Exercises on Chapter 1: Convexity

Mathematics of Data Science, M1 IDD

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Exercise 1.1: Partial sum

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two convex sets in $\mathbb{R}^{m+n}$. Show that the set

$$\mathcal{C} := \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 + \boldsymbol{y}_2 \end{bmatrix} \middle| \boldsymbol{x} \in \mathbb{R}^m, \boldsymbol{y}_1 \in \mathbb{R}^n, \boldsymbol{y}_2 \in \mathbb{R}^n, \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} \in \mathcal{C}_1, \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} \in \mathcal{C}_2 \right\}$$

is also convex.

Exercise 1.2: Characterizing convexity

Show that a set $\mathcal{C} \subseteq \mathbb{R}^n$ is convex if and only if $(\alpha + \beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$ for any $\alpha \geq 0$ and $\beta \geq 0$.

Exercise 1.3: Normal cone

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a arbitrary set (not necessarily convex) and let $\boldsymbol{x}_0 \in \mathcal{X}$. The *normal cone* of $\mathcal{X}$ at $\boldsymbol{x}_0$ is defined by

$$\mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0) := \left\{ \boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0 \ \forall \boldsymbol{x} \in \mathcal{X} \right\}.$$

Show that $\mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0)$ is a convex cone.

Exercise 1.4: Retraction set

Let $\mathcal{C} \in \mathbb{R}^n$ be a convex set and $r \geq 0$. The purpose of this exercise is to establish that the retraction set

$$\mathcal{C}_{-r} := \{ \boldsymbol{x} \in \mathbb{R}^n \mid \mathcal{B}_r(\boldsymbol{x}) \subseteq \mathcal{C} \}$$

is convex.

a) Given any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}_{-r}$ and $t \in [0, 1]$, show that $t\boldsymbol{x} + (1 - t)\boldsymbol{y} \in \mathcal{C}$.

b) Consider now $\boldsymbol{z} \in \mathcal{B}_r(t\boldsymbol{x} + (1-t)\boldsymbol{y})$, and write

$$\boldsymbol{z} = \boldsymbol{w} + t\boldsymbol{x} + (1-t)\boldsymbol{y}.$$

Show that $\boldsymbol{w} + \boldsymbol{x} \in \mathcal{B}_r(\boldsymbol{x})$ and $\boldsymbol{w} + \boldsymbol{y} \in \mathcal{B}_r(\boldsymbol{y})$.

c) Show finally that $\boldsymbol{z} \in \mathcal{C}$, and explain how it proves the desired result.

Exercise 1.5: Convex and affine hulls

Show that the following properties hold for any set $\mathcal{X} \in \mathbb{R}^n$:

a) $\mathrm{conv}(\mathcal{X}) \subseteq \mathrm{aff}(\mathcal{X})$;

b) $\mathrm{aff}(\mathcal{X}) = \mathrm{aff}(\mathrm{conv}(\mathcal{X}))$;

c) $\mathrm{aff}(\mathcal{X})$ is a closed set and $\mathrm{aff}(\mathcal{X}) = \mathrm{aff}(\mathrm{cl}(\mathcal{X}))$ (*Indication: Use the second definition of affine envelope given in class*).

Exercise 1.6: Convexity and upper bounds

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Suppose that there exists $M \in \mathbb{R}$ such that $f(\boldsymbol{x}) \leq M$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Show then that the gradient of f is zero at every point, which implies that the function f is constant.
Hint: For any linear function $g(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, g is bounded if and only if $\boldsymbol{A} = \boldsymbol{0}$.

Exercise 1.7: Convexity and extended values

Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a convex function. Show that

a) $\mathrm{dom}(f)$ is a convex set;

b) the sublevel sets of f are convex sets.

Exercise 1.8: Inequalities and convexity

a) Let $p > 1$. We define the ℓ_p norm by

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \quad \|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Show that the function $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|_p^p$ is convex.

b) Using the convexity of $x \mapsto -\ln(x)$, show that

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$$

for any $a, b \geq 0$ and any $\theta \in [0, 1]$.

c) Use the previous question to construct n inequalities that can be combined to show the so-called Hölder inequality:

$$\forall p > 1, \forall q > 1, \tfrac{1}{p} + \tfrac{1}{q} = 1,\ \forall (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^n)^2, \qquad \sum_{i=1}^{n} x_i y_i \leq \|\boldsymbol{x}\|_p \|\boldsymbol{y}\|_q.$$

Hint: Show that the result holds when $\sum_{i=1}^n x_i y_i$ *is replaced by* $\sum_{i=1}^n |x_i||y_i|$*, then conclude.*

Exercise 1.9: Log-concave and log-convex functions

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *log-concave* (short for logarithmically concave) if $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathrm{dom}(f)$ and $\boldsymbol{x} \mapsto \ln(f(\boldsymbol{x}))$ is concave (where we let $\ln(f(\boldsymbol{x})) = -\infty$ when $f(\boldsymbol{x}) = 0$). The concept of *log-convex* functions is defined in a similar way.

a) Given $a \in \mathbb{R}$, show that the function $x \mapsto x^a$ is log-concave on $\mathbb{R}_{++}$ when $a \geq 0$ and log-convex on $\mathbb{R}_{++}$ when $a \leq 0$.

b) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two log-concave functions. Show that the product function $f\,g$ is log-concave.

c) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two log-convex functions. Show that the sum $f + g$ is log-convex.
Hint: Consider the function $\boldsymbol{x} \mapsto (f(\boldsymbol{x}), g(\boldsymbol{x}))$ *and the function* $h : \mathbb{R}^2 \mapsto \mathbb{R}$ *defined by* $h(a, b) = \log(\exp(a) + \exp(b))$*, which is nondecreasing and convex.*

d) Show that the set $\mathcal{S}^n_{++}$ is convex, and that the function $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ is log-concave on $\mathcal{S}^n_{++}$.
Hint: Use the following properties of $\mathcal{S}^n_{++}$*:*

- *For any* $\boldsymbol{X}, \boldsymbol{Y}$ *in* $\mathcal{S}^n_{++}$*, the matrix* $\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{Y}^{-1}$ *belongs to* $\mathcal{S}^n_{++}$*.*
- *For any* $\boldsymbol{X} \in \mathcal{S}^n_{++}$*, the determinant of* $\boldsymbol{X}$ *has the expression* $\det(\boldsymbol{X}) = \prod_{i=1}^n \lambda_i^{\boldsymbol{X}}$*, where* $\lambda_1^{\boldsymbol{X}}, \dots, \lambda_n^{\boldsymbol{X}}$ *are positive real numbers (corresponding to the eigenvalues of* $\boldsymbol{X}$*.*

Solutions

Solution for Exercise 1.1: Partial sum

In order to show that $\mathcal{C}$ is a convex set, it suffices to show that any line segment formed by two points in $\mathcal{C}$ lies in $\mathcal{C}$.

Let $(\boldsymbol{z}, \bar{\boldsymbol{z}}) \in \mathcal{C}^2$. By definition of these sets, there exist $(\boldsymbol{x}, \bar{\boldsymbol{x}}) \in (\mathbb{R}^m)^2$ and $(\boldsymbol{y}_1, \boldsymbol{y}_2, \bar{\boldsymbol{y}}_1, \bar{\boldsymbol{y}}_2) \in (\mathbb{R}^n)^4$ such that

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 + \boldsymbol{y}_2 \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} \in \mathcal{C}_1, \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} \in \mathcal{C}_2,$$

and

$$\bar{\boldsymbol{z}} = \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2 \end{bmatrix}, \qquad \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{y}}_1 \end{bmatrix} \in \mathcal{C}_1, \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{y}}_2 \end{bmatrix} \in \mathcal{C}_2.$$

Let now $\alpha \in [0, 1]$, and consider the vector $\boldsymbol{z}^* = \alpha \boldsymbol{z} + (1 - \alpha)\bar{\boldsymbol{z}}$. Then,

$$\begin{aligned}
\boldsymbol{z}^* &= \alpha \boldsymbol{z} + (1-\alpha)\bar{\boldsymbol{z}} \\
&= \alpha \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 + \boldsymbol{y}_2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha \boldsymbol{x} + (1-\alpha)\bar{\boldsymbol{x}} \\ \alpha \boldsymbol{y}_1 + (1-\alpha)\bar{\boldsymbol{y}}_1 + \alpha \boldsymbol{y}_2 + (1-\alpha)\bar{\boldsymbol{y}}_2 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}_1^* + \boldsymbol{y}_2^* \end{bmatrix},
\end{aligned}$$

where we define $\boldsymbol{x}^* = \alpha \boldsymbol{x} + (1-\alpha)\bar{\boldsymbol{x}} \in \mathbb{R}^m$, $\boldsymbol{y}_1^* = \alpha \boldsymbol{y}_1 + (1-\alpha)\bar{\boldsymbol{y}}_1 \in \mathbb{R}^n$ and $\boldsymbol{y}_2^* = \alpha \boldsymbol{y}_2 + (1-\alpha)\bar{\boldsymbol{y}}_2 \in \mathbb{R}^n$.

With these notations, we have that

$$\begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}_1^* \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{y}}_1 \end{bmatrix},$$

hence this vector belongs to $\mathcal{C}_1$ as a convex combination of two elements of $\mathcal{C}_1$.

Similarly, the vector $\begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}_2^* \end{bmatrix}$ belongs to $\mathcal{C}_2$ as a convex combination of $\mathcal{C}_2$. Overall, we have shown that

$$\boldsymbol{z}^* = \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}_1^* + \boldsymbol{y}_2^* \end{bmatrix}, \qquad \boldsymbol{x}^* \in \mathbb{R}^n, (\boldsymbol{y}_1^*, \boldsymbol{y}_2^*) \in (\mathbb{R}^m)^2, \qquad \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}_1^* \end{bmatrix} \in \mathcal{C}_1, \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}_2^* \end{bmatrix} \in \mathcal{C}_2.$$

hence this vector belongs to $\mathcal{C}$, showing that the set is convex.

