# COMPUTATIONAL METHODS IN OPTIMIZATION

November 27, 2023

3 weeks remaining!

Today: Lecture / Solutions of exercises on conic programming
Dec 4 : Last lecture (implementation details)
Dec 5 3.30-6.45pm : Lab
Dec 11 8.30-11.45am : Lab
Dec 12 3.30-5pm : Lab
Dec 14 1.45-3.15pm : Lab

Exam date (TBC): Wednesday January 10, 10am-12pm

# EXERCISES ON CONIC PROGRAMMING

(Second exercise sheet)

## Ex 4: QCQPs and SOCPs

4.1. Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $Q \in \mathbb{R}^{n \times n}$ be symmetric ($Q^T = Q$) and positive semidefinite ($Q \succeq 0$).

Show that a vector $x \in \mathbb{R}^n$ satisfies $c^T x + \frac{1}{2} x^T Q x \le b$ if and only if

$$\left\| \begin{bmatrix} Q^{1/2} x \\ b - c^T x - 1/2 \end{bmatrix} \right\| \le b - c^T x + \frac{1}{2}$$

$$\begin{bmatrix} Q^{1/2} x \\ b - c^T x - \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{n+1}$$

where $Q^{1/2}$ is the matrix square root of $Q$.

$$(Q^{1/2})^T = Q^{1/2} \qquad Q^{1/2} Q^{1/2} = Q$$

Solution:

$$\left\| \begin{bmatrix} Q^{1/2} x \\ b - c^T x - 1/2 \end{bmatrix} \right\|^2 = \begin{bmatrix} Q^{1/2} x \\ b - c^T x - 1/2 \end{bmatrix}^T \begin{bmatrix} Q^{1/2} x \\ b - c^T x - 1/2 \end{bmatrix}$$

$$= (Q^{1/2} x)^T Q^{1/2} x + (b - c^T x - 1/2)^2$$

$$= x^T (Q^{1/2})^T Q^{1/2} x + b^2 + (c^T x)^2 + 1/4 - 2b c^T x - b + c^T x$$

$$= x^T Q x + b^2 + (c^T x)^2 + 1/4 - 2 b c^T x - b + c^T x$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

$$\left(b - c^T x + 1/2\right)^2 = b^2 + (c^T x)^2 + 1/4 - 2 b c^T x + b - c^T x$$

If $x$ satisfies the second-order inequality $\left\| \begin{bmatrix} Q^{1/2}x \\ b - c^T x - 1/2 \end{bmatrix} \right\| \leq b - c^T x + 1/2$, we have

$$\left\| \begin{bmatrix} Q^{1/2}x \\ b - c^T x - 1/2 \end{bmatrix} \right\|^2 \leq \left( b - c^T x + \frac{1}{2} \right)^2$$

Second-order cone constraint

$$\Leftrightarrow \quad x^T Q x + \cancel{b^2} + \cancel{(c^T x)^2} + \cancel{\tfrac{1}{4}} - \cancel{2bc^T x} - b + c^T x \leq \cancel{b^2} + \cancel{(c^T x)^2} + \cancel{\tfrac{1}{4}} - \cancel{2bc^T x} + b - c^T x$$

$$\Leftrightarrow \quad x^T Q x - b + c^T x \leq b - c^T x$$

$$\Leftrightarrow \quad 2c^T x + x^T Q x \leq 2b$$

$$\Leftrightarrow \quad c^T x + \frac{1}{2} x^T Q x \leq b$$

4.2. Consider the Quadratically Constrained Quadratic Program

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c_0^T x + \frac{1}{2} x^T Q_0 x \\ \text{s.t.} \quad & c_1^T x + \frac{1}{2} x^T Q_1 x \leq b \end{aligned}$$

where $Q_0 = Q_0^T \succeq 0$, $Q_1 = Q_1^T \succeq 0$, $c_0 \in \mathbb{R}^n$, $c_1 \in \mathbb{R}^n$, $b \in \mathbb{R}$, $Q_0 \in \mathbb{R}^{n \times n}$, $Q_1 \in \mathbb{R}^{n \times n}$

