Exercises on Chapter 2: Convex optimization

Mathematics of Data Science, M1 IDD

October-November 2024 (V6, fixed typos)



Exercise 2.1: Reformulations

a) Write the epigraph reformulation of the problem

$$\min_{oldsymbol{x} \in \mathbb{R}^n} \|oldsymbol{x}\|_{\infty},$$

where $\|\boldsymbol{x}\|_{\infty} = \max_{1 \leq i \leq n} |x_i|$.

- b) Using that $\|x\|_{\infty} \ge 0$, reformulate the epigraph problem so that it consists of a linear objective function and linear constraints.
- c) More generally, consider the Chebyshev approximation problem

$$\min_{\boldsymbol{y} \in \mathbb{R}^m} \|\boldsymbol{A}\boldsymbol{y} - \boldsymbol{b}\|_{\infty} \quad \text{where } \boldsymbol{A} \in \mathbb{R}^{n \times m} \text{ and } \boldsymbol{b} \in \mathbb{R}^n.$$
 (1)

Find a reformulation of problem (1) as a linear program with the same optimal value.

d) Consider now a function $\phi : \mathbb{R}^m \to \mathbb{R}^n$ defined by $\phi(y) = [\phi_i(y)]_{i=1}^n$, where every $\phi_i : \mathbb{R}^m \to \mathbb{R}$ is a nonnegative convex function.

Using the result of the previous question, reformulate the problem

$$\min_{\boldsymbol{y} \in \mathbb{R}^m} \|\boldsymbol{\phi}(\boldsymbol{y})\|_{\infty}$$

as a convex optimization problem in standard form.

Exercise 2.2: Nonlinear equality constraints

Consider the optimization problem

$$\begin{cases} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} & f(\boldsymbol{x}) := \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{a}\|_2^2 \\ \text{subject to} & \sum_{i=1}^{n-1} x_i^2 = 0. \end{cases}$$

where $\boldsymbol{a} \in \mathbb{R}^n$.

- a) Show that the feasible set of this problem is convex, and give a description of this set that only involves convex inequalities and/or linear equalities.
- b) Justify that the problem is convex.
- c) Find the unique solution to this problem by hand.
- d) Use the first-order optimality condition to compute the solution to this problem. *NB:* The gradient of the objective is $\nabla f(\mathbf{x}) = \mathbf{x} - \mathbf{a}$.

Exercise 2.3: Existence and optimality

We consider the following optimization problem:

 $\left\{ \begin{array}{ll} \mathrm{minimize}_{\boldsymbol{x} \in \mathbb{R}^3} & f(\boldsymbol{x}) := x_1^2 + x_2^2 + x_3^2 \\ \mathrm{subject \ to} & x_1 + 2x_2 - x_3 = 4 \\ & x_1 - x_2 + x_3 = -2. \end{array} \right.$

- a) Justify that the problem is convex, and give a standard form reformulation.
- b) Show that there is a unique solution to the problem. *Hint: Use that f is strictly convex.*
- c) Using the first-order optimality condition, check that the point $x^* = \begin{bmatrix} 2\\7, \frac{10}{7}, -\frac{6}{7} \end{bmatrix}^T$ is the solution.
- d) What happens to the solution set of this problem in absence of constraints (i.e. the problem becomes minimize $x \in \mathbb{R}^3 f(x)$) ?

Exercise 2.4: Convex conjugate

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. The convex conjugate of f, denoted by $\overline{f}(\cdot)$, is given by

$$orall oldsymbol{y} \in \mathbb{R}^n, \quad \overline{f}(oldsymbol{y}) = \sup_{oldsymbol{x} \in ext{dom}(f)} (oldsymbol{y}^{ ext{T}}oldsymbol{x} - f(oldsymbol{x})).$$

- a) Using that the supremum of convex functions is always convex, justify that the function \overline{f} is convex.
- b) Suppose that f is convex and differentiable on $dom(f) = \mathbb{R}^n$. Using the result of question a), show that for $z \in \mathbb{R}^n$, we have

$$\overline{f}(\nabla f(\boldsymbol{z})) = \boldsymbol{z}^{\mathrm{T}} \nabla f(\boldsymbol{z}) - f(\boldsymbol{z})$$

c) Give a closed-form expression for \overline{f} when f is linear on \mathbb{R}^n , i.e. $f(\boldsymbol{x}) = \boldsymbol{a}^T \boldsymbol{x} + b$ for some $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$;

- d) Show that if $f(x) = \frac{1}{2} ||x||^2 = \frac{1}{2} x^T x$ for all $x \in \mathbb{R}^n$, then $\overline{f} = f$.
- e) Consider the problem

