MATHEMATICS OF DATA SCIENCE

October 11, 2024

Today: Convex optimization Part 1 (1.45pm - 5pm)

Next week: Break!

Homework assignment TBA by Oct 25

Optimization problems and definitions

1) Definitions

Def: An optimization problem is a mathematical object of the form

$$(P)\left\{\begin{array}{lll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) & \\ \text{subject to} & g_i(x)\leq 0 & \forall\, i=1..m \\ & h_i(x)=0 & \forall\, i=1..l \end{array}\right.$$

separates the constraints from the objective

→ "minimize": we are looking for the value of $x\in\mathbb{R}^n$ that yields the lowest value for f

"maximize": also exists, corresponds to finding x giving the highest value of f

→ $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ objective function: quantifies how good a choice of x is

→ $x\in\mathbb{R}^n$: (vector of) variables / (decision) variables } parameters of your decision

→ $g_i:\mathbb{R}^n\to\overline{\mathbb{R}}$ $i=1..m$ inequality constraints
$h_i:\mathbb{R}^n\to\overline{\mathbb{R}}$ $i=1..l$ equality constraints } conditions that the variables x should satisfy

When $m=l=0$, the problem is called unconstrained (in which case we write $\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f(x)$), otherwise the problem is called constrained.

NB: We always assume $m \geq 1$

$h: \mathbb{R}^n \to (\overline{\mathbb{R}})^\ell$

↳ Letting $g: x \longmapsto \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix}$ ($\mathbb{R}^n \to (\overline{\mathbb{R}})^m$) and $h: x \longmapsto \begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}$, we can rewrite (P) as

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) \quad \text{subject to} \quad \begin{aligned} g(x) &\leq 0_{\mathbb{R}^m} \\ h(x) &= 0_{\mathbb{R}^\ell} \end{aligned}$$

$a \in \mathbb{R}^m \quad b \in \mathbb{R}^m$

$a \leq b \iff a_i \leq b_i \quad \forall i = 1..m$

Key concepts related to problem (P)

- Problem domain (Domain of (P))

$$D := \text{dom}(f) \cap \bigcap_{i=1}^{m} \text{dom}(g_i) \cap \bigcap_{i=1}^{\ell} \text{dom}(h_i)$$

(set of $x \in \mathbb{R}^n$ such that $f(x), g_1(x), \ldots, g_m(x), h_1(x), \ldots, h_\ell(x)$ are $< +\infty$)

- Feasible set of (P)

$$\mathcal{F} := \left\{ x \in \bigcap_{i=1}^{m} \text{dom}(g_i) \cap \bigcap_{i=1}^{\ell} \text{dom}(h_i) \;\middle|\; g(x) \leq 0_{\mathbb{R}^m}, h(x) = 0_{\mathbb{R}^\ell} \right\}$$

(set of $x \in \mathbb{R}^n$ that satisfy the constraints)

$x \in \mathcal{F}$: feasible point

- Optimal value of (P)

$$f^* := \inf \{ f(x) \mid x \in \mathcal{F} \} \in \overline{\mathbb{R}}$$

"inf"/"infimum"

Alternate notation

$$\inf_{x \in \mathbb{R}^n} \{ f(x) \mid x \in \mathcal{F} \}$$

NB: If $\mathcal{F} = \emptyset$, $f^* = +\infty$

- Solution of (P)

→ A point $x^* \in \mathbb{R}^n$ is called an optimal point of (P) or a solution of (P) if

$$\underbrace{x^* \in F}_{\text{feasibility}} \quad \text{and} \quad \underbrace{f(x^*) \leq f(x) \ \forall x \in F}_{\text{optimality}}$$

In that case, $f^* = f(x^*)$ and we write $f^* = \min\{f(x) \mid x \in F\}$

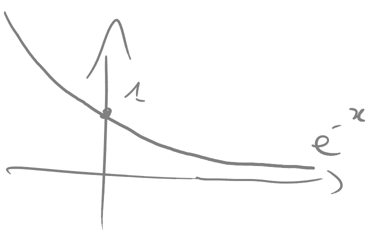
→ The set of solutions of (P) is denoted by

"argmin" = minimal argument → $\underset{x \in \mathbb{R}^n}{\text{argmin}} \left\{ f(x) \mid g(x) \leq 0_{\mathbb{R}^m},\ h(x) = 0_{\mathbb{R}^\ell} \right\} \subseteq \mathbb{R}^n$

NB: This set may be empty even when the optimal value is finite

Ex) $n = 1$, $f : x \mapsto e^{-x}$, $f^* = 0$

$F = \mathbb{R}$ $\qquad \underset{x \in \mathbb{R}^n}{\text{argmin}} \{f(x)\} = \emptyset$



Main questions about optimization problems

This course (
↳ What is the best way to formulate an optimization problem?
↳ How can we check that a given problem has a solution?

