STOCHASTIC PROGRAMMING

November 12, 2024

Today:

Linear two-stage programs (PK-2)

Back to the exercise

$$\underset{x\in\mathbb{R}}{\text{minimize}} \left\{ 2x + \mathbb{E}_{\xi}\left[Q(x,\xi)\right] \text{ s.t. } x \geq 0 \right\}$$

where $$Q(x,\xi) = \min_{y\in\mathbb{R}} \left\{ \xi y \text{ s.t. } \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

a) Explicit formula for $Q(x,\xi)$

$\xi = 0$ $$Q(x,0) = \min_{y\in\mathbb{R}} \left\{ 0 \text{ s.t. } \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

$\to$ Problem is feasible (e.g. $y = \max(1-x, 0)$ is feasible)

$\to$ objective is constant

Hence $Q(x,0) = 0$

$\xi > 0$ $$Q(x,\xi) = \min_{y\in\mathbb{R}} \left\{ \xi y \text{ s.t. } \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

$$= \xi \min_{y\in\mathbb{R}} \left\{ y \text{ s.t. } \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

$$= \xi\, Q(x,1)$$

$$= \xi \max(1-x, 0)$$

$\xi < 0$ $$Q(x,\xi) = \min_{y\in\mathbb{R}} \left\{ \xi y \text{ s.t. } \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

$$= \min_{y \in \mathbb{R}} \left\{ \underset{-\xi > 0}{(-\xi)}(-y) \quad \text{s.t.} \quad \begin{array}{l} y \geq 1 - x \\ y \geq 0 \end{array} \right\}$$

$$= (-\xi) \min_{y \in \mathbb{R}} \left\{ -y \quad \text{s.t.} \quad \begin{array}{l} y \geq 1 - x \\ y \geq 0 \end{array} \right\}$$

$$= -\xi \, Q(x, -1)$$

$\min_{y \in \mathbb{R}} \left\{ -y \ \text{s.t.} \ \begin{array}{l} y \geq 1 - x \\ y \geq 0 \end{array} \right\}$ is feasible but unbounded

For any $y \geq \max(1-x, 0)$, $2y \geq y \geq \max(1-x, 0)$ is feasible

and $-2y \leq -y$ with equality if and only if $y = 0$.

Hence $Q(x, -1) = -\infty$ and $Q(x, \xi) = \underbrace{(-\xi)}_{>0} Q(x, -1)$

$= -\infty$

Overall,

$$Q(x, \xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ \xi \max(1-x, 0) & \text{if } \xi > 0 \\ -\infty & \text{if } \xi < 0 \end{cases}$$

b) Formulate the scenario version of the problem with two scenarios $\xi_1 = 1 \quad \mathbb{P}(\xi = \xi_1) = 3/4$

$\xi_2 = 3 \quad \mathbb{P}(\xi = \xi_2) = 1/4$

Generic form:

$$\underset{x \in \mathbb{R}}{\text{minimize}} \left\{ 2x + \mathbb{E}_\xi \left[Q(x,\xi) \right] \text{ s.t. } x \geq 0 \right\}$$

$$Q(x,\xi) = \min_{y \in \mathbb{R}} \left\{ \xi y \quad \text{s.t.} \quad \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

From a), we have

$$Q(x,\xi_1) = Q(x,1) = \max(1-x,0)$$

and $Q(x,\xi_2) = \xi_2 Q(x,1) = 3\max(1-x,0)$

$$\mathbb{E}_\xi\left[Q(x,\xi)\right] = \mathbb{P}(\xi=\xi_1)\, Q(x,\xi_1) + \mathbb{P}(\xi=\xi_2)\, Q(x,\xi_2)$$

$$= \frac{3}{4}\max(1-x,0) + \frac{1}{4}\times 3\max(1-x,0)$$

$$= \frac{3}{2}\max(1-x,0)$$

The problem can be rewritten as

$$\min_{x \in \mathbb{R}} \left\{ 2x + \frac{3}{2}\max(1-x,0) \quad \text{s.t.} \quad x \geq 0 \right\}$$