Solution for Exercice 1.2: Characterizing convexity

We first show $\Leftarrow$*, i.e. that a set* $\mathcal{C}$ *satisfying* $(\alpha+\beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$ *is convex. We then show the other direction.*

$\boxed{\Leftarrow}$ For any nonnegative values α and β, one has $(\alpha+\beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$. Thus, for any $\alpha \in [0,1]$, letting $\beta = 1-\alpha$ leads to

$$\mathcal{C} = (\alpha + (1-\alpha))\mathcal{C} = \alpha\mathcal{C} + (1-\alpha)\mathcal{C}.$$

As a result, for any pair $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{C}^2$,

$$\underbrace{\alpha\boldsymbol{x}}_{\in\alpha\mathcal{C}} + \underbrace{(1-\alpha)\boldsymbol{y}}_{\in(1-\alpha)\mathcal{C}} \in \mathcal{C}.$$

Since this property holds for any $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{C}^2$ and any $\alpha \in [0,1]$, we have shown that $\mathcal{C}$ is a convex set.

$\boxed{\Rightarrow}$ When $\alpha = \beta = 0$ (and independently of the convexity of $\mathcal{C}$), we clearly have $(\alpha+\beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$. Suppose now that α and β are nonnegative real values with $\alpha + \beta > 0$ (i.e. one of these values is nonzero). We will show that

$$\mathcal{C} = \frac{\alpha}{\alpha+\beta}\mathcal{C} + \frac{\beta}{\alpha+\beta}\mathcal{C}, \tag{1}$$

which is equivalent to showing $(\alpha+\beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$. To prove (1), we establish that each set contains the other.

- $\boxed{\mathcal{C} \subseteq \frac{\alpha}{\alpha+\beta}\mathcal{C} + \frac{\beta}{\alpha+\beta}\mathcal{C}}$: Every $\boldsymbol{x} \in \mathcal{C}$ admits the decomposition

  $$\boldsymbol{x} = \frac{\alpha}{\alpha+\beta}\boldsymbol{x} + \frac{\beta}{\alpha+\beta}\boldsymbol{x},$$

  where $\frac{\alpha}{\alpha+\beta}\boldsymbol{x} \in \frac{\alpha}{\alpha+\beta}\mathcal{C}$ and $\frac{\beta}{\alpha+\beta}\boldsymbol{x} \in \frac{\beta}{\alpha+\beta}\mathcal{C}$. Therefore, $\boldsymbol{x} \in \frac{\alpha}{\alpha+\beta}\mathcal{C} + \frac{\beta}{\alpha+\beta}\mathcal{C}$.

- $\boxed{\frac{\alpha}{\alpha+\beta}\mathcal{C} + \frac{\beta}{\alpha+\beta}\mathcal{C} \subseteq \mathcal{C}}$ Let $\boldsymbol{x} \in \frac{\alpha}{\alpha+\beta}\mathcal{C} + \frac{\beta}{\alpha+\beta}\mathcal{C}$. Then $\boldsymbol{x} = \boldsymbol{x}^\alpha + \boldsymbol{x}^\beta$, where $\boldsymbol{x}^\alpha \in \frac{\alpha}{\alpha+\beta}\mathcal{C}$ and $\boldsymbol{x}^\beta \in \frac{\beta}{\alpha+\beta}\mathcal{C}$. Moreover, there exists $\boldsymbol{y} \in \mathcal{C}$ such that $\boldsymbol{x}^\alpha = \frac{\alpha}{\alpha+\beta}\boldsymbol{y}$ and $\boldsymbol{z} \in \mathcal{C}$ such that $\boldsymbol{x}^\beta = \frac{\beta}{\alpha+\beta}\boldsymbol{z}$. Thus,

  $$\boldsymbol{x} = \frac{\alpha}{\alpha+\beta}\boldsymbol{y} + \frac{\beta}{\alpha+\beta}\boldsymbol{z},$$

  showing that the vector $\boldsymbol{x}$ can be written as a convex combination of two elements of $\mathcal{C}$. Since $\mathcal{C}$ is a convex set, we conclude that $\boldsymbol{x} \in \mathcal{C}$.

Overall, we have shown that $(\alpha+\beta)\mathcal{C} = \alpha\mathcal{C} + \beta\mathcal{C}$ for any nonnegative real values α and β, which proves the implication.

Solution for Exercice 1.3: Normal cone

We begin by showing that the normal cone of $\mathcal{X}$ at $\boldsymbol{x}_0$ is a cone using the definition. We then show that this set is convex by the standard technique (it includes every line segment passing through two of its points).

We first show that $\mathcal{N}_{\mathcal{X}}(x_0)$ is a cone using the definition. Let $\boldsymbol{y} \in \mathcal{N}_{\mathcal{X}}(x_0)$ and $t > 0$. Since $\boldsymbol{y}$ belongs to $\mathcal{N}_{\mathcal{X}}(x_0)$, it satisfies

$$\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0 \quad \forall \boldsymbol{x} \in \mathcal{X}.$$

Since $t > 0$, we also have

$$(t\boldsymbol{y})^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) = t\left(\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0)\right) \leq 0 \quad \forall \boldsymbol{x} \in \mathcal{X}.$$

As a result, $t\boldsymbol{y} \in \mathcal{N}_{\mathcal{X}}(x_0)$, showing that this set is a cone.