Partly this course (
↳ How do we solve this problem in practice?

2) Reformulations

↳ Given an optimization problem, there exist infinitely many to ways to write it

subject to

Ex) (P) $\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f(x)$ s.t. $g(x) \leq 0_{\mathbb{R}^m}$, $h(x) = 0_{\mathbb{R}^\ell}$

$$(Q) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}}\ \alpha_0 f(x) + \beta_0 \quad \text{s.t.} \quad \begin{array}{ll} \alpha_i g_i(x) \leq 0 & i = 1..m \\ \beta_i h_i(x) = 0 & i = 1..\ell \end{array}$$

where $\alpha_0 > 0$, $\beta_0 \in \mathbb{R}$, $\alpha_i > 0 \quad i = 1..m$
$\beta_i \neq 0 \quad i = 1..\ell$

(P) and (Q) have the same set of optimal solutions

- x feasible for (P) $\iff g_i(x) \leq 0 \quad \forall i = 1..m$
  $h_i(x) = 0 \quad \forall i = 1..\ell$

  $\iff \begin{cases} \alpha_i g_i(x) \leq 0 & \forall i = 1..m \\ \beta_i h_i(x) = 0 & \forall i = 1..\ell \end{cases}$

  $\iff x$ feasible for (Q)

- $f(x^*) \leq f(x) \quad \forall$ feasible x

  $\iff \alpha_0 f(x^*) \leq \alpha_0 f(x) \quad \forall$ feasible x

  $\iff \alpha_0 f(x^*) + \beta_0 \leq \alpha_0 f(x) + \beta_0 \quad \forall$ feasible x

If f^* is the optimal value of (P), then $\alpha_0 f^* + \beta_0$ is the optimal value of (Q) (and conversely, if $\bar{f}$ is the optimal value of (Q), $\frac{\bar{f} - \beta_0}{\alpha_0}$ is the optimal value value of (P))

We say that (Q) is an equivalent reformulation of (P)

More generally: An equivalent reformulation of (P) is an optimization problem for which the solutions of (P) can be obtained, and vice-versa $\Rightarrow$ (P) and the reformulation are two "equivalent formulations"

⚠ Two equivalent formulations need not have the same variables, the same feasible set, the same objective, and need not both be minimization problems (see dual problem in a few lectures)

Other useful reformulations of (P)

$\rightarrow$ $\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x)$ s.t. $x \in \mathcal{F}$ $\quad$ $\mathcal{F} \subseteq \mathbb{R}^n$ that represents $\{x \in \mathbb{R}^n \mid g(x) \le 0_{\mathbb{R}^m}, h(x) = 0_{\mathbb{R}^\ell}\}$

$\rightarrow$ $\underset{x \in \mathbb{R}^n}{\text{maximize}} \; (-f)(x)$ s.t. $g(x) \le 0_{\mathbb{R}^m}$, $h(x) = 0_{\mathbb{R}^\ell}$

$\rightarrow$ $\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x)$ s.t. $\hat{g}(x) \le 0_{\mathbb{R}^{m+2\ell}}$ $\quad$ $\hat{g} : x \mapsto \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \\ h_1(x) \\ \vdots \\ h_\ell(x) \\ -h_1(x) \\ \vdots \\ -h_\ell(x) \end{bmatrix}$

$(h_i(x) \le 0 \text{ and } -h_i(x) \le 0 \iff h_i(x) = 0)$

$\rightarrow$ $\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) + \delta_{\mathcal{F}}(x)$ $\quad$] unconstrained problem

where $\delta_{\mathcal{F}}$ is the indicator function of $\mathcal{F}$

$$\delta_{\mathcal{F}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{F} \\ \infty & \text{otherwise} \end{cases}$$

→ Epigraph (re)formulation

problem variables are x and t

$$(P)\ \begin{cases} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{s.t.} & g_i(x)\leq 0 \quad i=1..m \\ & h_i(x)=0 \quad i=1..l \end{cases} \longrightarrow (E)\ \begin{cases} \underset{\substack{x\in\mathbb{R}^n \\ t\in\mathbb{R}}}{\text{minimize}} & t \\ \text{s.t.} & f(x)-t\leq 0 \\ & g_i(x)\leq 0 \quad i=1..m \\ & h_i(x)=0 \quad i=1..l \end{cases}$$

