COMPUTATIONAL METHODS IN OPTIMIZATION

October 23, 2023

This week:
- Introduction to QP (today)
- Lab session on IPMs and LP (tomorrow A211)

Coming up:
- Exercise sheet (notebook)
- Notes

Interior-point methods and quadratic programming

Motivating example

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad 8x_1 - x_2 \quad \text{s.t.} \quad \begin{aligned} x_1 - 2x_2 &\geq -2 \\ x_1 - x_2 &\geq -7 . \\ x &\geq 0 \end{aligned}$$

$\rightarrow$ short for $x_1 \geq 0$, $x_2 \geq 0$

$\hookrightarrow$ This problem is not in standard form
To apply IPMs, we put the problem in standard form by adding variables

$$\underset{x \in \mathbb{R}^4}{\text{minimize}} \quad 8x_1 - x_2 \quad \text{s.t.} \quad \begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ x_1 - x_2 - x_4 &= -7 \\ x &\geq 0 \end{aligned}$$

$\rightarrow$ $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$

$\Rightarrow$ Standard form

$$\underset{x \in \mathbb{R}^4}{\text{minimize}} \quad c^T x \quad \text{s.t.} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 4} \qquad b = \begin{bmatrix} -2 \\ -7 \end{bmatrix} \in \mathbb{R}^2 \qquad c = \begin{bmatrix} 8 \\ -1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

Using A, b, c, we can write the dual of the problem

$$\underset{\substack{y \in \mathbb{R}^2 \\ s \in \mathbb{R}^4}}{\text{maximize}} \quad b^T y \quad \text{s.t.} \quad \begin{aligned} A^T y + s &= c \\ s &\geq 0 \end{aligned}$$

$$\underset{\substack{y \in \mathbb{R}^2 \\ s \in \mathbb{R}^4}}{\text{maximize}} \quad -2y_1 - 7y_2 \quad \text{s.t.} \quad \begin{aligned} y_1 + y_2 + s_1 &= 8 \\ -2y_1 - y_2 + s_2 &= -1 \\ -y_1 + s_3 &= 0 \\ -y_2 + s_4 &= 0 \\ s &\geq 0 \end{aligned}$$

KKT optimality conditions

(x^*, y^*, s^*) is a primal-dual solution of the problem if (and only if)

$$A x^* = b \qquad x^* \geq 0$$

$$A^T y^* + s^* = c \qquad s^* \geq 0$$

$$x_i^* s_i^* = 0 \qquad \forall i = 1..4$$

↳ The simplex method works by "inspection"

- Fix some x_is and s_is to 0 $\quad$ ($x_i^* s_i^* = 0 \Leftrightarrow x_i^* = 0$ or $s_i^* = 0$)
- Try to find the remaining variables

$\Rightarrow$ the simplex method iterates over that process

$\Rightarrow$ Can be done manually by considering all possible cases

Sketch of inspection solve

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ x_1 - x_2 - x_4 &= -7 \end{aligned} \qquad x \geq 0$$

$$\begin{aligned} y_1 + y_2 + s_1 &= 8 \\ -2y_1 - y_2 + s_2 &= -1 \\ -y_1 + s_3 &= 0 \\ -y_2 + s_4 &= 0 \end{aligned} \qquad s \geq 0$$

$$x_1 s_1 = x_2 s_2 = x_3 s_3 = x_4 s_4 = 0$$

(1) $s_3 = s_4 = 0 \Rightarrow y_1 = y_2 = 0 \Rightarrow s_2 = -1$ ✗

(2) $s_3 \neq 0$, $s_4 \neq 0 \Rightarrow x_3 = x_4 = 0 \Rightarrow$ $\begin{aligned} x_1 - 2x_2 &= -2 \\ x_1 - x_2 &= -7 \\ x &\geq 0 \end{aligned}$ has no solution ✗

(3) $s_3 = 0$, $s_4 \neq 0$ ✗

(4) $s_3 \neq 0 \Rightarrow x_3 = 0$, $s_4 = 0$

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 6 \end{bmatrix} \quad y = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \quad s = \begin{bmatrix} 7.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

$n=4$

Interior-point approach

To avoid considering all possibilities for satisfying $x_i s_i = 0 \quad \forall i=1..n$, $x \geq 0$, $s \geq 0$

only consider strictly feasible points $x > 0$ ($x_i > 0 \ \forall i = 1..n$), $s > 0$

and compute a sequence (x^k, y^k, s^k) such that $\mu^k = \frac{(x^k)^T (s^k)}{n} \xrightarrow[k \to \infty]{} 0$

