve with PGD applied to $\boldsymbol{U}$ and $\boldsymbol{V}$.

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What about low-rank black-box attacks?

Black-box low-rank attacks: Setup

Four methods (applied to factors $\boldsymbol{U}$, $\boldsymbol{V}$)

- **DS** Deterministic, uses fixed directions.
  $\rightarrow \mathcal{O}(\dim(\boldsymbol{U}) + \dim(\boldsymbol{V}))$ evaluations/iteration.
- **DS-PD(1)** Probabilistic, 1-dimensional subspace.
  $\rightarrow \mathcal{O}(1)$ evaluations/iteration.
- **DS-PD(rank)** Probabilistic, r-dimensional subspace for rank-r attacks.
  $\rightarrow \mathcal{O}(r)$ evaluations/iteration.
- **GFM** Gradient-Free Method, based on zeroth-order formula

$$\boldsymbol{W}_{k+1} = \boldsymbol{W}_k + \sigma_k \frac{f(\boldsymbol{W}_k + \mu_k \boldsymbol{D}_k) - f(\boldsymbol{W}_k - \mu_k \boldsymbol{D}_k)}{2\mu_k}$$

  over 1-dimensional subspaces.
  $\rightarrow \mathcal{O}(1)$ evaluations/iteration.

Black-box low-rank attacks: Setup

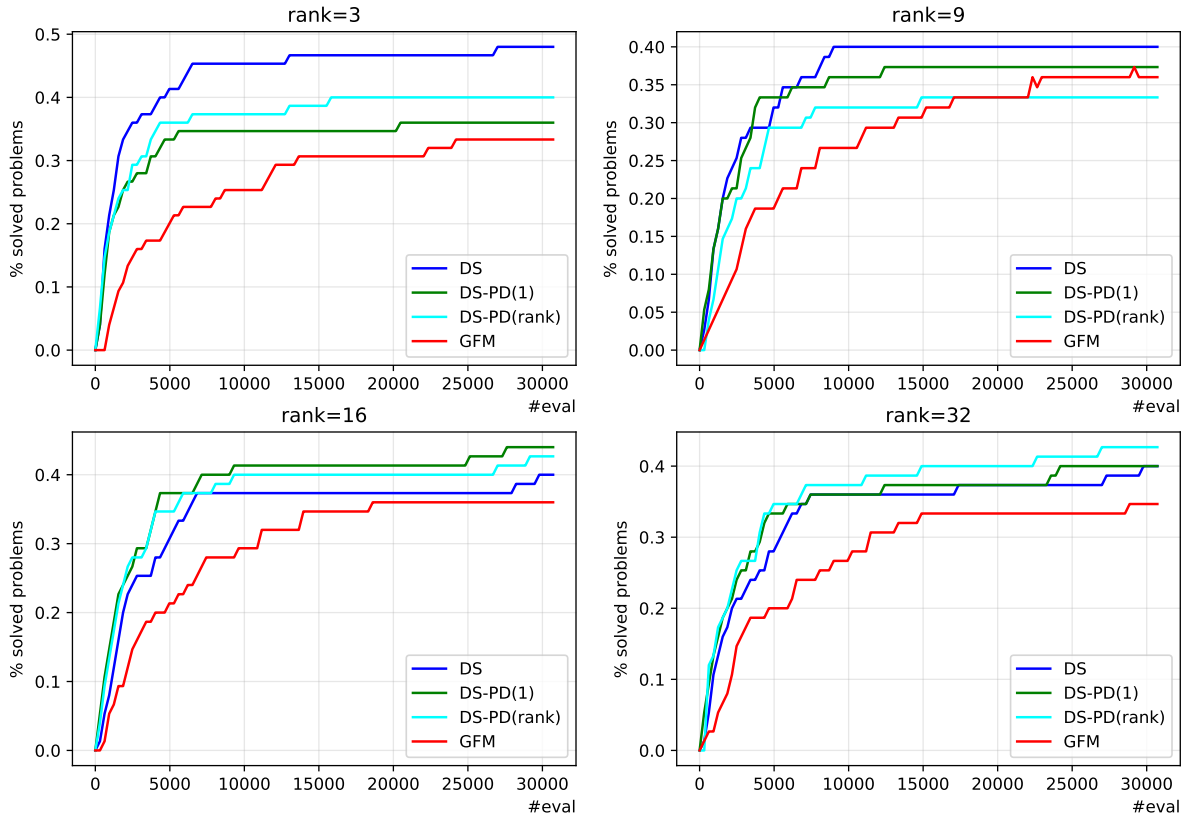
Model

- From RobustBench (Croce et al '21).
- WideResNet-28-10 trained on CIFAR-10.
- Problem dimension: $32 \times 32 \times 3 = 3072$.

Task

- Attack 75 images from CIFAR-10 (Target loss value).
- Budget: 10 PGD iterations with finite differences.
  $\rightarrow 32 \times 32 \times 3 \times 10 = 30720$ evaluations.

Black-box low-rank attacks: Results (fresh!)



Data profiles (top left is better)

- Small rank: Deterministic direct search is good.
- Increasing rank: Probabilistic variants for scalability!

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Black-box adversarial attacks

- Hugely popular topic.
- Challenge: High dimensionality (but structure).

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- Classical techniques for low dimensions.
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- Classical techniques for low dimensions.
- Recent effort to scale up through randomization.

Our approach: Direct search

- Theory for Riemannian+Randomized versions.
- Promising results with low-rank perturbation of images.

References

- B. Cavarretta, F. Goyens, C. W. Royer and F. Yger, *Complexity guarantees and polling strategies for Riemannian direct-search methods*, arXiv:2511.15360, under review.
- K. J. Dzahini, F. Rinaldi, C. W. Royer and D. Zeffiro, *Direct search methods in the year 2025: Theoretical guarantees and algorithmic paradigms*, EURO Journal on Computational Optimization, 2025.
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→ Stay tuned for more!

References

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Merci!

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