A feasible point for (E) is in particular in $\text{epi}(f)$ because $t\geq f(x)$

x^* solution of (P) $\iff$ $(x^*, f(x^*))$ solution of (E)

3) Convex optimization problems

Informal definition: convex objective function + convex feasible set

Definition: An optimization problem is called convex if there exists an equivalent formulation of the problem of the form:

standard form of convex optimization problems →

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{s.t.} \quad g_i(x)\leq 0 \quad i=1..m$$
$$a_i^T x - b_i = 0 \quad i=1..l$$

where $f, g_1, \ldots, g_m$ are convex functions and $(a_i, b_i)\in\mathbb{R}^n\times\mathbb{R}$ $\forall i=1..l$ (h_i → affine/linear functions)

Remark: If $g_1,\ldots,g_m$ are convex, $\{x\in\mathbb{R}^n \mid g_i(x)\leq 0 \ i=1..m,\ a_i^T x - b_i = 0 \ i=1..l\}$ is a convex set

Ex) Consider $\underset{x\in\mathbb{R}^2}{\text{minimize}}\ x_1^2+x_2^2$ s.t. $\dfrac{x_1}{1+x_2^2}\leq 0$

$(x_1+x_2)^2 = 0$

→ NOT a convex optimization problem in standard form

→ But the feasible set of the problem is

$$\left\{x \in \mathbb{R}^2 \;\middle|\; \frac{x_1}{1+x_2^2} \leq 0,\ (x_1+x_2)^2 = 0\right\}$$

$$= \left\{x \in \mathbb{R}^2 \;\middle|\; x_1 \leq 0,\ (x_1+x_2)^2 = 0\right\}$$

$$= \left\{x \in \mathbb{R}^2 \;\middle|\; x_1 \leq 0,\ x_1+x_2 = 0\right\}$$

$$\left.\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad x_1^2 + x_2^2 \quad \text{s.t.} \quad \begin{array}{l} x_1 \leq 0 \\ x_1 + x_2 = 0 \end{array}\right\} \text{ standard form}$$

Ex) $\underset{x \in \mathbb{R}^n}{\text{maximize}} \; f(x) \quad \text{s.t.} \quad x \in F$

f concave on F
F convex set
$F = \{x \mid g(x) \leq 0_{\mathbb{R}^m},\ a_i^T x - b_i = 0,\ i = 1..l\}$

is a convex optimization problem because it is equivalent to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; (-f)(x) \quad \text{s.t.} \quad x \in F$$

Important classes of optimization problems

- Linear programming (LP)

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c^T x \quad \text{s.t.} \quad \boxed{\begin{array}{l} Ax = b \\ x \geq 0_{\mathbb{R}^n} \end{array}}$$

$A \in \mathbb{R}^{l \times n}$, $b \in \mathbb{R}^l$, $c \in \mathbb{R}^n$

$m = n$

$-x_i \leq 0 \quad i = 1..m$

$a_i^T x - b_i = 0 \quad i = 1..l$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_l^T \end{bmatrix}$$

- Quadratic programming (QP)

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad c^T x + \frac{1}{2} x^T H x \quad \text{s.t.} \quad \begin{array}{l} Ax = b \\ x \geq 0 \end{array}$$

$$A \in \mathbb{R}^{l\times n} \quad b \in \mathbb{R}^l$$

$$c \in \mathbb{R}^n$$

If $H \succeq 0$, the objective function is convex otherwise, the problem may or may not be convex (that depends on the feasible set)

- Semidefinite programming (SDP)

$$\underset{X\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad \text{trace}(C^T X) \quad \text{s.t.} \quad \begin{array}{l} \text{trace}(A_i^T X) = b_i \quad i=1..l \\ X \succeq 0 \end{array}$$

$X = X^T$

$v^T X v \geq 0$
$\forall v \in \mathbb{R}^n$

where $C = C^T \in \mathbb{R}^{n\times n}$

$A_i = A_i^T \in \mathbb{R}^{n\times n} \; \forall i = 1..l$, $b \in \mathbb{R}^l$

4) Existence and characterization of solutions

↳ Several theorems guarantee that an optimization problem possesses a solution (possibly not unique)

Ex | $\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$
$x \mapsto -\ln(x)$
$-\ln(0) = \infty$

Theorem (Weierstrass)

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a continuous function on $\mathbb{R}^n$ and let $X \subseteq \mathbb{R}^n$ be a nonempty $\underbrace{\text{compact}}_{\text{closed, bounded}}$ set of $\mathbb{R}^n$.

Then, the problem $\underset{x\in\mathbb{R}^n}{\text{minimize}} \; f(x)$ s.t. $x \in X$ has at least one solution.