We now show that this cone is convex. To this end, let $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0)^2$ and $\alpha \in [0,1]$. For any $\boldsymbol{x} \in \mathcal{X}$, we have

$$\begin{aligned}
(\alpha\boldsymbol{y}_1 + (1-\alpha)\boldsymbol{y}_2)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) &= \alpha\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) + (1-\alpha)\boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \\
&\leq (1-\alpha)\boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \\
&\leq 0,
\end{aligned}$$

where we successively used the linearity of the inner product, the fact that $\alpha \geq 0$ and $\boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0$, and finally the fact that $(1-\alpha) \geq 0$ and $\boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0$. Therefore, the vector $\alpha\boldsymbol{y}_1 + (1-\alpha)\boldsymbol{y}_2$ satisfies

$$(\alpha\boldsymbol{y}_1 + (1-\alpha)\boldsymbol{y}_2)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \quad \forall \boldsymbol{x} \in \mathcal{X},$$

hence it belongs to $\mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0)$. This proves the convexity of $\mathcal{N}_{\mathcal{X}}(\boldsymbol{x}_0)$.

Solution for Exercice 1.4: Retraction set

a) Since $\mathcal{C}_{-r} \subseteq \mathcal{C}$, we have $\boldsymbol{x} \in \mathcal{C}$ and $\boldsymbol{y} \in \mathcal{C}$. Using the convexity of $\mathcal{C}$, we immediately conclude that $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathcal{C}$.

b) By definition of $\boldsymbol{w}$, we have

$$\|\boldsymbol{w}\| = \|\boldsymbol{z} - t\boldsymbol{x} - (1-t)\boldsymbol{y}\| \leq r,$$

where the inequality follows from $\boldsymbol{z} \in \mathcal{B}_r(t\boldsymbol{x} + (1-t)\boldsymbol{y})$. Moreover,

$$\|\boldsymbol{w} + \boldsymbol{x} - \boldsymbol{x}\| = \|\boldsymbol{w}\| \leq r \quad \text{and} \quad \|\boldsymbol{w} + \boldsymbol{y} - \boldsymbol{y}\| = \|\boldsymbol{w}\| \leq r.$$

It results from these two inequalities that $\boldsymbol{w} + \boldsymbol{x} \in \mathcal{B}_r(\boldsymbol{x})$ and $\boldsymbol{w} + \boldsymbol{y} \in \mathcal{B}_r(\boldsymbol{y})$.

c) Recalling the result of the previous question, we have that $\boldsymbol{w} + \boldsymbol{x} \in \mathcal{B}_r(\boldsymbol{x}) \subseteq \mathcal{C}$ and $\boldsymbol{w} + \boldsymbol{y} \in \mathcal{B}_r(\boldsymbol{y}) \subseteq \mathcal{C}$. By convexity of $\mathcal{C}$, we have

$$t(\boldsymbol{w} + \boldsymbol{x}) + (1-t)(\boldsymbol{w} + \boldsymbol{y}) \in \mathcal{C}.$$

Since $t(\boldsymbol{w} + \boldsymbol{x}) + (1-t)(\boldsymbol{w} + \boldsymbol{y}) = \boldsymbol{w} + t\boldsymbol{x} + (1-t)\boldsymbol{y} = z$, we have show that $\boldsymbol{z} \in \mathcal{C}$. Overall, the previous questions have established that for any $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{C}_{-r}^2$ and any $t \in [0,1]$, we have $\mathcal{B}_r(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \subseteq \mathcal{C}$ and thus $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathcal{C}_{-r}$. We conclude that $\mathcal{C}_{-r}$ is convex.

Solution for Exercice 1.5: Convex and affine hulls

a) Any vector $\boldsymbol{x} \in \mathrm{conv}(\mathcal{X})$ can be written as a convex combination of elements of $\mathcal{X}$. A convex combination is a special case of an affine combination, hence $\boldsymbol{x}$ can be written as an affine combination of elements in $\mathcal{X}$ and $\boldsymbol{x} \in \mathrm{aff}(\mathcal{X})$, from which we conclude that $\mathrm{conv}(\mathcal{X}) \subseteq \mathrm{aff}(\mathcal{X})$.

b) $\boxed{\mathrm{aff}(\mathcal{X}) \subseteq \mathrm{aff}(\mathrm{conv}(\mathcal{X}))}$ By definition of the convex hull, $\mathcal{X} \subseteq \mathrm{conv}(\mathcal{X})$, from which we directly obtain $\mathrm{aff}(\mathcal{X}) \subset \mathrm{aff}(\mathrm{conv}(\mathcal{X}))$ (any affine combination of elements in $\mathcal{X}$ is also an affine combination of elements in $\mathrm{conv}(\mathcal{X})$).
$\boxed{\mathrm{aff}(\mathrm{conv}(\mathcal{X})) \subset \mathrm{aff}(\mathcal{X})}$ Any vector $\boldsymbol{x} \in \mathrm{aff}(\mathrm{conv}(\mathcal{X}))$ can be written as

$$\boldsymbol{x} = \sum_i \alpha_i \boldsymbol{y}_i,$$

where $\sum_i \alpha_i = 1$, $\alpha_i \in \mathbb{R}$, and $\boldsymbol{y}_i \in \mathrm{conv}(\mathcal{X})$ for any i (the index set need not be finite). Besides, any $\boldsymbol{y}_i \in \mathrm{conv}(\mathcal{X})$ can be written as

$$\boldsymbol{y}_i = \sum_j \beta_{ij} \boldsymbol{z}_{ij},$$

where $\sum_j \beta_{ij} = 1$, $\beta_{ij} \geq 0$, and $\boldsymbol{z}_{ij} \in \mathcal{X}$ for any j. Combining both decompositions, the vector $\boldsymbol{x}$ can be decomposed as

$$\boldsymbol{x} = \sum_{i,j} \alpha_i \beta_{ij} \boldsymbol{z}_{ij},$$

where

$$\sum_{i,j} \alpha_i \beta_{ij} = \sum_i \alpha_i \sum_j \beta_{ij} = \sum_i \alpha_i = 1.$$