 $\begin{array}{ll} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{A} \boldsymbol{x} = \boldsymbol{b}. \end{array}$

where $A \in \mathbb{R}^{\ell \times n}$ et $b \in \mathbb{R}^{\ell}$. Show that the dual function of this problem, denoted by $d(\mu)$, is given by

$$d(\boldsymbol{\mu}) = -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\mu} - \overline{f}(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu}).$$

Exercise 2.5: Weak duality

Consider the problem

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^2} & f(\boldsymbol{x}) \\ \text{subject to} & \frac{x_1^2}{x_2} \le 0, \end{array}, \quad \text{where} \quad f(\boldsymbol{x}) = \begin{cases} \exp(-x_1) & \text{if } x_2 > 0 \\ \infty & \text{otherwise.} \end{cases}$$
(2)

The domain of the problem is $\mathcal{X} = \{(x_1, x_2) | x_2 > 0\}.$

- a) Justify that this problem is convex. What is its optimal value?
- b) Write the dual problem of (2), give a formula for its solution λ^* and its optimal value d^* . What is then the duality gap?

Exercise 2.6: KKT and optimality conditions

Let $f: x \mapsto 2x_1^2 + 4x_2^2 - x_1 - x_2$ be a function from \mathbb{R}^2 to \mathbb{R} and let

$$C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0 \text{ et } x_2 \ge 0\}.$$

- a) Justify that f is convex.
- b) Justify that C is a convex cone and express then the first-order optimality condition for the problem minimize_{$x \in C$} f(x). Give then the solution of this problem.
- c) Write the KKT conditions for this problem, and justify that they can be used to compute a solution.

Exercise 2.7: Trust-region subproblem (Bonus)

Let $H \in \mathbb{R}^{n \times n}$ be a symmetric matrix, $g \in \mathbb{R}^n$ and $\delta > 0$. We consider the so-called *trust-region* subproblem defined by

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} & q(\boldsymbol{x}) \coloneqq \boldsymbol{g}^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} \boldsymbol{H} \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} \leq \delta, \end{array} \tag{3}$$

where dom $(q) = \mathbb{R}^n$.

a) Find a diagonal matrix $D \in \mathbb{R}^{n \times n}$ and a vector $c \in \mathbb{R}^n$ such that problem (3) is equivalent to

 $\begin{cases} \text{minimize}_{\boldsymbol{x}\in\mathbb{R}^n} & f(\boldsymbol{x}) := \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \leq 1, \end{cases}$ (4)

where dom $(f) = \mathbb{R}^n$.

- b) Write the primal and dual problems corresponding to (4), and justify that the primal problem is equivalent to (4).
- c) Suppose now that D is not positive semidefinite, i.e. that there exists $i \in \{1, ..., n\}$ such that $[D]_{ii} < 0$. In that case, it can be shown that problem (4) is equivalent to

$$\begin{cases} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} = 1, \end{cases}$$
(5)

- i) Justify that problem (5) has a solution.
- ii) Show that for any $\lambda > 0$,

$$\inf_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ f(\boldsymbol{x}) \middle| \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} = 1 \right\} = -\lambda + \inf_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} (\boldsymbol{D} + \lambda \boldsymbol{I}) \boldsymbol{x} \middle| \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} = 1 \right\}.$$

iii) For any $\lambda \geq \max_{1 \leq i \leq n} |[D]_{ii}|$, show that strong duality holds for problem

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} & f_{\lambda}(\boldsymbol{x}) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} (\boldsymbol{D} + \lambda \boldsymbol{I}) \boldsymbol{x} - \lambda \\ \text{s.t.} & \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} = 1. \end{array}$$
(6)

How is strong duality expressed using the Lagrangian of problem (6)?

iv) Use the result from the previous question to conclude that strong duality holds between problem (5) and its dual.

Solutions

Solution for Exercise 2.1: Chebyshev approximation

Question a) By definition, the epigraph reformulation of the problem is

 $\begin{array}{ll} \underset{t \in \mathbb{R}^n}{\text{minimize}_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ t \in \mathbb{R}}}} & t\\ \text{subject to} & \|\boldsymbol{x}\|_{\infty} \leq t. \end{array}$

Question b) Using the definition of $||x||_{\infty}$, we have that

$$\begin{split} \|\boldsymbol{x}\|_{\infty} &\leq t \quad \Leftrightarrow \quad \max_{1 \leq i \leq n} |x_i| \leq t \\ &\Leftrightarrow \quad |x_i| \leq t \quad \forall i = 1, \dots, n \\ &\Leftrightarrow \quad t \geq 0 \text{ and } -t \leq x_i \leq t \quad \forall i = 1, \dots, n \end{split}$$