Another formulation: Use two second-stage variables y_1 and y_2

$$\min_{\substack{x \in \mathbb{R} \\ y_1 \in \mathbb{R} \\ y_2 \in \mathbb{R}}} \left\{ 2x + \underbrace{\frac{3}{4}}_{\mathbb{P}(\xi=\xi_1)} \times y_1 + \frac{1}{4}\times 3y_2 \quad \text{s.t.} \quad \begin{array}{l} x \geq 0 \\ y_1 \geq 1-x \\ y_1 \geq 0 \\ y_2 \geq 1-x \\ y_2 \geq 0 \end{array} \right\}$$

$$Q(x,\xi_1) = \min_{y_1 \in \mathbb{R}} \left\{ y_1 \text{ s.t. } \begin{array}{l} y_1 \geq 1-x \\ y_1 \geq 0 \end{array} \right\}$$

→ The second formulation has more variables and constraints but it is a linear program, unlike the first one which is only piecewise linear

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Solving linear two-stage stochastic programs

↳ Recall: LP duality

(P) $\underset{x \in \mathbb{R}^n}{\text{minimize}}$ $c^T x$ s.t. $Ax = b$, $x \geq 0$

$c \in \mathbb{R}^n$
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

The dual of (P) is

(D) $\underset{u \in \mathbb{R}^m}{\text{maximize}}$ $b^T u$ s.t. $A^T u \leq c$

Key result: Exactly one of three cases below occurs

i) Both (P) and (D) are infeasible

ii) One of the problems is infeasible, and the other is unbounded

iii) Both (P) and (D) are feasible, and for any (x^*, u^*) such that x^* solves (P) and u^* solves (D), $c^T x^* = b^T u^*$

while for any x feasible for (P) and any u feasible for (D), we have $c^T x \geq b^T u$.

↳ Consider a linear two-stage stochastic program

$c \in \mathbb{R}^n$
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$
$q \in \mathbb{R}^{n_1}$
$T \in \mathbb{R}^{m_1 \times n}$
$W \in \mathbb{R}^{m_1 \times n_1}$
$h \in \mathbb{R}^{m_1}$

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \left\{ c^T x + \mathbb{E}_\xi\left[Q(x,\xi)\right] \text{ s.t. } \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}$$

where $Q(x,\xi) = \underset{y \in \mathbb{R}^{n_1}}{\min} \left\{ q^T y \text{ s.t. } \begin{array}{l} Tx + Wy = h \\ y \geq 0 \end{array} \right\}$

$$\xi = (q, T, W, h)$$

↳ We assume that ξ is represented by K scenarios $\xi_1, \dots, \xi_K$ with $\mathbb{P}(\xi = \xi_k) = p_k$, $\sum_{k=1}^{K} p_k = 1$, $p_k \geq 0$

(NB: Equivalent to considering that ξ follows a discrete distribution).

Then, using one second-stage variable per scenario, the problem can be reformulated as

$$\underset{\substack{x \in \mathbb{R}^n \\ y_1, \dots, y_K \in \mathbb{R}^{n_1}}}{\text{minimize}} \quad c^T x + \sum_{k=1}^{K} p_k q_k^T y_k \quad \text{s.t.} \quad \begin{array}{ll} Ax = b & \\ \boxed{T_k x + W_k y_k = h_k} & k = 1..K \\ x \geq 0 & \\ y_k \geq 0 & k = 1..K \end{array}$$

Key observation: The y_k variables are only linked through x and the constraints $T_k x + W_k y_k = h_k$

