MATHEMATICS OF DATA SCIENCE

Sep 19, 2023

Program

Today: Convex sets

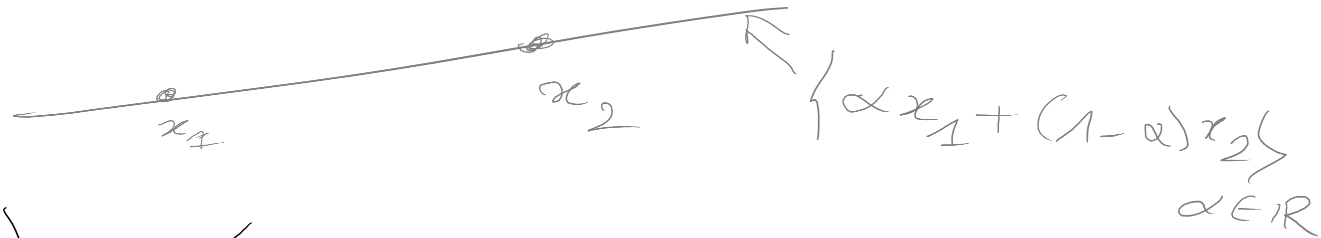
Thursday: First exercise session

1) Affine sets

Def: A set $\mathcal{X} \subseteq \mathbb{R}^n$ is called an affine set if $\forall (x_1, x_2) \in \mathcal{X}^2$, $\forall \alpha \in \mathbb{R}$,

$$\alpha x_1 + (1-\alpha) x_2 \in \mathcal{X}$$

$\hookrightarrow$ An affine set contains every line passing through any pair of points in the set



Ex)
- $\emptyset$ (empty set)
- $\mathbb{R}^n$
- Any line $\alpha \longmapsto \alpha x_1 + (1-\alpha) x_2$ where $(x_1, x_2) \in (\mathbb{R}^n)^2$ "maps to" $\mathbb{R} \longrightarrow \mathbb{R}^n$
- $\{x\}$, $x \in \mathbb{R}^n$

Def: Let $x_1, \dots, x_k \in \mathbb{R}^n$ $(k \geq 1)$

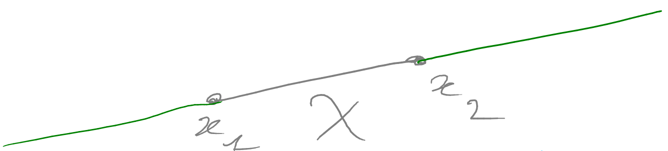
$x \in \mathbb{R}^n$ is an affine combination of $x_1, \dots, x_k$ if there exist k real values $\alpha_1, \dots, \alpha_k$ such that

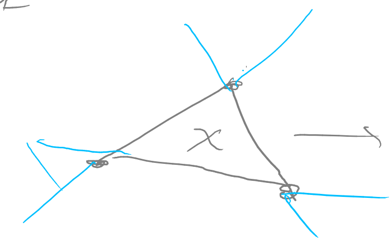
$\llcorner$ $x = \sum_{i=1}^{k} \alpha_i x_i$ and $\sum_{i=1}^{k} \alpha_i = 1$.

Def : (Affine hull)

Let $X \subseteq \mathbb{R}^n$. The affine hull of X, denoted by $\text{aff}(X)$, is the set of all affine combinations of elements of X.

$\hookrightarrow$ $\forall X \subseteq \mathbb{R}^n$, $\text{aff}(X)$ is an affine set and it is the smallest affine set that contains X. It is also the intersection of all affine sets containing X





2 Convex sets

Def : A set $C \subseteq \mathbb{R}^n$ is a convex set if

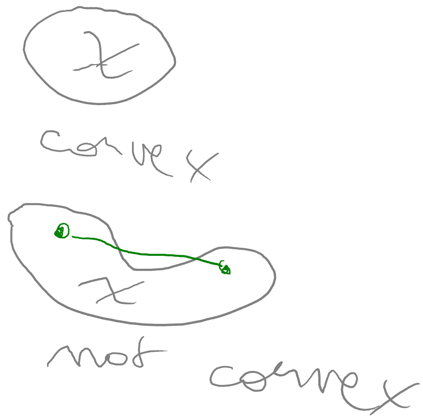
$\forall (x_1, x_2) \in C^2$, $\forall \alpha \in [0,1]$, $\alpha x_1 + (1-\alpha) x_2 \in C$

→ A set is convex if it contains every line segment passing through any pair of points within the set.

