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ORIGINAL PAPER

Pareto-efficiency of ordinal multiwinner voting rules

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Abstract

We investigate the Pareto-efficiency of ordinal multiwinner voting rules, that is, voting rules based on ordinal preference profiles over candidates. Defining Pareto-optimality of a committee requires relating the voters' rankings over individual candidates to their preferences over committees. We consider two well-known extension principles that extend rankings over candidates to preferences over committees: the responsive extension and the lexicographic extension. As the responsive extension outputs partial orders, we consider two Pareto-optimality notions: a committee is possibly (respectively, necessary) Pareto-optimal if it is Pareto-optimal for some (respectively, every) completion of these partial orders. As the lexicographic extension principle outputs a total order, it leads to only one Pareto-optimality notion. We then define several notions of Pareto-efficiency of multiwinner rules, depending on whether some (respectively, all) committees in the output are Pareto-optimal for one of the latter notions. We review what we believe to be a complete list of ordinal multiwinner rules that have been studied in the literature, and identify which Pareto-efficiency notions they satisfy. Our finding is that, somewhat surprisingly, these rules show a huge diversity: some satisfy the strongest notion, some do not even satisfy the weakest one, with many other rules at various intermediate levels.

Keywords Extension principle · Multi-winner rules · Committees

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1 Introduction

Multiwinner voting rules (also called committee rules) are natural generalizations of single-winner voting rules. They are useful in a variety of situations, from shortlisting to proportional representation and group recommendation. See (Faliszewski et al. 2017) for a survey.

A multiwinner voting rule outputs a set of k candidates, also called a *committee*, for some integer k. The literature distinguishes two important families of multiwinner rules, depending on the format of the input: those that are based on approval votes, and those that are based on ordinal votes (each voter ranking the candidates). *We focus on the latter family.*

The recent literature extensively discusses axiomatic (and computational) properties of multiwinner voting rules. One axiomatic property that has been neglected is *Pareto-efficiency*. The reasons for this negligence are easy to explain.

If the input of the rule consists of approval ballots, a (rough) way of measuring the satisfaction of a voter by a committee is to count how many approved candidates it contains. This is the path followed by Lackner and Skowron (2020), who show that most well-studied approval-based committee rules are Pareto-efficient in this sense, with the noticeable exception of rules defined by a greedy, sequential selection process.

If the input of the rule is ordinal, then things become more difficult. When considering *single-winner* voting rules with ordinal input, Pareto-efficiency is easy to define, since voters' preferences about candidates are given in the input (and it is usually easy to tell whether a rule satisfies it or not). With multiwinner rules, this is quite different: while the input allows to say how each voter ranks single candidates, *it does generally not allow to say how they rank committees*.[1]

A first way of going around this difficulty would consist in generalizing Pareto-efficiency to committees in the following straightforward way: a committee is Pareto-optimal if whenever a candidate x Pareto-dominates a candidate y, then y can be on the committee only if x is there too.[2] Although this is a very plausible necessary condition for Pareto-optimality to hold, as a definition it is too weak: consider the following profile P, with four candidates a, b, x, y and four voters whose preferences are respectively $a \succ x \succ b \succ y, b \succ x \succ a \succ y, a \succ y \succ b \succ x$ and $b \succ y \succ a \succ x$, then all four candidates are Pareto-efficient, yet the committee $\{a, b\}$ clearly dominates the committee $\{c, d\}$.

A second way consists in considering an extension principle lifting preferences from single alternatives to committees: a *preference extension* maps a ranking over candidates to a partial order over committees. Once we know how to order committees, we can apply the classical Pareto-dominance directly on committees. Such extension principles have been frequently used in various subdomains of social choice, including

[1] As there are exponentially many committees if k is variable, asking voters to rank committees explicitly is not usually considered an option.

[2] We are grateful to an anonymous reviewer for this suggestion, and the way of coping with it, which we quote almost verbatim.

irresolute rules, fair division of indivisible items, many-to-one matching, and hedonic games. Three different interpretations of sets of candidates (or alternatives) have been considered, and discussed in Barbera et al. (2004): final outcomes (all elements in the set are jointly obtained), complete uncertainty (only one of the elements in the set is obtained in the end, and nature will decide which one), or opportunities (only one of the elements in the set is obtained in the end, and the concerned agent can choose which one). The first of these three interpretations is *conjunctive* while the other two are *disjunctive*. In the committee election setting (as in fair division, matching and hedonic games), the interpretation that prevails is the conjunctive one: a subset of candidates S is seen as a *joint set of candidates* (as opposed to the choice, by nature or by the agent, of one alternative within S).