For any x such that $Ax=b$ and $x \geq 0$, we define the problems

$$(P_k^x) \quad \underset{y_k \in \mathbb{R}^{m_1}}{\text{minimize}} \; p_k q_k^T y_k \quad \text{s.t.} \quad W_k y_k = h_k - T_k x, \quad y_k \geq 0$$

and

$$P_k(x) = \underset{y_k \in \mathbb{R}^{m_1}}{\min} \left\{ p_k q_k^T y_k \;\; \text{s.t.} \;\; W_k y_k = h_k - T_k x, \; y_k \geq 0 \right\}$$

For fixed x, the problems $(P_1^x), \ldots, (P_K^x)$ can be solved independently $\Rightarrow$ Idea behind the L-shaped method

Algorithm: L-shaped method

- Pick $x^0 \in \mathbb{R}^n$, typically by solving

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; c^T x \quad \text{s.t.} \quad Ax=b, \; x \geq 0$$

- For $i = 0, 1, \ldots$

  $\rightarrow$ Solve $(P_1^{x^i}), \ldots, (P_K^{x^i})$ independently

  and set $UB = c^T x^i + \sum_{k=1}^{K} P_k(x^i)$

  ($\triangle$ Can be infinite if one of the problems is infeasible)

↳ Compute x^{i+1} and $\{z_k^{i+1}\}_{k=1..K}$ by solving

$$\underset{\substack{x\in\mathbb{R}^n \\ z_1,\dots,z_K\in\mathbb{R}^{n_1}}}{\text{minimize}} \quad c^T x + \sum_{k=1}^{K} z_k$$

subject to $\quad Ax = b$

$\qquad\qquad\qquad x \geq 0$

At iteration i, $K(i+1)$ constraints

One constraint per feasible problem $(P_k^{x^j})$

$$(u_k^j)^T (T_k x^j - T_k x) + P_k(x^j) \leq z_k$$

if $(P_k^{x^j})$ is feasible

$\forall k=1..K, \ \forall j=0...i$

one constraint per infeasible problem

$$(u_k^j)^T (h_k - T_k x) \leq 0$$

if $(P_k^{x^j})$ infeasible

$\forall k=1..K$

$\forall j=0...i$

↳ Set $LB = c^T x^{i+1} + \sum_{k=1}^{K} z_k^{i+1}$

↳ Stop when $UB - LB$ is small enough

↳ why those inequalities?

For every problem $(P_k^{x^i})$, u_k^i will be a dual variable associated with the problem.

$(P_h^{x^i})$ $\quad \underset{y_h \in \mathbb{R}^{m_1}}{\text{minimize}} \; p_h q_h^T y_h \quad$ s.t. $\quad W_h y_h = h_h - T_h x$

$$y_h \geq 0$$

The dual of $(P_h^{x^i})$ is

$$\underset{u \in \mathbb{R}^{m_1}}{\text{maximize}} \; (h_h - T_h x^i)^T u \quad \text{s.t.} \quad W_h^T u \leq p_h q_h$$

1) If $(P_h^{x^i})$ is infeasible, then the dual is unbounded and there exists $u_h^i \in \mathbb{R}^{m_1}$ such that $(h_h - T_h x^i)^T u_h^i > 0$ (and u_h^i is feasible)

$\Rightarrow$ To avoid this situation in future iterations, we add the constraint $(h_h - T_h x)^T u_h^i \leq 0$) "Feasibility cut"

$\Rightarrow$ Guarantees that $x^{i+1} \neq x^i$

2) If $(P_h^{x^i})$ is feasible, then if u_h^i is a dual solution, we have

$$\underbrace{P_h(x^i)}_{\substack{\text{optimal value} \\ \text{for } (P_h^{x^i})}} = \underbrace{u_h^{i\,T}(h_h - T_h x^i)}_{\substack{\text{optimal value for} \\ \text{the dual}}}$$

We add $\quad P_h(x) \geq (u_h^i)^T (h_h - T_h x)$ as a constraint, which can be rewritten as

$$P_h(x) \geq (u_h^i)^T (T_h x^i - T_h x) + P_h(x^i)$$

(NB: Subgradient inequality for the function $x \mapsto P_h(x)$)

→ "Optimality cut"

Metrics for solution quality of stochastic programs

↳ Consider a two-stage stochastic program

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + \mathbb{E}_\xi\left[Q(x,\xi)\right] \quad \text{s.t. } x \in X$$

where $Q(x,\xi)$ is defined as the optimal value of a second-stage problem

Q) How can we show that this formulation is better than other ways of handling uncertainty? Is this even the case for all problems?

