MATHEMATICS OF DATA SCIENCE

Oct 24, 2025

Today: Duality

November 11: Correction on constraint qualification
$\Rightarrow$ Only Slater's condition implies strong duality!

DUALITY

For every optimization problem, there exists a related problem called the dual that can help you solving the original problem.

1 Lagrangian, primal and dual problems

$$(P) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{s.t.} \quad \begin{array}{ll} g_i(x) \leq 0 & i = 1..m \\ h_i(x) = 0 & i = 1..l \end{array}$$

$$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

$$g : \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^m \qquad h = [h_i] : \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^l$$

Suppose that the problem domain is not empty:

$$\mathcal{X} = \operatorname{dom}(f) \cap \bigcap_{i=1}^{m} \operatorname{dom}(g_i) \cap \bigcap_{i=1}^{l} \operatorname{dom}(h_i) \neq \emptyset$$

$$\mathcal{F} = \{ x \in \bigcap_{i=1}^{m} \operatorname{dom}(g_i) \cap \bigcap_{i=1}^{l} \operatorname{dom}(h_i) \mid g(x) \leq 0, \ h(x) = 0 \} \quad : \text{feasible set}$$

$$p^* = \inf_{x \in \mathbb{R}^n} \{ f(x) \mid g(x) \leq 0, h(x) = 0 \} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

Definition: The Lagrangian function of (P) is

$$\begin{aligned} \mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l &\longrightarrow \overline{\mathbb{R}} \\ (x, \lambda, \mu) &\longmapsto f(x) + \lambda^T g(x) + \mu^T h(x) \end{aligned}$$

→ If $x \in \mathcal{F}$, $\mathcal{L}(x,\lambda,\mu) \le f(x)$ $\quad \forall \lambda \ge 0_{\mathbb{R}^m}, \forall \mu \in \mathbb{R}^\ell$

($\mathcal{L}$ ≈ lower bound of f for feasible x)

→ If $\mathcal{F} \neq \emptyset$, (P) is equivalent to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ p(x), \quad \overset{\text{primal function}}{\overbrace{p(x)}} = \begin{cases} \sup_{\lambda,\mu} \mathcal{L}(x,\lambda,\mu) \text{ s.t. } \lambda \ge 0 & \text{if } x \in \mathcal{X} \\ +\infty & \text{if } x \notin \mathcal{X} \end{cases}$$

↓

Primal problem

⊕ Unconstrained

⊖ Objective has a complicated form: optimal value to another optimization problem

Dual problem: Flip the roles of x and (λ,μ)

The dual problem associated with (P) is

$$\boxed{(D) \quad \underset{(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^\ell}{\text{maximize}}\ d(\lambda,\mu) \quad \text{s.t.} \quad \lambda \ge 0}$$ → Dual problem

with $\underbrace{d(\lambda,\mu)}_{\text{dual function}} = \begin{cases} \inf_{x \in \mathbb{R}^n} \mathcal{L}(x,\lambda,\mu) \text{ s.t. } x \in \mathcal{X} & \text{if } \lambda \ge 0 \\ -\infty & \text{otherwise} \end{cases}$

Theorem (D) is a convex optimization problem.
(even when (P) is not!)

Proof sketch

- Feasible set for D : $\{(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^\ell \mid \lambda \geq 0\}$ convex set
- (D) $\Leftrightarrow$ $\underset{(\lambda,\mu)}{\text{minimize}}$ $-d(\lambda,\mu)$ s.t. $\lambda \geq 0$
- $\forall (\lambda,\mu)$ feasible $(\lambda\geq 0)$,

$$-d(\lambda,\mu) = -\inf_{x\in X} \mathcal{L}(x,\lambda,\mu)$$
$$= \sup_{x\in X}\left[-\mathcal{L}(x,\lambda,\mu)\right]$$
$$= \sup_{x\in X}\left[-f(x) - \lambda^T g(x) - \mu^T h(x)\right]$$

- $\forall x \in X$, $(\lambda,\mu) \mapsto -f(x) - \lambda^T g(x) - \mu^T h(x)$ is convex because linear in λ and μ
- The supremum of convex functions is convex (in an extended-value sense), so

$$(\lambda,\mu) \mapsto \sup_{x\in X}\left[-f(x) - \lambda^T g(x) - \mu^T h(x)\right]$$

is convex on $\{(\lambda,\mu) \mid \lambda \geq 0\}$

② Duality results

Q) What is the link between (P) and (D)?

