Exercises on Chapter 1: Convexity

Mathematics of Data Science, M1 IDD

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Convex sets

Exercise 1.1: Partial sum

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two convex sets in $\mathbb{R}^{m+n}$. Show that the set

$$\mathcal{C} := \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 + \boldsymbol{y}_2 \end{bmatrix} \middle| \boldsymbol{x} \in \mathbb{R}^m, \boldsymbol{y}_1 \in \mathbb{R}^n, \boldsymbol{y}_2 \in \mathbb{R}^n, \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} \in \mathcal{C}_1, \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} \in \mathcal{C}_2 \right\}$$

is also convex.

Exercise 1.2: Characterizing convexity

Show that a set $\mathcal{C} \subseteq \mathbb{R}^n$ is convex if and only if $(\alpha + \beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$ for any $\alpha \geq 0$ and $\beta \geq 0$.

Exercise 1.3: Normal cone

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a arbitrary set (not necessarily convex) and let $\boldsymbol{x}_0 \in \mathcal{X}$. The *normal cone* of $\mathcal{X}$ at $\boldsymbol{x}_0$ is defined by

$$\mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0) := \left\{ \boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0 \ \forall \boldsymbol{x} \in \mathcal{X} \right\}.$$

Show that $\mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0)$ is a convex cone.

Exercise 1.4: Convex and affine hulls

Show that the following properties hold for any set $\mathcal{X} \in \mathbb{R}^n$:

a) $\mathrm{conv}(\mathcal{X}) \subseteq \mathrm{aff}(\mathcal{X})$;

b) $\mathrm{aff}(\mathcal{X}) = \mathrm{aff}(\mathrm{conv}(\mathcal{X}))$.

c) $\mathrm{aff}(\mathcal{X})$ is a closed set and $\mathrm{aff}(\mathcal{X}) = \mathrm{aff}(\mathrm{cl}(\mathcal{X}))$

Exercise 1.5: Retraction set

Let $\mathcal{C} \in \mathbb{R}^n$ be a convex set and $r \geq 0$. The purpose of this exercise is to establish that the retraction set

$$\mathcal{C}_{-r} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathcal{B}_r(\boldsymbol{x}) \subseteq \mathcal{C}\}$$

is convex.

a) Given any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}_{-r}$ and $t \in [0, 1]$, show that $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathcal{C}$.

b) Consider now $\boldsymbol{z} \in \mathcal{B}_r(t\boldsymbol{x} + (1-t)\boldsymbol{y})$, and write

$$\boldsymbol{z} = \boldsymbol{w} + t\boldsymbol{x} + (1-t)\boldsymbol{y}.$$

Show that $\boldsymbol{w} + \boldsymbol{x} \in \mathcal{B}_r(\boldsymbol{x})$ and $\boldsymbol{w} + \boldsymbol{y} \in \mathcal{B}_r(\boldsymbol{y})$.

c) Show finally that $\boldsymbol{z} \in \mathcal{C}$, and explain how it proves the desired result.

Convex functions

Exercise 1.6: Convexity and upper bounds

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Suppose that there exists $M \in \mathbb{R}$ such that $f(\boldsymbol{x}) \leq M$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Show then that the gradient of f is zero at every point, which implies that the function f is constant.
Hint: For any linear function $g(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, g is bounded if and only if $\boldsymbol{A} = \boldsymbol{0}$.

Exercise 1.7: Convexity and extended values

Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a convex function. Show that

a) $\mathrm{dom}(f)$ is a convex set;

b) the sublevel sets of f are convex sets.

Exercise 1.8: Inequalities and convexity

a) Let $p > 1$. We define the ℓ_p norm by

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \quad \|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Show that the function $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|_p^p$ is convex.

b) Using the convexity of $x \mapsto -\ln(x)$, show that

$$a^{\theta} b^{1-\theta} \leq \theta a + (1-\theta) b$$

for any $a, b \geq 0$ and any $\theta \in [0,1]$.

c) Let $p > 1$ and $q > 1$ be two values such that $\frac{1}{p} + \frac{1}{q} = 1$. Use the previous question to construct n inequalities that can be combined to show the so-called Hölder inequality:

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^n)^2, \qquad \sum_{i=1}^{n} x_i y_i \leq \|\boldsymbol{x}\|_p \|\boldsymbol{y}\|_q.$$

Hint: Show that the result holds when $\sum_{i=1}^{n} x_i y_i$ is replaced by $\sum_{i=1}^{n} |x_i||y_i|$, then conclude.

Exercise 1.9: Log-concave functions

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *log-concave* (short for logarithmically concave) if $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathrm{dom}(f)$ and $\boldsymbol{x} \mapsto \ln(f(\boldsymbol{x}))$ is concave (where we let $\ln(f(\boldsymbol{x})) = -\infty$ when $f(\boldsymbol{x}) = 0$). The concept of *log-convex* functions is defined in a similar way.

a) Given $a \in \mathbb{R}$, show that the function $x \mapsto x^a$ is log-concave on $\mathbb{R}_{++}$ when $a \geq 0$ and log-convex on $\mathbb{R}_{++}$ when $a \leq 0$.

b) We will now show that the determinant function is log-concave on $\mathcal{S}^n_{++}$. For any $\boldsymbol{X} \in \mathcal{S}^n_{++}$, the determinant of $\boldsymbol{X}$ has the expression $\det(\boldsymbol{X}) = \prod_{i=1}^{n} \lambda_i^{\boldsymbol{X}}$, where $\lambda_1^{\boldsymbol{X}}, \ldots, \lambda_n^{\boldsymbol{X}}$ are positive real numbers corresponding to the eigenvalues of $\boldsymbol{X}$.

(i) Show that the set $\mathcal{S}^n_{++}$ is convex.

(ii) For any $\boldsymbol{X}$,$\boldsymbol{Y}$ in $\mathcal{S}^n_{++}$, the matrix $\boldsymbol{Z} = \boldsymbol{Y}\boldsymbol{X}^{-1}$ belongs to $\mathcal{S}^n_{++}$. Using this property, show that

$$\ln(\det(\alpha \boldsymbol{X} + (1-\alpha)\boldsymbol{Y})) \;=\; \sum_{i=1}^{n} \ln(\lambda_i^{\boldsymbol{X}}) + \sum_{i=1}^{n} \ln(\alpha + (1-\alpha)\lambda_i^{\boldsymbol{Z}}) \qquad (1)$$

for any $\alpha \in [0,1]$.

(iii) With the same notations than in the previous question, show that

$$\alpha \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln(\det(\boldsymbol{Y})) \;=\; \sum_{i=1}^{n} \ln(\lambda_i^{\boldsymbol{X}}) + (1-\alpha)\sum_{i=1}^{n} \ln(\lambda_i^{\boldsymbol{Z}}). \qquad (2)$$

(iv) Conclude that $\ln\det$ is concave on $\mathcal{S}^n_{++}$.