Notation: Given any uncertainty vector ξ, define

$$x(\xi) \in \underset{x \in \mathbb{R}^n}{\text{argmin}} \left\{ f(x) + Q(x,\xi) \text{ s.t. } x \in X \right\}$$

ξ is fixed to a given value instead of being considered uncertain

↳ The two-stage approach decides x ahead of knowing the value of ξ, by solving $\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) + \mathbb{E}_\xi\left[Q(x,\xi)\right]$ s.t. $x \in X$

⇒ Defines the value of the recourse problem (VRP or RP)

$$RP = \min_{x \in \mathbb{R}^n} f(x) + \mathbb{E}_\xi\left[Q(x,\xi)\right] \text{ s.t. } x \in X$$

↳ If one waits to observe ξ before solving the problem, the average optimal value of that problem is

$$WS = \mathbb{E}_{\xi}\left[f(x(\xi)) + Q(x(\xi), \xi) \right]$$

optimal value of

solution of $\left|\text{minimize } f(x) + Q(x,\xi) \text{ s.t. } x \in X\right.$

This is called the (expected) wait-and-see solution or WS.

↳ $WS \leq RP$ in general (typically for convex problems) but equality is rare!

Def: the expected value of perfect information (EVPI) is defined as $EVPI = RP - WS$

→ The EVPI represents the cost of not knowing the value of ξ at decision time.

→ Hard to assess the EVPI from the formula of the problem, even in simple cases

→ But computing the EVPI for different choices of scenarios helps understanding the interest of those choices.

↳ The expected value solution (EV) is defined as

$$EV = \min_{x \in \mathbb{R}^n} \{ f(x) + Q(x, \underbrace{\mathbb{E}_\xi[\xi]}_{=\bar{\xi}}) \quad s.t. \; x \in X \}$$

→ Alternative to two-stage formulation: If can compute $\bar{\xi}$, replace the uncertainty by its average

$$Q(x, \mathbb{E}_\xi[\xi]) \neq \mathbb{E}_\xi[Q(x,\xi)]$$

→ Comparing EV and RP directly does not reflect the fact that

- With EV, you compute a solution $x(\bar{\xi})$ and it will be used later on
- With RP, you compute a solution knowing its impact for all values of ξ

→ For this reason, we compare RP to

$$\underline{EEV} = \mathbb{E}_\xi \left[f(x(\bar{\xi})) + Q(x(\bar{\xi}), \xi) \right]$$

"Expected result of the Expected Value solution"

Average of the actual objective value

solution computed by fixing the uncertainty to $\bar{\xi}$

NB: $RP = f(x^*) + \mathbb{E}_\xi[Q(x^*, \xi)]$

with $x^* \in \arg\min_x f(x) + \mathbb{E}_\xi[Q(x,\xi)] \; s.t. \; x \in X$

↳ $RP \leq EEV$ (but not equality in general)

Def: The value of the stochastic solution (VSS) is defined as $VSS = EEV - RP$

→ Measures how much you loose by approximating the uncertainty with its average as opposed to using a two-stage formulation

→ Like the EVPI, hard to distinguish instances with large VSS from instances with small VSS.