Aziz et al. (2016); Aziz and Monnot (2020) consider several extension principles. For each of them, they study the computational complexity of determining whether a committee is Pareto-optimal, of computing *some* Pareto-optimal committee, and when possible, they give simple characterisations of Pareto-optimal committees. They do not, however, consider the following question: given a multiwinner rule f and a preference extension E, does f always output committees that are Pareto-optimal with respect to E?

Some of the common preference extension principles extend rankings over singletons to *partial* orders over committees. In such a case, we are sometimes not able to say whether a committee is preferred by a voter to another one. It is however possible to consider, for such extension principles, two modal notions: *possible* and *necessary Pareto-optimality* and *efficiency*.[3] A committee is possibly Pareto-optimal if it is Pareto-optimal for *some* completion of these partial preferences, and necessarily Pareto-optimal if it is Pareto-optimal for *all* completions of these partial preferences. (Obviously, when an extension principle outputs a complete preference relation then both notion coincide.) These notions of possible and necessary Pareto-optimality with respect to an extension principle carry on to multiwinner voting rules: a voting rule is necessarily (respectively, possibly) Pareto-efficient if all the committees it outputs are necessarily (respectively, possibly) Pareto-optimal. As the rules we consider are irresolute, we also introduce a weaker notion: a rule if weakly possibly Pareto-efficient if some of its output committees is possibly Pareto-optimal.

Our aim is to study the Pareto-efficiency of most well-studied ordinal multiwinner voting rules under two classical preference extension principles that are especially relevant for our setting.

The central preference extension principle we consider is the *responsive extension*, which is particular suitable to the context of multiwinner elections, since it amounts to assume that voters have additively decomposable preferences over committees. The responsive extension can be seen as the ordinal counterpart of additivity. It has been introduced for the first time by [6], in the context of one-to-many matching, and studied further in Roth and Sotomayor (1990); Bossert (1995). It is arguably the most suitable preference extension principle under the conjunctive interpretation, and has been used several times in this context, especially in matching (Khare et al. 2021; Belahcène et al. 2021), fair division (Aziz et al. 2015; Bouveret et al. 2010;

[3] We use the term *optimality* for committees and the term *efficiency* for rules.

Aziz et al. 2019; Segal-Halevi et al. 2020), committee selection (Aziz et al. 2016; Aziz and Monnot 2020) and coalition formation (Lucchetti et al. 2022; Kerkmann et al. 2020) As it produces a partial order, we will consider its "possible" and "necessary" versions, defined by quantifying over complete extensions of these partial preferences. On the profile P introduced above, $\{a, b\}$ necessarily Pareto-dominates $\{x, y\}$: if every voter has an additively decomposable preference consistent with their ordinal preferences over single candidates, then whatever the choice of the utility values, $\{a, b\}$ Pareto-dominates $\{x, y\}$. If we replace the fourth vote by $x \succ y \succ b \succ a$, then $\{b, y\}$ is not necessarily Pareto-optimal: if the second voter has utility values $u_i(b) = 4, u_i(x) = 3, u_i(a) = 2, u_i(y) = 0$, the third voter has utility values $u_i(a) = 4, u_i(y) = 2, u_i(b) = 1, u_i(x) = 0$, and the fourth voter has utility values $u_i(x) = 4, u_i(y) = 2, u_i(b) = 1, u_i(a) = 0$, then $\{a, x\}$ Pareto-dominates $\{b, y\}$; however, it is possibly Pareto-optimal: if the second voter has utility values $u_i(b) = 4, u_i(x) = 2, u_i(a) = 1, u_i(y) = 0$ and the third voter has utility values $u_i(a) = 5, u_i(y) = 4, u_i(b3 = 1, u_i(x) = 0$, then $\{b, y\}$ is Pareto-optimal (whatever the utility values for voters 1 and 4).

Beyond the responsive extension, we also consider the *lexicographic* (or *leximax*) extension principle. The lexicographic extension has been introduced in Bossert (1995), and used in voting contexts in Klamler et al. (2012); Lang et al. (2018); Aziz et al. (2019, 2016); Aziz and Monnot (2020). As it is complete, Pareto-efficiency is directly applicable (so that we do not need to distinguish between possible and necessary Pareto-optimality) and as it is a refinement of the responsive extension, Pareto-efficiency for the lexicographic extension is stronger than possible Pareto-efficiency and weaker than necessary Pareto-efficiency.

We give below a few explanations about why we do not consider other extension principles.