As a result, the problem can be equivalently reformulated as the linear program

$$\begin{array}{ll} \underset{t \in \mathbb{R}}{\text{minimize}} x_{t \in \mathbb{R}} & t \\ \text{subject to} & -t - x_i \leq 0 \quad \forall i = 1, \dots, n \\ & -t + x_i \leq 0 \quad \forall i = 1, \dots, n \\ & -t \leq 0. \end{array}$$

Question c) Let $[a_i^T]_{i=1}^n$ denote the rows of the matrix A, and let $\phi_i(\mathbf{y}) = a_i^T \mathbf{y} - b_i$, where b_i is the corresponding coefficient of the vector \mathbf{b} . Applying the same reasoning as in question b), we obtain the following reformulation:

$$\begin{array}{ll} \underset{t \in \mathbb{R}}{\operatorname{minimize}} \boldsymbol{y}_{\substack{t \in \mathbb{R}}\\ t \in \mathbb{R}} & t \\ \text{subject to} & -t + \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{y} - b_i \leq 0 \quad \forall i = 1, \dots, n \\ & -t - \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{y} + b_i \leq 0 \quad \forall i = 1, \dots, n \\ & -t \leq 0. \end{array}$$

This problem has a linear objective as well as linear constraints.

The solutions of the problem are that of the epigraph formulation of the original problem. As a result, if (y^*, t^*) is a solution of the problem, then y^* is a solution of the original problem, and we must have $t^* = ||Ay^* - b||_{\infty}$, otherwise this point would not be an optimal solution.

Question d) *NB: The problem is already in standard form, but we seek another formulation.* By using the epigraph form of the problem and applying the same reasoning as in question b), we obtain the formulation

$$\begin{array}{ll} \underset{t \in \mathbb{R}}{\operatorname{minimize}_{\substack{\boldsymbol{y} \in \mathbb{R}^m \\ t \in \mathbb{R}}}} & t \\ \text{subject to} & -t - \phi_i(\boldsymbol{y}) \leq 0 \quad \forall i = 1, \dots, n \\ & -t + \phi_i(\boldsymbol{y}) \leq 0 \quad \forall i = 1, \dots, n \\ & -t < 0. \end{array}$$

Since $\phi_i(\mathbf{y}) \ge 0$ for any \mathbf{y} , the constraint $-t - \phi_i(\mathbf{y}) \le 0$ is always satisfied for $t \ge 0$. As a result, the problem can be equivalently reformulated as

minimize
$$\mathbf{y} \in \mathbb{R}^m$$
 t
subject to $-t + \phi_i(\mathbf{y}) \le 0 \quad \forall i = 1, \dots, n$
 $-t \le 0.$

Since a sum of convex functions is convex, all functions of the form $(\boldsymbol{y},t) \mapsto -t + \phi_i(\boldsymbol{y})$ involved in the inequality constraints are convex, hence the problem is a convex problem in standard form.

Solution for Exercise 2.2: Nonlinear equality constraints

Question a) Let \mathcal{F} denote the feasible set. For any $x \in \mathcal{F}$, we have $x_1 = x_2 = \cdots = x_{n-1}$, and this property is preserved by convex combinations. As a result,

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^n)^2, \forall \alpha \in [0, 1], \quad \sum_{i=1}^{n-1} (\alpha x_i + (1 - \alpha) y_i)^2 = 0,$$

hence $\alpha x + (1 - \alpha)y \in \mathcal{F}$, proving that the set is convex. Our reasoning provides the following description of \mathcal{F} :

$$\mathcal{F} = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_1 = 0, \dots, x_{n-1} = 0 \}.$$

Question b) General justification: The objective function is a sum of quadratic functions of the entries of x, that are all convex functions. As a result, the objective function is convex, hence the problem is a convex optimization problem.

Detailed justification for the convexity of f: There are several ways to justify that the function f is convex on \mathbb{R}^n . One possibility is to express it as a sum of n functions of one variable, then to use the fact that $x \mapsto \frac{1}{2}(x-a)^2$ is a convex function. The proof of the latter observation follows from that in the general case.

The general case is to show convexity of $f : x \mapsto \frac{1}{2} ||x - a||^2$ directly. For any $(x, y) \in (\mathbb{R}^n)^2$ and any $\alpha \in [0, 1]$, our goal is to show that

$$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y}) \quad \Leftrightarrow \quad \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \ge 0.$$

Focusing on the latter expression, we have

$$\begin{aligned} \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &= \frac{1}{2} \left(\alpha \|\boldsymbol{x} - \boldsymbol{a}\|^2 + (1-\alpha)\|\boldsymbol{y} - \boldsymbol{a}\|^2 - \|(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) - \\ &= \frac{1}{2} \left(\alpha \|\boldsymbol{x} - \boldsymbol{a}\|^2 + (1-\alpha)\|\boldsymbol{y} - \boldsymbol{a}\|^2 - \|\alpha(\boldsymbol{x} - \boldsymbol{a}) + (1-\alpha)\|\boldsymbol{y}\|^2 \right) \end{aligned}$$