→ $VSS \leq EV$ and $EVPI \leq EEV - EV$

Examples of EVPI / VSS calculations

1) Consider the exercise problem

$$\underset{x \in \mathbb{R}}{\text{minimize}} \left\{ 2x + \mathbb{E}_{\xi}[Q(x,\xi)] \text{ s.t. } x \geq 0 \right\}$$

$$Q(x,\xi) = \min_{y \in \mathbb{R}} \left\{ \xi y \text{ s.t. } \begin{array}{l} y \geq 1-x \\ y \geq 0 \end{array} \right\}$$

$$= \begin{cases} \xi \max(1-x, 0) & \text{if } \xi \geq 0 \\ -\infty & \text{if } \xi < 0 \end{cases}$$

Suppose that $\mathbb{P}(\xi = 1) = 3/4$ and $\mathbb{P}(\xi = 3) = 1/4$

$$WS = \mathbb{E}_\xi \left[\min_{x\in\mathbb{R}} \{ 2x + Q(x,\xi) \ \text{s.t.}\ x \geq 0 \} \right]$$

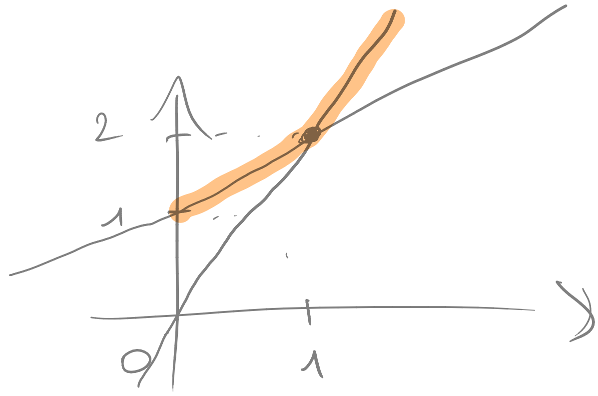
$$= \mathbb{P}(\xi = 1)\, V(1) + \mathbb{P}(\xi = 3)\, V(3)$$

where $V(\xi) = \min_{x\in\mathbb{R}} \{ 2x + Q(x,\xi) \ \text{s.t.}\ x\geq 0 \}$

$$V(1) = \min_{x\in\mathbb{R}} \{ 2x + \max(1-x, 0) \ \text{s.t.}\ x \geq 0 \}$$

$$= \min_{x\geq 0} \{ \max(1+x, 2x) \ \text{s.t.}\ x\geq 0 \}$$

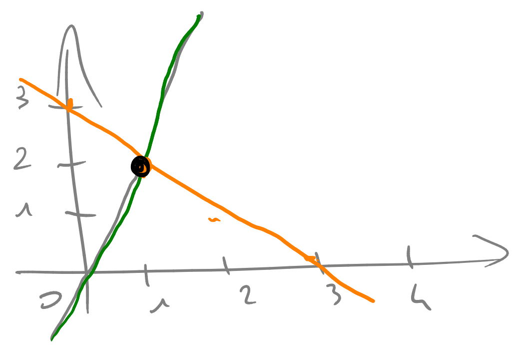
$$= 1$$



$$V(3) = \min_{x\in\mathbb{R}} \{ 2x + \max(1-x, 0) \ \text{s.t.}\ x\geq 0 \}$$

$$= \min_{x\in\mathbb{R}} \{ \max(3-x, 2x) \ \text{s.t.}\ x \geq 0 \}$$

$$= 2$$



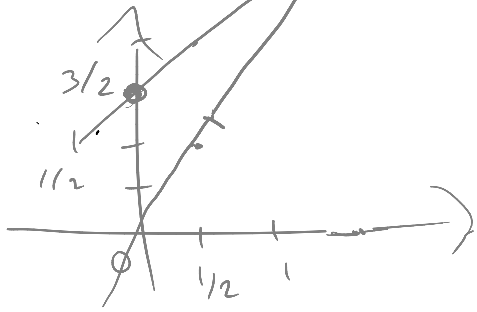
$$WS = \mathbb{P}(\xi = 1) \times 1 + \mathbb{P}(\xi = 3) \times 2 = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$$

$$RP = \min_{x \in \mathbb{R}} \left\{ 2x + \mathbb{E}_\xi\left[Q(x,\xi)\right] \text{ s.t. } x \geq 0 \right\}$$