Ex) $\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad x^T A x \quad \text{s.t.} \quad \underbrace{\|x\|^2 = 1}_{\text{Unit hypersphere in } \mathbb{R}^n} \qquad A = A^T \in \mathbb{R}^{n \times n}$

→ this problem has at least one solution (which is an eigenvector associated with the minimum eigenvalue of A)

↳ Weierstrass' theorem does not apply to unbounded sets (like $\mathbb{R}^n$)
⇒ In particular, it is not applicable to unconstrained problems.

↳ challenge with unbounded feasible sets (think about $x \mapsto e^{-x}$):
the function may decrease at infinity

Definition: Let $f : \mathcal{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ where $\mathcal{X} \subseteq \mathbb{R}^n$ is unbounded nonempty

f is called coercive if $\displaystyle \lim_{\substack{x \in \mathcal{X} \\ \|x\| \to \infty}} f(x) = +\infty$

Theorem Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ be coercive and continuous on a set $\mathcal{X} \subseteq \mathbb{R}^n$ nonempty (and unbounded) closed set.
Then $\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) \quad \text{s.t.} \quad x \in \mathcal{X}$ has at least one solution.

↳ With convexity, we can get stronger guarantees.

Theorem: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ be convex on a nonempty convex set $\mathcal{X} \subseteq \mathbb{R}^n$.
Then, either $\underset{x \in \mathbb{R}^n}{\text{argmin}} \{ f(x) \mid x \in \mathcal{X} \}$ is empty,

or it is a nonempty, convex set

Ex) • $\underset{x \in \mathbb{R}^n}{\text{minimize}} \ c^T x \qquad c \in \mathbb{R}^n$

$\underset{x \in \mathbb{R}^n}{\text{argmin}} \{c^T x\} = \emptyset$ when $c \neq 0_{\mathbb{R}^n}$

$\underset{x \in \mathbb{R}^n}{\text{argmin}} \{c^T x\} = \mathbb{R}^n$ convex set when $c = 0_{\mathbb{R}^n}$

• $\underset{x \in \mathbb{R}^2}{\text{minimize}} \ x_1^2$

$\underset{x \in \mathbb{R}^2}{\text{argmin}} \{x_1^2\} = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \middle| x_2 \in \mathbb{R} \right\}$

nonempty, convex set

(Equivalent formulation $\underset{x \in \mathbb{R}^2}{\text{minimize}} \ x_1^2 \quad \text{s.t. } x_1 = 0$)

↳ The results above are "existence" results: they guarantee (when the assumptions are satisfied) that a solution exists, but they do not provide a way to compute a solution or to check that a point is a solution

⚠ We cannot use the definition of a solution in general because it involves checking $f(x^*) \le f(x) \ \forall x \in \mathcal{F}$ (i.e. infinitely many function value comparisons)

↳ For differentiable problems, we can find conditions/certificates of optimality that can be checked in finite time.

⇒ For convex optimization, these certificates are used in practice to compute solutions.

↳ These conditions are called optimality conditions (or 1st-order optimality conditions because of the use of the derivative)

Theorem: Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty convex set.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable on an open set containing $\mathcal{X}$.

$$\left[x^* \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \{ f(x) \mid x \in \mathcal{X} \} \right] \iff \left[\underbrace{x^* \in \mathcal{X}}_{\text{feasibility}} \text{ and } \underbrace{\nabla f(x^*)^T (z - x^*) \geq 0 \quad \forall z \in \mathcal{X}}_{\text{optimality (using the derivative)}} \right]$$

Proof idea: x^* solution $\Rightarrow f(x^*) \leq f(z) \ \forall z \in \mathcal{X}$

By convexity $f(z) \geq f(x^*) + \nabla f(x^*)^T (z - x^*) \quad \forall z \in \mathcal{X}$

$$\Leftarrow \quad f(z) \geq f(x^*) + \underbrace{\nabla f(x^*)^T (z - x^*)}_{\geq 0} \geq f(x^*) \quad \forall z \in \mathcal{X}$$

$(\Rightarrow)$ If $\exists z \in \mathcal{X}$ such that $\nabla f(x^*)^T (z - x^*) < 0$

then $\underbrace{f(x^*) + \nabla f(x^*)^T (z - x^*)}_{\approx f(z) \text{ for } \|z - x^*\| \text{ sufficiently small}} < f(x^*)$