Using then the identity $\| m{u} + m{v} \|^2 = \| m{u} \|^2 + 2 m{u}^{\mathrm{T}} m{v} + \| m{v} \|^2$ leads to

$$\begin{aligned} \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &= \frac{1}{2} \left(\alpha \|\boldsymbol{x} - \boldsymbol{a}\|^2 + (1-\alpha)\|\boldsymbol{y} - \boldsymbol{a}\|^2 - \alpha^2 \|\boldsymbol{x} - \boldsymbol{a}\|^2 \\ &- 2\alpha (1-\alpha)(\boldsymbol{x} - \boldsymbol{a})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{a}) - (1-\alpha)^2 \|vy - \boldsymbol{a}\|^2 \right) \\ &= \frac{1}{2} \left(\alpha (1-\alpha)\|vx - \boldsymbol{a}\|^2 - 2\alpha (1-\alpha)(\boldsymbol{x} - \boldsymbol{a})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{a}) \\ &+ (1-\alpha - (1-\alpha)^2)\|\boldsymbol{y} - \boldsymbol{a}\|^2 \right). \end{aligned}$$

Noticing that $1 - \alpha - (1 - \alpha)^2 = \alpha(1 - \alpha)$, we can factorize $\alpha(1 - \alpha)$ in the last expression, and we obtain

$$\begin{aligned} \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &= \frac{\alpha(1-\alpha)}{2} \left(\|\boldsymbol{x} - \boldsymbol{a}\|^2 - (\boldsymbol{x} - \boldsymbol{a})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{a}) + \|\boldsymbol{y} - \boldsymbol{a}\|^2 \right) \\ &= \frac{\alpha(1-\alpha)}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2, \end{aligned}$$

where we again used the identity (but with v = -v). Since the last value is always nonnegative, we have established convexity (and even 1-strong convexity).

Note that f is differentiable on \mathbb{R}^n with $\nabla f(x) = x - a$, hence we can also use the first-order characterization. For any $(x, y) \in (\mathbb{R}^n)^2$, we obtain that

$$\begin{split} f(\boldsymbol{y}) &= \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{a}\|^2 \\ &= \frac{1}{2} \|(\boldsymbol{x} - \boldsymbol{a}) + (\boldsymbol{y} - \boldsymbol{x})\|^2 \\ &= \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{a}\|^2 + (\boldsymbol{x} - \boldsymbol{a})^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{x}) + \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \\ &\geq \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{a}\|^2 + (\boldsymbol{x} - \boldsymbol{a})^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{x}) \\ &= f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{x}), \end{split}$$

showing the desired result.

Question c) For any $x \in \mathcal{F}$, $f(x) = \frac{1}{2}(x_n - a_n)^2 + \frac{1}{2}\sum_{i=1}^n a_i^2$. As a result, for any $x \in \mathcal{F}$, we have $f(x) \ge \frac{1}{2}\sum_{i=1}^n a_i^2$, and this value is only attained for $x = [0 \cdots 0 a_n]^T$. As a result, we have

$$\inf \left\{ f(\boldsymbol{x}) | \boldsymbol{x} \in \mathcal{F} \right\} = \frac{1}{2} \sum_{i=1}^{n} a_i^2 \quad \text{and} \quad \operatorname{argmin} \left\{ f(\boldsymbol{x}) | \boldsymbol{x} \in \mathcal{F} \right\} = \left\{ [0 \ \cdots \ 0 \ a_n]^{\mathrm{T}} \right\}.$$

Question d) Since the problem is convex and the objective is differentiable, we can characterize solutions via the first-order optimality conditions. Since here the feasible set is a linear subspace, we know that a point \bar{x} is a solution if and only if

$$ar{m{x}}\in \mathcal{F}$$
 and $abla f(ar{m{x}})^{\mathrm{T}}m{z}=0$ $orall m{z}\in \mathcal{F}_{+}$

Thanks to the description of \mathcal{F} , we know that any vector in \mathcal{F} has its first n-1 components equal to 0. Thus, the conditions above translate to

$$\begin{cases} x_1 = 0 \\ \vdots \\ x_{n-1} = 0 \\ [\nabla f(\bar{x})]_n (z_n - \bar{x}_n) = 0 \quad \forall z_n \in \mathbb{R}. \end{cases}$$

Using $\nabla f(\bar{x}) = \bar{x} - a$, the last condition becomes $(\bar{x}_n - a_n)(z_n - \bar{x}_n) = 0$, which only holds for all $z_n \in \mathbb{R}$ if $\bar{x}_n - a_n = 0$. As a result, we obtain the system

$$\begin{cases} x_1 &= 0\\ \vdots & \\ x_{n-1} &= 0\\ x_n - a_n &= 0, \end{cases}$$

confirming the result of Question c).