$$= \min_{x \in \mathbb{R}} \left\{ 2x + \frac{3}{2}\max(1-x,0) \text{ s.t. } x \geq 0 \right\}$$

$$= \min_{x \in \mathbb{R}} \left\{ \max\left(\frac{3}{2} + \frac{x}{2}, 2x\right) \text{ s.t. } x \geq 0 \right\}$$

$$= \frac{3}{2} > WS$$



$$EVPI = \frac{3}{2} - \frac{5}{4} = \frac{1}{4}$$

$$\hookrightarrow \quad \bar{\xi} = \mathbb{E}_\xi[\xi] = \mathbb{P}(\xi = 1) \times 1 + \mathbb{P}(\xi = 3) \times 3 = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$$

$$EV = V(3/2) = \min_{x \in \mathbb{R}} \left\{ 2x + \frac{3}{2}\max(1-x,0) \text{ s.t. } x \geq 0 \right\}$$

$$= RP = 3/2$$

$$x(\bar{\xi}) = \operatorname{argmin}_{x \in \mathbb{R}} \left\{ 2x + \frac{3}{2}\max(1-x,0) \text{ s.t. } x \geq 0 \right\} = 0$$

$$EEV = \mathbb{E}_\xi\left[2x(\bar{\xi}) + Q(x(\bar{\xi}), \xi) \right]$$

$$= 2x(\bar{\xi}) + \mathbb{E}_\xi\left[Q(x(\bar{\xi}), \xi) \right]$$

$$= 2 \times 0 + \mathbb{P}(\xi = 1) \times \max(1 - x(\bar{\xi}), 0)$$

$$+ \mathbb{P}(\xi = 3) \times 3\max(1 - x(\bar{\xi}), 0)$$

$$= \frac{3}{4} \times 1 + \frac{1}{4} \times 3 \times 1 = 3/2 = RP$$

Conclusion: $VSS = 0$ No added value in considering the recourse problem vs replacing ξ with its expected value.

NB: Here $Q(x, \bar{\xi}) = E_\xi[Q(x,\xi)]$

2) "Pathological case" from "Introduction to Stochastic Programming" J.R. Birge, F. Louveaux 2011 (Second Edition)

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad x_1 + 4x_2 + Q(x,\xi) \quad \text{s.t.} \quad x_1 + x_2 = 1, \; x \geq 0$$

$$Q(x,\xi) = \min_{y \in \mathbb{R}^3} \left\{ y_1 + 10y_2 + 10y_3 \quad \text{s.t.} \quad y_1 + y_2 - y_3 = \xi + x_1 - 2x_2, \; y_1 \leq 2, \; y \geq 0 \right\}$$

① ξ has a discrete distribution

$$\mathbb{P}(\xi = 0) = \mathbb{P}(\xi = 3/2) = \mathbb{P}(\xi = 3) = 1/3$$

then $EVPI = 3$ $\quad VSS = 0$

$EEV = 13/2 \neq EV = 7/2$

Key: For fixed ξ, the problem has multiple solutions

(2) ξ is uniformly distributed in $[1, 3]$

Then $EVPI = 0$ and $VSS = \frac{11}{4}$

Key: $x^* = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ is a solution for any ξ

Exercise

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad \mathbb{E}_\xi \left[Q(x, \xi) \right] \quad \text{s.t.} \quad x_1 + x_2 = 1, \quad x \geq 0$$

$$Q(x, \xi) = \begin{cases} -x_1 & \text{if } \xi = \xi_1 \\ 0 & \text{if } \xi \neq \xi_1 \end{cases}$$

$\mathbb{P}(\xi = \xi_1) = p$, $\mathbb{P}(\xi = \xi_2) = 1 - p$ $\qquad \xi \in \{\xi_1, \xi_2\}$

Compute the EVPI and the VSS of the problem.