Theorem (Weak duality)

- $\forall (x,\lambda,\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$, $d(\lambda,\mu) \leq p(x)$
- $\forall x \in \mathcal{X}$, $\forall (\lambda,\mu) \in \{(\lambda,\mu) \mid \lambda \geq 0\}$,

$$d(\lambda,\mu) \leq f(x)$$

- $$\underbrace{\sup_{(\lambda,\mu)} \{d(\lambda,\mu) \mid \lambda \geq 0\}}_{\text{"}d^*\text{"}} \leq \underbrace{\inf_{x \in \mathbb{R}^n} \{f(x) \mid g(x) \leq 0,\ h(x) = 0\}}_{\text{"}p^*\text{"}}$$

d^*: Optimal value for the dual problem

p^*: Optimal value for the primal / the original problem

→ Weak duality: $d^* \leq p^*$: the optimal value of the dual approximates p^*

→ In general, $d^* < p^*$: we say there is a duality gap between (P) and (D)

→ $d^* < p^*$ ⇒ the two problems are not equivalent, and in particular you cannot solve (P) by solving (D)

Strong duality: Strong duality holds for (P) and (D) when $p^* = d^*$

→ Under strong duality, we can write

$$d^* = \sup_{(\lambda,\mu)} \inf_x \mathcal{L}(x,\lambda,\mu) = \inf_x \sup_{(\lambda,\mu)} \mathcal{L}(x,\lambda,\mu) = p^*$$

"Min-max Theorem"

→ Under strong duality, if x^* solution of (P) and (λ^*, μ^*) solution of (D), we have

$$\forall (x, \lambda, \mu), \quad \mathcal{L}(x^*, \lambda, \mu) \leq \underset{\substack{\| \\ p^* \\ \| \\ d^*}}{\mathcal{L}(x^*, \lambda^*, \mu^*)} \leq \mathcal{L}(x, \lambda^*, \mu^*)$$

(x^*, λ^*, μ^*) is a saddle point of the Lagrangian (minimum w.r.t. x, maximum w.r.t. (λ, μ))

<u>Key question</u>: when does strong duality holds?

- An example for which strong duality does not hold

$$(P) \quad \underset{x \in \mathbb{R}^2}{\text{minimize}} \; f(x) \quad \text{s.t.} \quad x_1 \leq 0$$

$$f(x) = \begin{cases} e^{-\sqrt{x_1 x_2}} & \boxed{\text{if } x \geq 0} \\ +\infty & \text{otherwise} \end{cases}$$

$$\boxed{X = \{x \geq 0\}}$$

$$\forall (x, \lambda), \quad \mathcal{L}(x, \lambda) = f(x) + \lambda x_1$$

<u>Optimal value of (P)</u>

$$\inf_{x \in X} \sup_{\lambda \geq 0} \mathcal{L}(x, \lambda) = \inf_{x \in X} \sup_{\lambda \geq 0} \Big[\overset{\text{independent of } \lambda}{f(x)} + \lambda x_1 \Big]$$

$$= \inf_{x \in X} \Big[f(x) + \sup_{\lambda \geq 0} \lambda x_1 \Big] \qquad x_1 \geq 0 \;\; \forall x \in X$$

$\forall\, x_1 \geq 0$

$$\sup_{\lambda \geq 0} \lambda x_1 = \begin{cases} +\infty & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 = 0 \end{cases}$$

it suffices to take the infimum over $x \in X$ such that $x_1 = 0$

Thus
$$\inf_{x \in X} \sup_{\lambda \geq 0} \mathcal{L}(x, \lambda) = \inf_{\substack{x \in X \\ x_1 = 0}} f(x)$$