Solution for Exercise 2.3: Existence and optimality

Question a) The objective function is convex on \mathbb{R}^3 as the sum of three convex functions. Moreover, the feasible set is an affine set in \mathbb{R}^3 , hence it is convex. As a result, the problem is a convex optimization problem. Up to the right-hand sides of the linear equality constraints, it is already in standard form.

Question b) The objective function is coercive since it is a quadratic function based on a positive definite matrix (the identity). Moreover, the feasible set is not empty as it contains the vector $[0 - 2 \ 0]^T$ and it is closed because it is a linear subspace. As a result, we are guaranteed that there exists at least one solution to the problem.

Since f is strictly convex and the feasible set is convex, the problem has at most one solution, from which we conclude that the solution is unique.

Question c) Given that the feasible set of the problem is an affine set, we know that the solution x^* of the problem is the unique vector such that

$$\begin{array}{rcl} x_1^* + 2x_2^* - x_3^* & = & 4 \\ x_1^* - x_2^* + x_3^* & = & -2 \\ \nabla f(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}^*) & = & 0 \quad \forall \boldsymbol{y} \in \mathcal{F}, \end{array}$$

where \mathcal{F} denotes the feasible set. It is clear that the choice $\boldsymbol{x}^* = \begin{bmatrix} \frac{2}{7}, \frac{10}{7}, -\frac{6}{7} \end{bmatrix}^T$ satisfies the first two conditions. Using $\nabla f(\boldsymbol{x}) = 2\boldsymbol{x}$, the latter condition can be written (after division by 2) as

$$x_1^*(y_1 - x_1^*) + x_2^*(y_2 - x_2^*) + x_3^*(y_3 - x_3^*) = 0 \Leftrightarrow 14y_1 + 70y_2 - 42y_3 = 140$$

Since y is feasible, it satisfies the two constraints

implying that

 $14y_1 + 70y_2 - 42y_3 = 14y_1 + 140 - 140y_1 + 126y_1 = 140,$

as desired. This shows that x^* is the solution to the problem.

Question d) In absence of constraints, the problem still has a unique solution, which is immediately found to be the zero vector.

Solution for Exercise 2.4: Convex conjugate

Question a) For any $x \in \text{dom}(f)$, the function $y \mapsto y^T x - f(x)$ is a linear function of y hence it is a convex function of y. Since the supremum of convex functions is convex, this shows that \overline{f} is convex. *N.B. The proof of convexity of* $\sup_{i \in \mathcal{I}} f_i$ with f_i convex follows from the definition of the supremum. **Question b)** Computing $\overline{f}(\nabla f(z)) = \sup_{z \in \mathbb{R}^n} \{\nabla f(z)^T z - f(z)\}$ amounts to computing the optimal value for the problem

$$\underset{\boldsymbol{x} \in \mathbb{R}^n}{\operatorname{maximize}} \nabla f(\boldsymbol{z})^{\mathrm{T}} \boldsymbol{x} - f(\boldsymbol{x}),$$

which is the opposite of that of

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \phi(\boldsymbol{x}) = f(\boldsymbol{x}) - \nabla f(\boldsymbol{z})^{\mathrm{T}} \boldsymbol{x}$$

This problem is convex and differentiable. As a result, we know that the solutions to this problem are characterized by the equation

$$\nabla \phi(\boldsymbol{x}) = \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{z}) = \boldsymbol{0}.$$

The point x = z is a solution of this equation, hence it gives the optimal value for the problem. As a result,

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n}\phi(\boldsymbol{x})=f(\boldsymbol{z})-\nabla f(\boldsymbol{z})^{\mathrm{T}}\boldsymbol{z},$$

and thus

$$\overline{f}(\nabla f(\boldsymbol{z})) = \sup_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ \nabla f(\boldsymbol{z})^{\mathrm{T}} \boldsymbol{x} - f(\boldsymbol{x}) \right\} = -\inf_{\boldsymbol{x} \in \mathbb{R}^n} \phi(\boldsymbol{x}) = \nabla f(\boldsymbol{z})^{\mathrm{T}} \boldsymbol{z} - f(\boldsymbol{z}).$$

Question c) If $f(x) = a^{T}x - b$, then $\overline{f}(y) = \sup_{x \in \mathbb{R}^{n}} y^{T}x - a^{T}x - b$, hence \overline{f} is defined as the supremum of linear functions. Given that a linear function is only bounded when it is constant (recall Exercise 1.6), we have $\overline{f}(a) = -b$ and $\overline{f}(y) = \infty$ for any $y \neq a$.