Ex) $\emptyset$, $\mathbb{R}^n$

open balls

(closed) balls



__Def.__ Let $x \in \mathbb{R}^n$ and $r > 0$

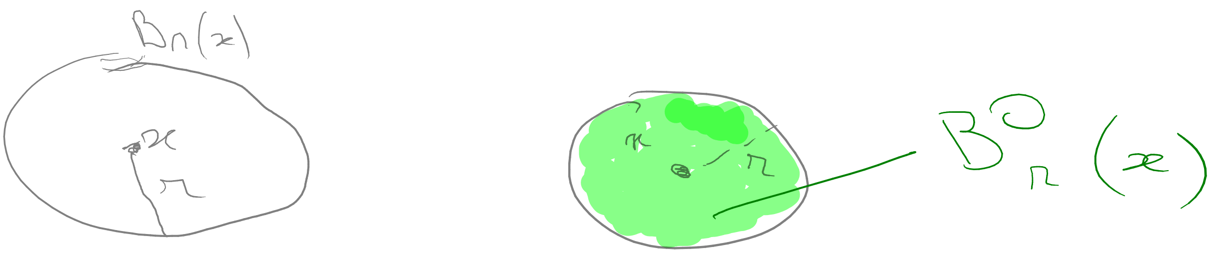
- The ball centered at x of radius r is the set

$$B_r(x) = \{ y \in \mathbb{R}^n \mid \|y - x\| \leq r \}$$

- The open ball centered at x of radius r is

$$B^{\circ}_r(x) = \{ y \in \mathbb{R}^n \mid \|y - x\| < r \}$$

Both $B_r(x)$ and $B^{\circ}_r(x)$ are convex sets.



(3) Aside: topology and convexity

Def: A set $X \subseteq \mathbb{R}^n$ is an open set if $\forall x \in X$, $\exists r > 0$,
$$B_r^{o}(x) \subseteq X$$

Def: A set $X \subseteq \mathbb{R}^n$ is closed if $\mathbb{R}^n \setminus X$ is open. (NB: $B_r^{o}(x)$ is open)

Def: The closure of $X \subseteq \mathbb{R}^n$ is the smallest closed set containing X. We denote it by $cl(X)$.

$$cl(B_r^{o}(x)) = B_r(x)$$

Def: The interior of X is the largest open set contained in X. We denote by $int(X)$

$$int(B_r(x)) = B_r^{o}(x)$$

↳ For any nonempty convex set $C \subseteq \mathbb{R}^n$:
- $cl(C)$ is convex and nonempty
- $int(C)$ is convex

4) Back to convexity

Recall: $C \subseteq \mathbb{R}^n$ convex if
$\forall (x_1, x_2) \in (\mathbb{R}^n)^2$, $\forall \alpha \in [0,1]$,

$$\alpha x_1 + (1-\alpha) x_2 \in C$$

Def: (Convex combination)
Let $x_1, \dots, x_k \in \mathbb{R}^n$
A vector $x \in \mathbb{R}^n$ is a convex combination of $x_1, \dots, x_k$ if there exist k nonnegative real values $\alpha_1, \dots, \alpha_k$ such that

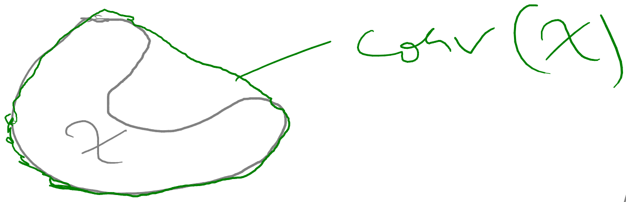
$$x = \sum_{i=1}^{k} \alpha_i x_i, \quad \sum_{i=1}^{k} \alpha_i = 1$$