- Reformulate the problem so that the objective becomes a linear function
- Reformulate it further as an SOCP in standard form

Solution

- To make the objective linear, we add a variable that represents the original, quadratic objective

(→ epigraph form)

$$\underset{\substack{x \in \mathbb{R}^n \\ t \in \mathbb{R}}}{\text{minimize}} \quad t \qquad \text{s.t.} \quad c_0^T x + \frac{1}{2} x^T Q_0 x - t \leq 0$$

$$c_1^T x + \frac{1}{2} x^T Q_1 x \leq b$$

linear in $x$ and $t$

(other possibility: $\underset{\substack{x \in \mathbb{R}^n \\ t \in \mathbb{R}}}{\text{minimize}} \quad c_0^T x + t$

$$\text{s.t.} \quad \frac{1}{2} x^T Q_0 x - t \leq 0$$

$$c_1^T x + 1/2\, x^T Q_1 x \leq b \quad )$$

- Using 4.1., we obtain that the constraint

$$c_1^T x + \frac{1}{2} x^T Q_1 x \leq b$$

can be reformulated as the second-order cone constraint

$$\left\| \begin{bmatrix} Q_1^{1/2} x \\ -b - c_1^T x - 1/2 \end{bmatrix} \right\| \leq b - c_1^T x + 1/2$$

<u>Recall</u>: A second-order cone constraint in standard form

$$\| \bar{A} x + \bar{b} \| \leq \bar{c}^T x + \bar{d}$$

Here $\bar{A} = \begin{bmatrix} (Q_1^{1/2})^T \\ -c_1^T \end{bmatrix} \in \mathbb{R}^{(n+1) \times n}$ $\quad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b - 1/2 \end{bmatrix} \in \mathbb{R}^{n+1}$

$$\bar{c} = -c_1 \qquad \bar{d} = b + 1/2$$

→ This second-order cone constraint in $x$ does not depend on $t$, hence we also consider it as a second-order cone constraint in $(x, t)$.

Technically, one shall write

$$\left\| \begin{bmatrix} \bar{A} & 0_{\mathbb{R}^{n+1}} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \bar{b} \right\| \leq \begin{bmatrix} \bar{c} & 0 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} + \bar{d}$$

$\hookrightarrow$ The constraint

$$c_0^T x + \frac{1}{2} x^T Q_0 x - t \leq 0$$

can be rewritten as

$$c_2^T \begin{bmatrix} x \\ t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x \\ t \end{bmatrix}^T Q_2 \begin{bmatrix} x \\ t \end{bmatrix} \leq 0$$

with $c_2 = \begin{bmatrix} c_0 \\ -1 \end{bmatrix} \in \mathbb{R}^{n+1}$ and $Q_2 = \begin{array}{c|c} \overset{\longleftarrow n \longrightarrow}{} & \overset{\leftarrow 1 \rightarrow}{} \\ \end{array} \left[\begin{array}{c|c} Q_0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \end{array}\right] \begin{matrix} \updownarrow n \\ \updownarrow 1 \end{matrix}$

$(Q_2 = Q_2^T,\ Q_2 \succeq 0)$

Hence the constraint is equivalent to the SOC constraint

$$\left\| \begin{bmatrix} Q_2^{1/2} \begin{bmatrix} x \\ t \end{bmatrix} \\ -c_2^T \begin{bmatrix} x \\ t \end{bmatrix} - 1/2 \end{bmatrix} \right\| \leq -c_2^T \begin{bmatrix} x \\ t \end{bmatrix} + \frac{1}{2}$$

More generally, any QCQP with positive semidefinite matrices can be reformulated as an SOCP

$\longrightarrow$ When the matrices are not positive semidefinite, this is not always possible because those matrices do not possess a square root in general

Ex 5 QCQPs and SDPs

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c_0^T x + \frac{1}{2} x^T Q_0 x$$

$$\text{s.t.} \quad c_1^T x + \frac{1}{2} x^T Q_1 x \leq b$$

$c_0 \in \mathbb{R}^n$, $c_1 \in \mathbb{R}^n$, $Q_0 = Q_0^T \in \mathbb{R}^{n \times n}$, $Q_1 = Q_1^T \in \mathbb{R}^{n \times n}$
$b \in \mathbb{R}$