Question d) For any $x, y \in (\mathbb{R}^n)^2$, we have

$$y^{\mathrm{T}}x - f(x) = y^{\mathrm{T}}x - \frac{1}{2}||x||^{2} \le ||y|| ||x|| - \frac{1}{2}||x||^{2}$$

by Cauchy-Schwarz inequality. The function $t\mapsto \|m{y}\|t-\frac{t^2}{2}$ is maximized at $t=\|m{y}\|$, hence

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} - f(\boldsymbol{x}) \leq \frac{1}{2} \|\boldsymbol{y}\|^2 \Rightarrow \overline{f}(\boldsymbol{y}) \leq \frac{1}{2} \|\boldsymbol{y}\|^2.$$

It suffices then to notice that the first inequality is an equality for x=y to obtain

$$\frac{1}{2} \|\boldsymbol{y}\|^2 = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{y} - f(\boldsymbol{y}) \leq \sup_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ \boldsymbol{y}^{\mathrm{T}} \boldsymbol{x} - f(\boldsymbol{x}) \right\} = \overline{f}(\boldsymbol{y}) \leq \frac{1}{2} \|\boldsymbol{y}\|^2,$$

hence the result.

Question e) By definition of the dual function, we have for any $\mu \in \mathbb{R}^{\ell}$:

$$d(\boldsymbol{\mu}) = \inf_{\boldsymbol{x} \in \mathbb{R}^n} \left(f(\boldsymbol{x}) + \boldsymbol{\mu}^{\mathrm{T}} (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \right)$$

$$= -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{\mu} + \inf_{\boldsymbol{x} \in \mathbb{R}^n} \left(\left[\boldsymbol{A}^{\mathrm{T}} \boldsymbol{\mu} \right]^{\mathrm{T}} \boldsymbol{x} + f(\boldsymbol{x}) \right)$$

$$= -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{\mu} - \sup_{\boldsymbol{x} \in \mathbb{R}^n} \left(\left[-\boldsymbol{A}^{\mathrm{T}} \boldsymbol{\mu} \right]^{\mathrm{T}} \boldsymbol{x} - f(\boldsymbol{x}) \right)$$

$$= -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{\mu} - \overline{f} (-\boldsymbol{A}^{\mathrm{T}} \boldsymbol{\mu}).$$

Solution for Exercise 2.5: Weak duality

Question a) The function f is convex (it satisfies the function inequality for every pair of points in the domain) on the feasible set of the problem $\{(0, x_2) \mid x_2 > 0\}$. The feasible set is a convex set, hence the problem is convex. Given that the objective function is constant on the problem domain, we have $p^* = f(0) = 1$.

Question b) The Lagrangian of the problem is given by

$$\mathcal{L}(\boldsymbol{x},\lambda) = \exp(-x_1) + \lambda \frac{x_1^2}{x_2}$$

where $\lambda \in \mathbb{R}$. As a result, the dual function of the problem is defined for every $\lambda \in \mathbb{R}$ as

$$d(\lambda) = \inf_{\boldsymbol{x} \in \mathcal{X}} \left\{ \exp(-x_1) + \lambda \frac{x_1^2}{x_2} \right\}.$$

When $\lambda < 0$, we have $d(\lambda) = -\infty$ (taking x_2 to be constant and taking x_1 to $+\infty$). When $\lambda = 0$, we obtain $d(0) = \inf_{x \in \mathcal{X}} \exp(-x_1) = 0$. Finally, when $\lambda > 0$, we get

$$\inf_{x_1 \in \mathbb{R}, x_2 > 0} \exp(-x_1) + \lambda \frac{x_1^2}{x_2} = \inf_{x_1 \in \mathbb{R}} \inf_{x_2 > 0} \exp(-x_1) + \lambda \frac{x_1^2}{x_2} = \inf_{x_1 \in \mathbb{R}} \exp(-x_1) = 0$$

hence $d(\lambda) = 0$ in that case as well. Overall, we obtain that

$$d(\lambda) = \begin{cases} -\infty & \text{if } \lambda < 0\\ 0 & \text{if } \lambda \ge 0, \end{cases}$$

hence $d^* = \sup_{\lambda \ge 0} d(\lambda) = 0$. As a result, the duality gap is $p^* - d^* = 1$, and strong duality does not hold.

Solution for Exercise 2.6: KKT and optimality conditions

Question a) The function f is a quadratic defined by the positive definite matrix $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$, hence it is convex (even strictly convex). Another possibility to prove convexity relies on computing the gradient

$$orall oldsymbol{x} \in \mathbb{R}^2,
abla f(oldsymbol{x}) = \left[egin{array}{c} 4x_1 - 1 \ 8x_2 - 1 \end{array}
ight]$$

and checking the characterization of convexity for differentiable functions.