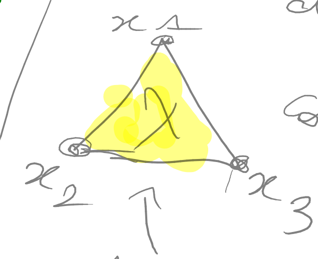
$\alpha_i \geq 0 \; \forall i = 1..k$

$\Rightarrow$ A convex combination is a special case of an affine combination.

Def: (Convex hull)
For any $\mathcal{X} \subseteq \mathbb{R}^n$, the convex hull of $\mathcal{X}$, denoted by $\mathrm{conv}(\mathcal{X})$, is the smallest convex set containing $\mathcal{X}$

$\Rightarrow$ Equivalently, $\mathrm{conv}(\mathcal{X})$ is the set of all convex combinations of elements of $\mathcal{X}$





$\mathrm{aff}(X) = \mathbb{R}^2$

$\mathrm{conv}(X) = X$

Any point in the triangle can be written as $\alpha x_1 + \beta x_2 + (1-\alpha-\beta)x_3$

Remark

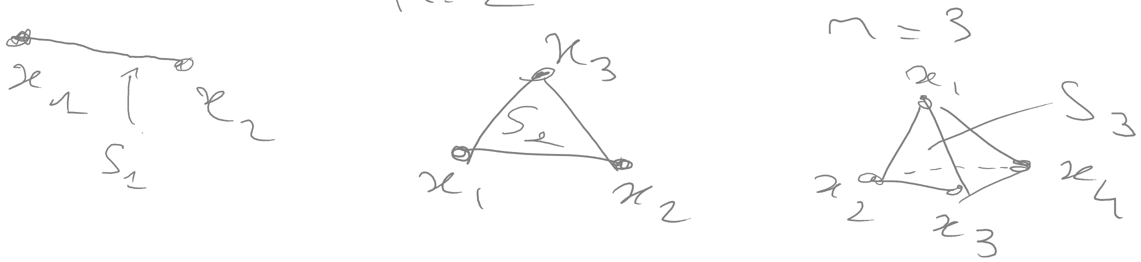
Given any $(x_1, \dots, x_{m+1}) \in (\mathbb{R}^m)^{m+1}$ the set

$$S_m = \left\{ \sum_{i=1}^{m+1} \alpha_i x_i \;\middle|\; \sum_{i=1}^{m+1} \alpha_i = 1,\ \alpha_i \geq 0 \ \ i = 1 \dots m+1 \right\}$$

is a convex set called a simplex (generalizes triangles from $\mathbb{R}^2$)

$$S_m = \mathrm{conv}(\{x_1, \dots, x_{m+1}\})$$

$m = 1$ $\qquad$ $m = 2$ $\qquad$ $m = 3$



Notation:

$(x, y) \in (\mathbb{R}^n)^2$ short for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$

Ex) $(x_1, \dots, x_k) \in (\mathbb{R}^n)^k$

(4) Cones

Def A set $K \subseteq \mathbb{R}^m$ is a cone if $\forall x \in K$, $\forall t > 0$, $tx \in K$

K is called a convex cone if it is both a cone and a convex set.

Def: Let $x_1, \dots, x_k \in \mathbb{R}^n$.
$x \in \mathbb{R}^n$ is a conic combination of $x_1, \dots, x_k$ if there exist k nonnegative real values $\alpha_1, \dots, \alpha_k$ such that

$$x = \sum_{i=1}^{k} \alpha_i x_i \qquad \alpha_i \geq 0 \ \forall i$$

(Any convex combination is a conic combination)

Def: (Conic hull)
Let $X \subseteq \mathbb{R}^n$. The conic hull of X, denoted by cone(X), is the set of all conic combinations of elements of X.

Ex)