Mathematics for Data Science *

Homework Assignment

M1 IDD, 2024-2025

Ðauphine | PSL
UNIVERSITÉ PARIS

Assignment

- This homework consists of presenting a mathematical tool for solving polynomial optimization problems. This is a difficult concept, and creativity is encouraged to present this in the friendliest way possible. In particular, the homework format is open (notebook, report, slides, etc).

- Students should organize as groups of n people with $n \in \{1, 2, 3\}$. In addition to the general assessment, each group will be given a particular instance to work on. To this end, each group should let themselves known to C. Royer. **Any student who does not contact C. Royer by December 1st will be given an individual assessment.**

- Students are expected to send their homework to `clement.royer@lamsade.dauphine.fr`. The email and homework must clearly indicate the students' first and last names.

- The deadline is set to **February 7, 2025 AOE** (Anywhere On Earth).

Major updates to the document

- 2024.12.08: Added instance for group 13.

- 2024.12.06: Added instances for groups 11-12.

- 2024.11.22: Added instances for groups 8-10.

- 2024.11.19: Clarified the main requirements of the project. Added instances for groups 1-7.

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*Version of December 6, 2024. The last version of this document can be found at:
https://www.lamsade.dauphine.fr/~croyer/ensdocs/MDS/DMMDS.pdf

General topic: Moment sum-of-squares (SOS) hierarchy

The sum-of-squares hierarchy is a mathematical tool to solve polynomial optimization problems even when those are not convex problems. To this end, one can build a hierarchy of semidefinite programs of increasing dimensions, that forms a sequence of tighter approximations to the original problem. Although this concept is quite technical, it has found a number of applications and has been the subject of numerous books and monographs [1].

> **Assignment:** *Explain the main principles behind this hierarchy at a level that best matches that of your fellow students.*

Your explanation should include (but is not limited to) the following items:

 (i) A formulation for the dual problem of a semidefinite program.

(ii) A definition of sum-of-squares polynomials.

(iii) A reformulation of a general polynomial optimization problem as a problem with infinitely many constraints.

(iv) A description of the moment-SOS hierarchy as relaxations of the problem from (iii) that involve SOS polynomials.

 (v) A formulation of the relaxations from (iv) as semidefinite programs.

Any concept that you deem new to other students should be introduced in your homework (possibly with references to freely available resources online).

Dedicated topic: Instance to be analyzed

> **Assignment:** *For the instance assigned to your group, provide the first two relaxations in the associated moment-SOS hierarchy, formulated as semidefinite programs.*

- Please send an email to `clement.royer@lamsade.dauphine.fr` with the names of your group members to receive an example of polynomial optimization problem. This example will then appear in the list below.

- Your homework should detail how the two relaxations were obtained.

References

[1] J.-B. Lasserre. The Moment-SOS hierarchy: Applications and related topics. *Acta Numer.*, 33:841–908, 2024.

List of dedicated instances

Group 1 (ACHOUR, FERNANDES MACEDO, PERIANAYAGASSAMY)

$$\underset{\boldsymbol{x}\in\mathbb{R}^4}{\text{minimize}}\ x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 4x_1^2 x_2^2 x_3^2 \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

Group 2 (ALLAL, ROUIBI, ZOUAD)

$$\underset{\boldsymbol{x}\in\mathbb{R}^4}{\text{minimize}}\ x_1^2 x_2^2 + 2x_1^2 x_3^2 + 3x_2^2 x_3^2 + x_4^4 - 5x_1 x_2 x_3 x_4 \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

Group 3 (JANIN, LEMAITRE, MATHIOT)

$$\underset{\boldsymbol{x}\in\mathbb{R}^4}{\text{minimize}}\ x_1^3 x_4 + x_2^3 x_4 + x_3^3 x_4 - (x_1^2 x_2 x_4 + x_1 x_2^2 x_4 + x_2^2 x_3 x_4 + x_2 x_3^2 x_4) \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

Group 4 (ERNADOTE, HANNACHI, LAHMADI)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1^3 + x_2^3 + x_3^3 + 6x_1 x_2 x_3 - 2(x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2) \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 = 1.$$

Group 5 (CHEN, GRAJEWSKA, YOU)

$$\underset{\boldsymbol{x}\in\mathbb{R}^4}{\text{minimize}}\ x_1 x_2 x_3 + x_2 x_3 x_4 - (x_1^3 + x_2^3 + x_3^3 + x_4^3) \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1.$$

Group 6 (ABDELOUHAB, AIT ELDJOUDI, FARZOLLAHI ZANJANI)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1^6 + x_2^6 + x_3^6 + 3x_1^2 x_2^2 x_3^2 - x_1^4(x_2^2 + x_3^2) - x_2^4(x_3^2 + x_1^2) - x_3^4(x_1^2 + x_2^2) \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 = 1.$$

Group 7 (DESCROIX, MICHEL, SHI)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1 x_2 x_3 - x_1 x_2 - x_2 x_3 + x_1^4 + x_2^3 + x_3^3 \quad \text{subject to} \quad x_1^2 \leq 1,\ x_2^2 \leq 1,\ x_3^2 \leq 1.$$

Group 8 (HOERTER, NASR, QUATREBOEUFS)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 3x_1^2 x_2^2 x_3^2 + x_2^2 \quad \text{subject to} \quad x_1 - x_2 x_3 \geq 0,\ -x_2 + x_3^2 \geq 0.$$

Group 9 (BERREBI, MEDUS)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 + x_1^4 + x_2^4 + x_3^4 \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 \geq 1.$$

Group 10 (ALDARWISH, MILED, RABIA)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1^3 + x_2^3 + x_3^3 + 4x_1 x_2 x_3 - x_1(x_2^2 + x_3^2) - x_2(x_1^2 + x_3^2) - x_3(x_1^2 + x_2^2) \quad \text{subject to} \quad x_1 \geq 0,\ x_1 x_2 \geq 1, x_2 x_3 \geq 1.$$

Group 11 (WU)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ x_1^4 x_2^2 + 3x_2^4 x_3^2 + 5x_3^4 x_1 - 7x_1^2 x_2^2 x_3^2 \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 = 1.$$

Group 12 (VO)

$$\underset{\boldsymbol{x}\in\mathbb{R}^2}{\text{minimize}}\ x_1^2 + 50x_2^2 \quad \text{subject to} \quad \begin{array}{rcl} x_1^2 - \frac{1}{2} & \geq & 0 \\ x_2^2 - 2x_1 x_2 - \frac{1}{8} & \geq & 0 \\ x_2^2 + 2x_1 x_2 - \frac{1}{8} & \geq & 0. \end{array}$$

Group 13 (SAIS)

$$\underset{\boldsymbol{x}\in\mathbb{R}^3}{\text{minimize}}\ (2x_1 + 3x_2 - 4x_3)x_1 x_2 + x_3^3 \quad \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 = 1.$$