Question b) The set C is a cone since $tx \ge 0$ for any $x \in C$ and t > 0, and it is also convex since $\alpha x + (1 - \alpha)y \ge 0$ for any $(x, y) \in C^2$ and $\alpha \in [0, 1]$. As a result, a solution $x^* = (x_1^*, x_2^*)$ of the problem satisfies $x_1^* \ge 0$, $x_2^* \ge 0$ as well as

$$abla f(oldsymbol{x}^*)^{\mathrm{T}}oldsymbol{x}^* = 0$$
 and $orall oldsymbol{y} \in \mathcal{C},
abla f(oldsymbol{x}^*)^{\mathrm{T}}oldsymbol{y} \geq 0.$

Putting all conditions together, we obtain

$$\begin{cases} x_1^* \ge 0\\ x_2^* \ge 0\\ (4x_1^* - 1)x_1^* + (8x_2^* - 1)x_2^* = 0\\ \forall y_1 \ge 0, \forall y_2 \ge 0, (4x_1^* - 1)y_1 + (8x_2^* - 1)y_2 \ge 0. \end{cases}$$

Letting $y_1 = 0$ and $y_2 = 0$ in the last condition gives $4x_1^* - 1 \ge 0$ and $8x_2^* - 1 \ge 0$. Combining with the previous two, we obtain

$$(4x_1^* - 1)x_1^* = (8x_2^* - 1)x_2^* = 0.$$

If $x_1^* = 0$, then $4x_1^* - 1 < 0$, contradicting what we had before. Similarly, $x_2^* = 0$ leads to the contradiction $8x_2^* - 1 < 0$. Overall, we must have

$$4x_1^* - 1 = 8x_2^* - 1 = 0,$$

hence $x_1^* = \frac{1}{4}$ and $x_2^* = \frac{1}{8}$.

Question c) Since the constraints defining C are linear, constraint qualification and strong duality hold for this problem. A solution of the problem is then also a solution of the KKT equations

$$4x_{1}^{*} - 1 - \lambda_{1}^{*} = 0$$

$$8x_{2}^{*} - 1 - \lambda_{2}^{*} = 0$$

$$x_{1}^{*} \ge 0$$

$$\lambda_{2}^{*} \ge 0$$

$$\lambda_{1}^{*} \ge 0$$

$$\lambda_{1}^{*}x_{1}^{*} = 0$$

$$\lambda_{2}^{*}x_{2}^{*} = 0,$$

where $\lambda^* = [\lambda_1^* \ \lambda_2^*]^T$ is a vector of dual variables, and the first two conditions correspond to the gradient of the Lagrangian function

$$\nabla_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \nabla f(\boldsymbol{x}) - \boldsymbol{\lambda}.$$

Solving this system leads to $x_1^* = \frac{1}{4}$, $x_2^* = \frac{1}{8}$ en $\lambda_1^* = \lambda_2^* = 0$, recovering the solution computed in question b).

Solution for Exercise 2.7: Trust-region subproblem

Question a) The matrix H is a symmetric real matrix, hence there exist an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ (i.e. such that $Q^{\mathrm{T}} = Q^{-1}$) and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$H = Q\Lambda Q^{\mathrm{T}},$$

where the diagonal elements of D correspond to the eigenvalues of H. The objective function of problem (3) can thus be written as

$$q(\boldsymbol{x}) = \boldsymbol{g}^{\mathrm{T}} \boldsymbol{x} + rac{1}{2} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{Q} \boldsymbol{x}.$$

Since Q is orthogonal, for any $x \in \mathbb{R}^n$, there is a unique y such that y = Qx. Using $y^T y = x^T Q^T Q x = x^T x$, it follows that the problem

$$\begin{cases} \text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} & q(\boldsymbol{x}) := \boldsymbol{g}^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} \boldsymbol{H} \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} \leq \delta. \end{cases}$$

can equivalently be reformulated as

$$\left\{\begin{array}{ll} \mathrm{minimize}_{\boldsymbol{y} \in \mathbb{R}^n} & \boldsymbol{g}^{\mathrm{T}} \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{y} + \boldsymbol{y}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{y} \\ \mathrm{s.t.} & \boldsymbol{y}^{\mathrm{T}} \boldsymbol{y} \leq \delta. \end{array}\right.$$

Letting $m{z}=rac{x}{\sqrt{\delta}}$, $m{c}=\sqrt{\delta}m{Q}m{g}$ and $m{D}=\deltam{\Lambda}$, the problem can be further reformulated as

$$\begin{cases} \text{ minimize}_{\boldsymbol{z} \in \mathbb{R}^n} & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{z} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{z} \\ \text{ s.t. } & \boldsymbol{z}^{\mathrm{T}} \boldsymbol{z} \leq 1, \end{cases}$$

which is the desired result.