We have thus established that $\boldsymbol{x}$ can be written as an affine combination of elements in $\boldsymbol{x}$, hence $\boldsymbol{x} \in \mathrm{aff}(\mathcal{X})$, i.e. that $\mathrm{aff}(\mathrm{conv}(\mathcal{X})) \subseteq \mathrm{aff}(\mathcal{X})$.
Combining both results gives

$$\mathrm{aff}(\mathrm{conv}(\mathcal{X})) = \mathrm{aff}(\mathcal{X}).$$

c) The affine hull of $\mathcal{X}$ can be defined as the intersection of all affine sets including $\mathcal{X}$. Meanwhile, an affine set is closed, and the intersection of closed sets is a closed set. As a result, the affine set of $\mathcal{X}$ is closed.
$\boxed{\mathrm{aff}(\mathcal{X}) \subseteq \mathrm{aff}(\mathrm{cl}(\mathcal{X}))}$ Since $\mathcal{X} \subseteq \mathrm{cl}(\mathcal{X})$, we have $\mathrm{aff}(\mathcal{X}) \subset \mathrm{aff}(\mathrm{cl}(\mathcal{X}))$.
$\boxed{\mathrm{aff}(\mathrm{cl}(\mathcal{X})) \subseteq \mathrm{aff}(\mathcal{X})}$ Since $\mathrm{aff}(\mathcal{X})$ is closed, it is a closed set containing $\mathcal{X}$. It also contains the smallest closed set containing $\mathcal{X}$, i.e. $\mathrm{cl}(\mathcal{X})$. Therefore, we have $\mathrm{cl}(\mathcal{X}) \subseteq \mathrm{aff}(\mathcal{X})$, and thus

$$\mathrm{aff}(\mathrm{cl}(\mathcal{X}) \subseteq \mathrm{aff}(\mathrm{aff}(\mathcal{X})) = \mathrm{aff}(\mathcal{X}).$$

Putting all results together, we obtain $\mathrm{aff}(\mathrm{cl}(\mathcal{X})) = \mathrm{aff}(\mathcal{X})$.

Solution for Exercice 1.6: Convexity and upper bounds

The function f is differentiable and convex. We then now that (see Theorem 1.7 in the lecture notes)

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^n)^2, \quad f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}).$$

Since f is assumed to be bounded, there exists $M \in \mathbb{R}$ such that $f(\boldsymbol{y}) \leq M$ for any $\boldsymbol{y} \in \mathbb{R}^n$. Combining this property with the above inequality leads to

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^n)^2, f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}) \leq M.$$

The left-hand side of this inequality is a linear function of $\boldsymbol{y}$, that can only be bounded if it is constant, i.e. if $\nabla f(\boldsymbol{x}) = \mathbf{0}$. We thus conclude that $\nabla f(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x} \in \mathbb{R}^n$, implying that the function is constant.

Solution for Exercice 1.7: Convexity and extended value functions

a) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two points in $\mathrm{dom}(f) \subset \mathcal{X}$, and let $\alpha \in [0,1]$. We aim at showing that the point $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$ belongs to $\mathrm{dom}(f)$. Since f is convex, we can use the inequality that characterizes convexity for any pair of points in $\mathrm{dom}(f)$ (this inequality is not ambiguous). Indeed, since both $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to $\mathrm{dom}(f) \subset \mathcal{X}$, we know that $f(\boldsymbol{x}) < \infty$ and $f(\boldsymbol{y}) < \infty$, hence

$$f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) < \infty.$$

As a result, we have $f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) < \infty$, and thus $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \mathrm{dom}(f)$, from which we conclude that $\mathrm{dom}(f)$ is a convex set.
Remark: This result applies even when f takes the value $-\infty$.

b) We proceed similarly to the previous question. Let $\alpha \in \mathbb{R}$, and consider two points $\boldsymbol{x}, \boldsymbol{y}$ in $\mathcal{S}_\alpha$, where we recall that

$$\mathcal{S}_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \leq \alpha\}.$$

By definition of the sublevel sets, $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to the domain of f. For any $\beta \in [0,1]$, convexity of f implies that

$$f(\beta\boldsymbol{x} + (1-\beta)\boldsymbol{y}) \leq \beta f(\boldsymbol{x}) + (1-\beta)\boldsymbol{y} \leq \beta\alpha + (1-\beta)\alpha = \alpha,$$

where the second inequality follows from the definition of a sublevel set. As a result, the point $\beta\boldsymbol{x} + (1-\beta)\boldsymbol{y}$ belongs to $\mathcal{S}_\alpha$, proving that this set is indeed convex.