Corollaries

i) If $\mathcal{X}$ is a linear subspace of $\mathbb{R}^n$

$$x^* \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \{ f(x) \mid x \in \mathcal{X} \} \iff \left[x^* \in \mathcal{X} \text{ and } \nabla f(x^*)^T y = 0 \quad \forall y \in \mathcal{X} \right]$$

ii) If $\mathcal{X}$ is an affine set in $\mathbb{R}^n$ (e.g. $\{x \mid Ax = b\}$)

$$\left(x^* \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \{ f(x) \mid x \in \mathcal{X} \} \right) \iff \left[x^* \in \mathcal{X} \text{ and } \nabla f(x^*)^T (y - x^*) = 0 \quad \forall y \in \mathcal{X} \right]$$

$\{y - x^* \mid y \in \mathcal{X}\}$ is a subspace

iii) If X is a convex cone in $\mathbb{R}^n$ with $0_{\mathbb{R}^n} \in X$ (pointed cone)

$$\left(x^* \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \{ f(x) \mid x \in X \}\right) \Leftrightarrow \left[x^* \in X, \ \nabla f(x^*)^T x^* = 0 \quad \nabla f(x^*)^T y \geq 0 \ \forall y \in X \right]$$

Illustration for quadratic programs

Consider $n=2$ and $\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad c^T x + \frac{1}{2} x^T H x$

where $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ and $H = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

with $\lambda_1 \geq 0 \quad \lambda_2 \geq 0$

$\rightarrow$ $H \succeq 0$ hence the objective function is convex

It is also coercive

We thus know that the problem has at least one solution and that the set of solutions is nonempty and convex.

$\hookrightarrow$ Consider $\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad c^T x + \frac{1}{2} x^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} x = q(x)$

$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-\lambda_2 < 0$

$\lambda_1 \geq 0$

The objective function is not coercive

$$\lim_{\left\|\begin{bmatrix} 0 \\ t \end{bmatrix}\right\| \to \infty} q\left(\begin{bmatrix} 0 \\ -t \end{bmatrix}\right) = -\infty \quad \left(q\left(\begin{bmatrix} 0 \\ -t \end{bmatrix}\right) = -t - \lambda_2 \frac{t^2}{2} \underset{t \to \infty}{\longrightarrow} -\infty \right)$$

The problem has no solution because

$$\inf_{x \in \mathbb{R}^2} q(x) = -\infty$$

↳ Consider finally
$$\begin{cases} \underset{x \in \mathbb{R}^2}{\text{minimize}} & c^T x + \frac{1}{2} x^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} x \quad (= q(x)) \\ \text{subject to} & x_2 = 0 \end{cases}$$

$\lambda_1 > 0 > -\lambda_2$

$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

q is convex on $\{x \mid x_2 = 0\} \leftarrow$ convex set

q is coercive on $\{x \mid x_2 = 0\}$ when $\lambda_1 > 0$

$\Rightarrow \exists$ solution to the problem (and the set of solutions is a convex set)

Moreover, q is differentiable (polynomial) on $\mathbb{R}^2$ (hence on an open set containing the feasible set) and the feasible set is a linear subspace. Therefore,

$$\left[x^* \in \operatorname{argmin}_{x \in \mathbb{R}^2} \{ f(x) \mid x_2 = 0 \}\right] \iff \left[x_2^* = 0 \text{ and } \nabla q(x^*)^T y = 0 \ \forall \text{ feasible } y \right]$$

$$\iff \left[x_2^* = 0 \text{ and } \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} x^* + c \right)^T y = 0 \ \forall y \text{ such that } y_2 = 0 \right]$$

$$\iff \begin{cases} x_2^* = 0 \\ (\lambda_1 x_1^* + 1) y_1 + (-\lambda_2 x_2^* + 1) y_2 = 0 \quad \forall y, y_2 = 0 \end{cases}$$

$$\iff \begin{cases} x_2^* = 0 \\ (\lambda_1 x_1^* + 1) y_1 = 0 \quad \forall y_1 \in \mathbb{R} \end{cases}$$

True for $y_1 = 1 \rightarrow \lambda_1 x_1^* + 1 = 0$

$$\iff \boxed{\begin{cases} x_2^* = 0 \\ \lambda_1 x_1^* + 1 = 0 \end{cases}}$$) linear equations that characterize the solution(s)

Since $\lambda_1 > 0$, $\underset{x \in \mathbb{R}^2}{\operatorname{argmin}} \{ q(x) \mid x_2 = 0 \} = \left\{ \begin{bmatrix} -1/\lambda_1 \\ 0 \end{bmatrix} \right\}$ (1 solution)

NB When $\lambda_1 = 0$: → No coercivity (no existence of solutions guaranteed)

→ the set of solutions is either empty or convex nonempty

→ the optimality conditions become $\begin{cases} x_2^* = 0 \\ 1 = 0 \end{cases}$

hence the solution set is empty