Question b) The Lagrangian function associated with problem (4) is given by

$$\forall (\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}, \quad \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) := f(\boldsymbol{x}) + \lambda (\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} - 1) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{x} + \lambda (\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} - 1)$$

The dual problem of (4) is given by

$$\underset{\lambda \in \mathbb{R}}{\operatorname{maximize}} d(\lambda), \quad d(\lambda) := \left\{ \begin{array}{ll} \inf_{\boldsymbol{x} \in \mathbb{R}^n} L(\boldsymbol{x}, \lambda) & \text{if } \lambda \geq 0 \\ -\infty & \text{otherwise.} \end{array} \right.$$

The primal problem corresponds to

$$\underset{\boldsymbol{x} \in \mathbb{R}^n}{\operatorname{minimize}} p(\boldsymbol{x}), \quad p(\boldsymbol{x}) := \left\{ \begin{array}{ll} \sup_{\lambda \geq 0} \mathcal{L}(\boldsymbol{x}, \lambda) & \text{if } \boldsymbol{x} \in \operatorname{dom}(f) \\ \infty & \text{otherwise.} \end{array} \right.$$

Since dom $(f) = \mathbb{R}^n$, we have $p(x) = \sup_{\lambda \ge 0} \mathcal{L}(x, \lambda)$ for any $x \in \mathbb{R}^n$. Finally, since the feasible set is not empty (unit ball in \mathbb{R}^n), the primal problem is equivalent to (4).

Question c) $D \not\succeq 0$.

- i) The function f is continuous and the feasible set is a nonempty compact set (unit sphere), hence the problem has a solution.
- ii) Using the properties of the feasible set, we have for any $\lambda > 0$ that

$$\begin{split} \inf_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ f(\boldsymbol{x}) \big| \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1 \right\} &= \inf_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ f(\boldsymbol{x}) + \lambda \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - \lambda \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \big| \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1 \right\} \\ &= \inf_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ f(\boldsymbol{x}) + \lambda \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - \lambda \big| \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1 \right\} \\ &= -\lambda + \inf_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ f(\boldsymbol{x}) + \lambda \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \big| \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1 \right\} \\ &= -\lambda + \inf_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{D} + \lambda \boldsymbol{I})\boldsymbol{x} \big| \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1 \right\} \end{split}$$

iii) Let $\lambda > 0$ such that $D + \lambda I \succeq 0$. Then, the problem (6) is convex and satisfies LICQ at every feasible point $(\nabla(\|\cdot\|^2 - 1)(\boldsymbol{x}) = 2\boldsymbol{x} \neq 0$ for any feasible point). Therefore, strong duality holds for that problem. Letting $\mathcal{L}_{\lambda}(\boldsymbol{x}, \mu) := f_{\lambda}(\boldsymbol{x}) + \mu(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - 1)$ denote the Lagrangian function of problem (6), we have

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n}\sup_{\boldsymbol{\mu}\in\mathbb{R}}\mathcal{L}_{\lambda}(\boldsymbol{x},\boldsymbol{\mu}) = \sup_{\boldsymbol{\mu}\in\mathbb{R}}\inf_{\boldsymbol{x}\in\mathbb{R}^n}\mathcal{L}_{\lambda}(\boldsymbol{x},\boldsymbol{\mu}) = \mathcal{L}(\boldsymbol{x}^*,\boldsymbol{\mu}^*).$$
(7)

iv) Letting \mathcal{L} denote the Lagrangian function of problem (5), we have

$$\mathcal{L}(\boldsymbol{x}, \nu) := f(\boldsymbol{x}) + \nu(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - 1),$$

For any $\lambda > 0$, we thus have $\mathcal{L}_{\lambda}(\boldsymbol{x}, \mu) = \mathcal{L}(\boldsymbol{x}, \lambda + \mu)$. It follows that

$$\sup_{\mu \in \mathbb{R}} \mathcal{L}_{\lambda}(\boldsymbol{x},\mu) = \sup_{\nu \in \mathbb{R}} \mathcal{L}(\boldsymbol{x},
u),$$

while (7) implies

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n}\sup_{\nu\in\mathbb{R}}\mathcal{L}(\boldsymbol{x},\nu)=\sup_{\nu\in\mathbb{R}}\inf_{\boldsymbol{x}\in\mathbb{R}^n}\mathcal{L}(\boldsymbol{x},\nu)=\mathcal{L}(\boldsymbol{x}^*,\lambda+\mu^*),$$

which matches the definition of strong duality.