5.1 Introduce $Y = xx^T$ and show that the problem can be reformulated as

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c_0^T x + \frac{1}{2}(Q_0 \bullet Y)$$

$$\text{s.t.} \quad c_1^T x + \frac{1}{2}(Q_1 \bullet Y) \leq b$$

$$Y = xx^T$$

Solution: It suffices to justify that $Q_0 \bullet Y = x^T Q_0 x$

$$Q_0 \bullet Y = Q_0 \bullet (xx^T) = \text{trace}\big(\overset{n \times n}{Q_0} \, \overset{n \times n}{\underbrace{x}_{n \times 1} \underbrace{x^T}_{1 \times n}}\big) = \text{trace}\big(\overset{\in \mathbb{R}}{x^T Q_0 x}\big) = x^T Q_0 x$$

$\forall (A, B) \in \mathbb{R}^{m_1 \times m_2} \times \mathbb{R}^{m_2 \times m_1}$, $\text{trace}(AB) = \text{trace}(BA)$

Ex) $\text{trace}(\underbrace{xx^T}_{n \times n}) = \text{trace}(\underbrace{x^T x}_{1 \times 1}) = \|x\|^2 \quad \forall x \in \mathbb{R}^n$

$$xx^T = \begin{bmatrix} x_1^2 & & x_1 x_n \\ & x_2^2 & \\ & & \ddots \\ x_1 x_n & & x_n^2 \end{bmatrix} = [x_i x_j]_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

5.2. Suppose that we relax the constraint $Y = xx^T$ into

$$Y \succeq xx^T \iff Y - xx^T \succeq 0$$

$$\iff \begin{bmatrix} Y & x \\ x^T & z \end{bmatrix} \succeq 0 \quad \text{and} \quad z = 1$$

Show that the relaxed problem is an SDP in the variable

$$X = \begin{bmatrix} Y & x \\ x^T & z \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

Solution

We consider

$$\begin{aligned} \underset{\substack{x \in \mathbb{R}^n \\ Y \in \mathbb{S}^{n\times n}}}{\text{minimize}} \quad & c_0^T x + \frac{1}{2}(Q_0 \bullet Y) \\ \text{s.t.} \quad & c_1^T x + \frac{1}{2}(Q_1 \bullet Y) \le b \\ & Y \succeq xx^T \end{aligned}$$

Y now treated as a variable because it is no longer fixed by the constraints

Introducing the variable $z$, we obtain

$$\begin{cases} \underset{\substack{x \in \mathbb{R}^n \\ Y \in \mathbb{S}^{n\times n} \\ z \in \mathbb{R}}}{\text{minimize}} \quad & c_0^T x + \frac{1}{2}(Q_0 \bullet Y) \\ \text{s.t.} \quad & c_1^T x + 1/2(Q_1 \bullet Y) \le b \\ & z = 1 \\ & \begin{bmatrix} Y & x \\ x^T & z \end{bmatrix} \succeq 0 \end{cases}$$

This is an SDP in terms of $X = \begin{bmatrix} Y & x \\ x^T & z \end{bmatrix} \in \mathbb{S}^{(n+1)\times(n+1)}$, and we can rewrite it in standard form as

$$
\begin{cases}
\underset{X \in S^{(n+1)\times(n+1)}}{\text{minimize}} & C \bullet X \\
\text{s.t.} & A_1 \bullet X \leq b_1 \\
 & A_2 \bullet X = b_2 \\
 & X \succeq 0
\end{cases}
$$

$$C = \frac{1}{2}\begin{bmatrix} Q_0 & c_0 \\ c_0^T & 0 \end{bmatrix}$$

$$A_1 = \frac{1}{2}\begin{bmatrix} Q_1 & c_1 \\ c_1^T & 0 \end{bmatrix} \quad b_1 = b$$

$$A_2 = \begin{bmatrix} 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 1 \end{bmatrix} \quad b_2 = 1$$