Solution for Exercice 1.8: Inequalities and convexity

a) For any $i = 1, \ldots, n$, the function $x_i \mapsto |x_i|^p$ is convex as a composition of the absolute value function (convex in $\mathbb{R}$) with the pth power function, which is convex and nondecreasing on $\mathbb{R}^+ +$. Consequently, the function $\boldsymbol{x} \mapsto \|\mathbf{x}\|_p^p$ is a conic combination of convex functions, thus it is a convex function.
N.B. An alternate proof uses the fact that the ℓ_p function is a norm for $p > 1$, and thus it is convex. Composing with the pth power function then gives the desired result.

b) Using that $x \mapsto -\ln x$ is convex on $\mathbb{R}_{++}$, we have

$$\forall a, b \in \mathbb{R}_{++},\ \forall \theta \in [0,1], -\ln(\theta a + (1-\theta)b) \leq -\theta \ln(a) - (1-\theta)\ln(b)$$

Using properties of the logarithm, we can rewrite this inequality as

$$\forall a, b \in \mathbb{R}_{++},\ \forall \theta \in [0,1], \ln\left(a^{\theta} b^{(1-\theta)}\right) \leq \ln(\theta a + (1-\theta)b)\,.$$

Taking the exponential on both sides of the inequality (and noting that exponential is monotonically increasing on $\mathbb{R}_{++}$, we obtain

$$\forall a, b \in \mathbb{R}_{++},\ \forall \theta \in [0,1], a^{\theta} b^{(1-\theta)} \leq \theta a + (1-\theta)b,$$

which is the desired result.

c) Let $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. For any $i = 1, \ldots, n$, applying the result from question b) to

$$a = \left(\frac{|x_i|}{\|\boldsymbol{x}\|_p}\right)^p, \quad b = \left(\frac{|y_i|}{\|\boldsymbol{y}\|_q}\right)^q, \quad \theta = \frac{1}{p} = 1 - \tfrac{1}{q}$$

gives

$$\begin{aligned} a^{\theta} b^{1-\theta} &= \frac{|x_i|}{\|\boldsymbol{x}\|_p} \frac{|y_i|}{\|\boldsymbol{y}\|_q} \\ &\leq \theta a + (1-\theta) b \\ &= \frac{1}{p}\left(\frac{|x_i|}{\|\boldsymbol{x}\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|\boldsymbol{y}\|_q}\right)^q. \end{aligned}$$

Overall, we have

$$\forall i = 1, \ldots, n, \qquad \frac{|x_i|}{\|\boldsymbol{x}\|_p} \frac{|y_i|}{\|\boldsymbol{y}\|_q} \leq \frac{1}{p}\left(\frac{|x_i|}{\|\boldsymbol{x}\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|\boldsymbol{y}\|_q}\right)^q.$$

Summing these n inequalities then gives

$$\begin{aligned} \sum_{i=1}^{n} \frac{|x_i|}{\|\boldsymbol{x}\|_p} \frac{|y_i|}{\|\boldsymbol{y}\|_q} &\leq \sum_{i=1}^{n} \left(\frac{1}{p}\left(\frac{|x_i|}{\|\boldsymbol{x}\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|\boldsymbol{y}\|_q}\right)^q\right) \\ \frac{1}{\|\boldsymbol{x}\|_p \|\boldsymbol{y}\|_q} \sum_{i=1}^{n} |x_i||y_i| &\leq \frac{1}{p} \sum_{i=1}^{n} \frac{x_i^p}{\|\boldsymbol{x}\|_p^p} + \frac{1}{q} \sum_{i=1}^{n} \frac{y_i^q}{\|\boldsymbol{y}\|_q^q}. \end{aligned} \tag{2}$$

Now, by definition of the ℓ_p and ℓ_q norms, the right-hand side of (2) simplifies as follows:

$$\begin{aligned} \frac{1}{p} \sum_{i=1}^{n} \frac{x_i^p}{\|\boldsymbol{x}\|_p^p} + \frac{1}{q} \sum_{i=1}^{n} \frac{y_i^q}{\|\boldsymbol{y}\|_q^q} &= \frac{1}{p} \frac{\sum_{i=1}^{n} x_i^p}{\|vx\|_p^p} + \frac{1}{q} \frac{\sum_{i=1}^{n} y_i^q}{\|vy\|_q^q} \\ &= \frac{1}{p} \frac{\|\boldsymbol{x}\|_p^p}{\|\boldsymbol{x}\|_p^p} + \frac{1}{q} \frac{\|\boldsymbol{y}\|_q^q}{\|\boldsymbol{y}\|_q^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

where we recall that the last equality is by definition of p and q. Plugging this result in (2) leads to

$$\frac{1}{\|\boldsymbol{x}\|_p\|\boldsymbol{y}\|_q}\sum_{i=1}^{n}|x_i||y_i| \leq 1 \quad \Leftrightarrow \quad \sum_{i=1}^{n}|x_i||y_i| \leq \|\boldsymbol{x}\|_p\|\boldsymbol{y}\|_q.$$

It then suffices to observe that $x_i y_i \leq |x_i||y_i|$ for any $i = 1, \ldots, n$ to arrive at the desired conclusion.