↳ Given (x^k, y^k, s^k), $(x^{k+1}, y^{k+1}, s^{k+1})$ is computed as

$$(x^k + \alpha^k \Delta x^k, \ y^k + \alpha^k \Delta y^k, \ s^k + \alpha^k \Delta s^k)$$

where

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{bmatrix} = - \begin{bmatrix} A x^k - b \\ A^T y^k + s^k - c \\ X^k S^k e - \sigma^k e \end{bmatrix}$$

$$X^k = \begin{bmatrix} x_1^k & & 0 \\ & \ddots & \\ 0 & & x_n^k \end{bmatrix} \qquad S^k = \begin{bmatrix} s_1^k & & 0 \\ & \ddots & \\ 0 & & s_n^k \end{bmatrix} \qquad e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \sigma^k \leq \mu^k$$

and $\alpha^k > 0$ is chosen so that $(x^{k+1}, y^{k+1}, s^{k+1})$ is strictly feasible

(The choice of $\{\alpha^k, \sigma^k\}$ defines the particular algorithm that is used)

↳ IPMs compute an approximate solution (in finitely many iterations)

↳ The exact solution can often be deduced from the approximate one if accurate enough

CVX outputs

$$x = \begin{bmatrix} -2.10^{-10} \\ 0.9999999 \\ 2.10^{-9} \\ 6.000000 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 6 \end{bmatrix}$$

$$y = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

(some solvers output $-y$ instead of y because $Ax = b \iff -Ax = -b$)

In practice, y may be recomputed using the outputs for x and s

$$s = \begin{bmatrix} 7.50000000 \\ 3 \cdot 10^{-10} \\ 0.500000000 \\ 1 \cdot 10^{-11} \end{bmatrix} \Rightarrow s = \begin{bmatrix} 7.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

Quadratic programming

Motivating example (again)

$$\underset{x \in \mathbb{R}^4}{\text{minimize}} \quad \alpha x_1^2 + \beta x_2^2 + 8x_1 - x_2 \quad \text{s.t.} \quad \begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ x_1 - x_2 - x_4 &= -7 \\ x &\geq 0 \end{aligned}$$

Quadratic function of x (polynomial of degree 2 in the coefficients of x)

$\alpha \geq 0, \beta \geq 0$

$\alpha = \beta = 0 \Rightarrow LP$

Q) what happens when $\alpha\beta > 0$?
- → solution?
- → dual?
- → IPM?

Def. A quadratic program in standard form is an optimization problem of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2} x^T Q x + c^T x \quad \text{s.t.} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$

$$\frac{1}{2} x^T Q x = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} x_i x_j$$

Quadratic program (QP) = Quadratic objective + linear constraints

Our example:
$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ -7 \end{bmatrix} \quad c = \begin{bmatrix} -8 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2\alpha & 0 & 0 & 0 \\ 0 & 2\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Big) \quad \frac{1}{2} x^T Q x = \alpha x_1^2 + \beta x_2^2$$

$c \Rightarrow$ linear part of the objective

$Q \Rightarrow$ quadratic part

Assumption: Q is symmetric ($Q_{ij} = Q_{ji} \; \forall (i,j) \iff Q = Q^T$)

$Q \succeq 0$ "Q is positive semidefinite"

$\not\geq$ $\quad (\forall x \in \mathbb{R}^n, \; x^T Q x \geq 0)$

Properties of the QP

- Dual:

$$\underset{\substack{y \in \mathbb{R}^m \\ s \in \mathbb{R}^n \\ x \in \mathbb{R}^n}}{\text{maximize}} \; -\frac{1}{2} x^T Q x + b^T y \quad \text{s.t.} \quad -Qx + A^T y + s = c, \quad s \geq 0$$

⚠ The primal variables also appear in the dual (but not with ≥ 0 constraints)

- KKT conditions: (x^*, y^*, s^*) solution if (and only if)

$$A x^* = b \qquad x^* \geq 0$$

only ≠ with LP (still a linear equation) $\rightarrow$

$$-Qx^* + A^T y^* + s^* = c \qquad s^* \geq 0$$

$$x_i^* s_i^* = 0 \quad \forall i = 1..n$$

$\rightarrow$ still the difficult part!

↳ No straightforward extension of the simplex method

↳ Inspection becomes more difficult because of the constraint $-Qx^* + A^T y^* + s^* = c$ that links the three vectors of variables

↳ IPMs extend "easily" to that case

Only change in the generic algorithm: linear system

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{bmatrix} = - \begin{bmatrix} Ax^k - b \\ -Qx^k + A^T y^k + s^k - c \\ X^k S^k e - \sigma^k e \end{bmatrix}$$