$e^{-\sqrt{0 \times x_2}} = 1$

$$= \inf_{\substack{x \in X \\ x_1 = 0}} 1$$

$$= 1$$

Primal value
optimal value of (P) : 1

Dual value

$$\sup_{\lambda \geq 0} \inf_{x \in X} \mathcal{L}(x, \lambda) = \sup_{\lambda \geq 0} \left[\inf_{x \in X} \left[f(x) + \lambda x_1 \right] \right]$$

$$= \sup_{\lambda \geq 0} \left[\inf_{x \in X} \underbrace{\left(e^{-\sqrt{x_1 x_2}} + \lambda x_1 \right)}_{\geq 0} \right] = 0$$

$x_1 \geq 0$
$x_2 \geq 0$

subset of X

$$0 \leq \inf_{x \in X} \left(e^{-\sqrt{x_1 x_2}} + \lambda x_1 \right) \leq \inf_{x \in X} \left\{ e^{-\sqrt{x_1 x_2}} + \lambda x_1 \;\middle|\; \begin{array}{l} x_2 = \frac{1}{x_1^2} \\ x_1 > 0 \end{array} \right\}$$

$\forall\, X_1 \subseteq X_2$, $\inf_{x \in X_2} f(x) \leq \inf_{x \in X_1} f(x)$

$$= \inf_{x \in X} \left\{ \underbrace{e^{-1/\sqrt{x_1}} + \lambda x_1}_{\to 0 \text{ as } x_1 \to 0} \;\middle|\; x_2 = 1/x_1,\ x_1 > 0 \right\}$$

$$= 0$$

Dual value $0 < 1$ (primal value)

- Dual variables provide useful information under constraint qualifications (CQ)

⚠ Some formulations may satisfy constraint qualifications while others do not

Two examples of CQ:

- If $f: \mathbb{R}^n \to \mathbb{R}$ (real-valued function) and all constraints are linear, we say that linear CQ holds

Ex) The example without strong duality can be reformulated as

Both satisfy linear CQ but not strong duality

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \; e^{-\sqrt{x_1 x_2}} \quad \text{s.t.} \quad x_1 \leq 0, \; x \geq 0$$

$$\Leftrightarrow \underset{x \in \mathbb{R}^2}{\text{minimize}} \; 1 \quad \text{s.t.} \quad x_1 = 0, \; x_2 \geq 0$$

→ linear CQ will hold for linear programs, quadratic programs.

- A more involved CQ: Slater's CQ (implies strong duality for convex problems!)

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) \quad \text{s.t.} \quad g_i(x) \leq 0, \; a_i^T x - b_i = 0$$

f, g_i convex

X domain

$ri(X)$ "$\{x \mid \exists \varepsilon > 0 \; B^o_\varepsilon(x) \cap aff(X) \subseteq X\}$"

open ball

If $\exists x \in ri(X)$ (relative interior) such that

strictly feasible point → $g_i(x) < 0$, $a_i^T x - b_i = 0$, then Slater CQ holds

Illustration : Duality for linear programs

$$(P) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c^T x \quad \text{s.t.} \quad Ax = b \qquad (Ax - b = 0)$$
$$x \geq 0 \qquad (-x \leq 0)$$

$c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \mu^T (Ax - b) - \lambda^T x$$

$$\inf_x \mathcal{L}(x, \lambda, \mu) = \begin{cases} -b^T \mu & \text{if } \lambda \geq 0 \text{ and } A^T \mu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual can be formulated as:

$$(D) \quad \underset{(\lambda, \mu)}{\text{maximize}} \quad -b^T \mu \quad \text{s.t.} \quad A^T \mu - \lambda + c = 0$$
$$\lambda \geq 0$$

$$\underset{(\lambda, \mu)}{\text{minimize}} \quad b^T \mu \quad \text{s.t.} \quad A^T \mu - \lambda + c = 0$$
$$\lambda \geq 0$$

$\Rightarrow$ Another linear program

Key result from (LP)

Exactly one of the three cases occurs

(1) (P) and (D) are both infeasible

(2) (P) is infeasible and (D) is unbounded
or (D) is infeasible and (P) is unbounded

(3) (P) and (D) are both feasible, and there exist (x^*, λ^*, μ^*) such that x^* solution of (P), (λ^*, μ^*) solution of (D)

and $-b^T\mu^* = c^T x^*$