Solution for Exercice 1.9: Log-concave and log-convex functions

a) Let $f : x \to x^a$ defined for $a \in \mathbb{R}$. By convexity of the function $x \to -\ln(x)$ on $\mathbb{R}_{++}$, we know that

$$\forall x, y \in R_{++}, \forall \alpha \in [0,1], \quad -\ln(\alpha x + (1-\alpha)y) \leq -\alpha\ln(x) - (1-\alpha)\ln(y).$$

Suppose first than $a \geq 0$. Then, the previous inequality leads to

$$\begin{aligned}
-\ln(\alpha x + (1-\alpha)y) &\leq -\alpha\ln(x) - (1-\alpha)\ln(y) \\
-a\ln(\alpha x + (1-\alpha)y) &\leq -a\alpha\ln(x) - a(1-\alpha)\ln(y) \\
-\ln\left((\alpha x + (1-\alpha)y)^a\right) &\leq -\alpha\ln(x^a) - (1-\alpha)\ln(y^a) \\
-\ln(f(\alpha x + (1-\alpha)y)) &\leq -\alpha\ln(f(x)) - (1-\alpha)\ln(f(y)).
\end{aligned}$$

The last inequality shows that the function $x :\to -\ln(f(x))$ is convex, hence $x :\to \ln(f(x))$ is concave and f is log-concave on $\mathbb{R}_{++}$.

Suppose now that $a \leq 0$. Proceeding as in the case $a \geq 0$, we obtain for any $x, y \in \mathbb{R}_{++}$ and any $\alpha \in [0,1]$ that

$$-\ln(f(\alpha x + (1-\alpha)y)) \geq -\alpha\ln(f(x)) - (1-\alpha)\ln(f(y)),$$

hence $x \mapsto \ln(f(x))$ is convex and f is log-convex on $\mathbb{R}_{++}$.

b) Let f and g be two log-concave functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Then, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\alpha \in [0,1]$, we have

$$\begin{aligned}
-\ln(f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})) &\leq -\alpha\ln(f(\boldsymbol{x})) - (1-\alpha)\ln(f(\boldsymbol{y})) \\
-\ln(g(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})) &\leq -\alpha\ln(g(\boldsymbol{x})) - (1-\alpha)\ln(g(\boldsymbol{y})).
\end{aligned}$$

Summing both inequalities then gives

$$\begin{aligned}
&-\ln(f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})) - \ln(g(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})) \\
\leq\ &-\alpha\ln(f(\boldsymbol{x})) - (1-\alpha)\ln(f(\boldsymbol{y})) - \alpha\ln(g(\boldsymbol{x})) - (1-\alpha)\ln(g(\boldsymbol{y})) \quad ,
\end{aligned}$$

hence

$$-\ln(f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})g(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})) \leq -\alpha\ln(f(\boldsymbol{x})g(\boldsymbol{x})) - (1-\alpha)\ln(f(\boldsymbol{y})g(\boldsymbol{y})),$$

showing that fg is log-concave on $\mathbb{R}^n$.

c) We apply a result from the lectures (item 6 from Proposition 1.10, that would have been provided to the students in an exam). Given any family of convex functions $f_i : \mathbb{R}^n \to \bar{\mathbb{R}}, i = 1, \dots, m$, and $h : \mathbb{R}^m \to \bar{\mathbb{R}}$ such as h is nondecreasing in all variables, then the function $h \circ f$ is convex. In our example, let $m = 2$, $f_1 = \ln(f)$, $f_2 = \ln(g)$ and $h : (x, y) \mapsto \log(\exp(x) + \exp(y))$. Since f and g are log-convex, the functions $\ln(f)$ and $\ln(g)$ are convex, and thus so is $h \circ (\ln f, \ln g) = \ln(\exp(\ln(f)) + \exp(\ln(g))) = \ln(f + g)$. Overall, we have shown that $\ln(f + g)$ is convex, hence $f + g$ is log-convex.

d) We first show that $\mathcal{S}^n_{++}$ is a convex set. Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{S}^n_{++}$ and $\alpha \in [0, 1]$. Since $\boldsymbol{X}$ and $\boldsymbol{Y}$ are both elements of $\mathcal{S}^n_{++}$, we know that

$$\forall \boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}, \quad \boldsymbol{v}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{v} > 0 \text{ and } \boldsymbol{v}^{\mathrm{T}} \boldsymbol{Y} \boldsymbol{v} > 0.$$

As a result,

$$\begin{aligned} \forall \boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}, \boldsymbol{v}^{\mathrm{T}}(\alpha \boldsymbol{X} + (1-\alpha)\boldsymbol{Y})\boldsymbol{v} &= \alpha \boldsymbol{v}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{v} + (1-\alpha)\boldsymbol{v}^{\mathrm{T}} \boldsymbol{Y} \boldsymbol{v} \\ &> \min\{\boldsymbol{v}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{v}, \boldsymbol{v}^{\mathrm{T}} \boldsymbol{Y} \boldsymbol{v}\} > 0, \end{aligned}$$

hence $\alpha \boldsymbol{X} + (1-\alpha)\boldsymbol{Y} \in \mathcal{S}^n_{++}$, showing that this set is convex.
We now show that the determinant function is log-concave on $\mathcal{S}^n_{++}$. Given two matrices $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{S}^n_{++}$ and $\alpha \in [0, 1]$, our goal is to prove that

$$\ln\left(\det\left(\alpha \boldsymbol{X} + (1-\alpha)\boldsymbol{Y}\right)\right) \geq \alpha \ln\left(\det(\boldsymbol{X})\right) + (1-\alpha) \ln\left(\det(\boldsymbol{Y})\right). \tag{3}$$

Using properties of the inverse of a matrix as well as that of the logarithm function, we have

$$\begin{aligned} \ln\left(\det\left(\alpha \boldsymbol{X} + (1-\alpha)\boldsymbol{Y}\right)\right) &= \ln\left(\det(\boldsymbol{X}) \det\left(\alpha \boldsymbol{I}_n + (1-\alpha) \underbrace{\boldsymbol{Y}\boldsymbol{X}^{-1}}_{\boldsymbol{Z} \in \mathcal{S}^n_{++}}\right)\right) \\ &= \ln(\det(\boldsymbol{X})) + \ln(\det(\alpha \boldsymbol{I}_n + (1-\alpha)\boldsymbol{Z})). \end{aligned}$$

The matrices $\boldsymbol{X}$ and $\alpha \boldsymbol{I}_n + (1-\alpha)\boldsymbol{Z}$ both belong to $\mathcal{S}^n_{++}$, hence their determinant is equal to the product of their eigenvalues, i.e. $\det(\boldsymbol{X}) = \prod_{i=1}^n \lambda_i^{\boldsymbol{X}}$ and

$$det\left(\alpha \boldsymbol{I}_n + (1-\alpha)\boldsymbol{Z}\right) = \prod_{i=1}^n \left(\alpha + (1-\alpha)\lambda_i^{\boldsymbol{Z}}\right).$$

As a result,

$$\begin{aligned} \ln\left(\det\left(\alpha \boldsymbol{X} + (1-\alpha)\boldsymbol{Y}\right)\right) &= \ln(\det(\boldsymbol{X})) + \ln(\det(\alpha \boldsymbol{I}_n + (1-\alpha)\boldsymbol{Z})) \\ &= \ln\left(\prod_{i=1}^n \lambda_i^{\boldsymbol{X}}\right) + \ln\left(\prod_{i=1}^n (\alpha + (1-\alpha)\lambda_i^{\boldsymbol{Z}})\right) \\ &= \sum_{i=1}^n \ln(\lambda_i^{\boldsymbol{X}}) + \sum_{i=1}^n \ln\left(\alpha + (1-\alpha)\lambda_i^{\boldsymbol{Z}}\right). \end{aligned} \tag{4}$$

Meanwhile, we have

$$\begin{aligned}
\alpha \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln(\det(\boldsymbol{Y})) &= \alpha \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln\left(\det(\boldsymbol{Y}\boldsymbol{X}^{-1}\boldsymbol{X})\right) \\
&= \alpha \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln\left(\det(\boldsymbol{Y}\boldsymbol{X}^{-1})\det(\boldsymbol{X})\right) \\
&= \alpha \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln\left(\det(\boldsymbol{Y}\boldsymbol{X}^{-1})\right) \\
&\quad +(1-\alpha)\ln(\det(\boldsymbol{X})) \\
&= \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln(\det(\boldsymbol{Z})) \\
&= \sum_{i=1}^{n} \ln(\lambda_i^{\boldsymbol{X}}) + (1-\alpha)\sum_{i=1}^{n}\ln\left(\lambda_i^{\boldsymbol{Z}}\right), \qquad (5)
\end{aligned}$$

where we successively used the properties of the determinant for product of positive definite matrices, the properties of the logarithm function and finally the formula for the determinant. Using finally the concavity of $x \to \ln(x)$ on $\mathbb{R}_{++}$, we have

$$\ln(\alpha + (1-\alpha)\lambda_i^{\boldsymbol{Z}})) \geq \alpha \ln(1) + (1-\alpha)\ln(\lambda_i^{\boldsymbol{Z}})) = (1-\alpha)\ln\left(\lambda_i^{\boldsymbol{Z}}\right). \qquad (6)$$

Putting (4), (6) and (5) together leads to

$$\begin{aligned}
\ln(\det(\alpha\boldsymbol{X} + (1-\alpha)\boldsymbol{Y}) &= \sum_{i=1}^{n}\ln\left(\lambda_i^{\boldsymbol{X}}\right) + \sum_{i=1}^{n}\ln\left(\alpha + (1-\alpha)\lambda_i^{\boldsymbol{Z}}\right) \\
&\geq \sum_{i=1}^{n}\ln\left(\lambda_i^{\boldsymbol{X}}\right) + (1-\alpha)\sum_{i=1}^{n}\ln\left(\lambda_i^{\boldsymbol{Z}}\right) \\
&= \alpha \ln(\det(\boldsymbol{X})) + (1-\alpha)\ln(\det(\boldsymbol{Y})).
\end{aligned}$$

This final equality proves that the function $\det$ is indeed log-concave on $\mathcal{S}_{++}^n$.