- The lexicographic extension principle is a refinement of the 'best' extension principle (Aziz et al. 2016), which orders committees only according to their best element. This extension principle is rougher, less sensitive to voters' preferences, than the lexicographic extension principle and is therefore less interesting. Still, as it is a coarsening of it, Pareto-efficiency for the lexicographic extension implies Pareto-efficiency for the 'best' extension, so all our positive results about lexicographic extension implies Pareto-efficiency for the 'best' extension.
- Instead of focusing on the best element we could focus on the worst element and define a *leximin* extension principle, defined exactly as the (or *leximax*) principle, but starting from the worst committee members instead of the best ones; this is a refinement of the 'worst' extension principle (Aziz et al. 2016). It has been argued in several places (see, *e.g.*, Skowron et al. (2016) and the references therein) that in multiwinner voting, focusing on the best element (the best representative of a voter in a committee, her best item or nearest facility in a set) generally makes more sense than focusing on the worst element. Also, beyond focusing on the best or the worst elements, there's a continuum of possibilities (Skowron et al. (2016); see in particular the discussion in Section 3); as they are less common we will not consider them here.

- We do not consider extension principles that make sense for the disjunctive interpretation of sets (used, for instance, for studying the axiomatic properties of irresolute rules), but that make little or no sense for the "sets of final outcomes" interpretation: Fishburn's, Gärdenfors' and Kelly's (Barbera et al. 2004).

Our results allow us to classify rules into six classes, according to their level of Pareto-efficiency:

- *Class 6*: those for which all winning committees are necessarily Pareto-optimal for the responsive extension principle (from now on we will simply say "necessarily Pareto-optimal"): they are said to be *necessarily Pareto efficient* (NPE). Such rules are rare; still, the *perfectionist* rule (Faliszewski et al. 2018), that outputs the set S of k candidates with the largest number of voters whose set of preferred k candidates is exactly S (in any order), is NPE, together with a set of rules that are highly similar to it, in a sense that will be made clear further.
- *Class 5*: those for which winning committees are lexicographically Pareto-optimal (they are said to be *lexicographically Pareto efficient*, or LPE) but not always necessarily Pareto-optimal. This is the case for some committee scoring rules, as well as for sequential rules such as Single Transferable Vote (STV), sequential plurality and sequential Chamberlin-Courant.
- *Class 4*: those that may output committees that are not lexicographically Pareto-optimal, but ensure that for any input profile, at least one of the committees in the output is lexicographically Pareto-optimal (these rules are said to be *weakly lexicographically Pareto-efficient*).
- *Class 3*: those that are not LPE, but for which all winning committees are possibly Pareto-optimal for the responsive extension principle (from now on we will simply say "possibly Pareto-optimal"); they are said to be *possibly Pareto efficient*, or PPE. This class contains compromise rules, all committee scoring rules with a strictly decreasing scoring function that fail lexicographic Pareto-efficiency, and more generally a large fraction of committee scoring rules.
- *Class 2*: those that may output committees that are not possibly Pareto-optimal, but ensure that for any input profile, at least one of the committees in the output is possibly Pareto-optimal (these rules are said to be *weakly possibly Pareto-efficient*). This is notably the case for committee scoring rules that are not PPE, including Single Non-Transferable Vote (SNTV), Bloc, and most variants of Chamberlin-Courant rules.
- *Class 1*: finally, some rules are not even weakly possibly Pareto-efficient. This class includes, typically, Condorcetian rules such as Number of External Defeats (NED) or Minimum Size of External Opposition (SEO).

The paper is organized as follows. In Sect. 2 we give the necessary background and provide a brief taxonomy of known ordinal multiwinner voting rules. Preference extensions are introduced in Sect. 3. In Sect. 4, we formally define the properties of possible, lexicographic and necessary Pareto-optimality (respectively efficiency) of committees (respectively multiwinner rules). Results dealing with the lexicographic preference extension are given in Sect. 5. The possible and the necessary Pareto-efficiency of multiwinner rules under the responsive extension are respectively investigated in Sects. 6

and 7. Section 8 concludes the paper with a summary of results together with comments on further research.

2 Background

2.1 Basic notions and notation

Consider a set of voters N with $|N| = n \geq 2$ confronting a set of candidates A with $|A| = m \geq 3$. $\mathcal{L}(A)$ is the set of linear orders (or rankings) over A.

A profile P is a collection of *votes*, each vote being a linear order over candidates: $P = (\succ_1, \ldots, \succ_n) \in \mathcal{L}(A)^n$.

To avoid overloaded notation, when writing votes we often omit the $\succ$ symbol: for instance, the vote $a \succ b \succ c \succ d$ is simply written $abcd$. Also, when several votes in a profile are identical, we sometimes use the following notation: $P = (2 \times abcd, 3 \times dcba)$ is the profile containing 3 votes $a \succ b \succ c \succ d$ and 2 votes $d \succ c \succ b \succ a$.

If $k \in \{1, ..., m-1\}$ then $\mathcal{S}_k(A) = \{X \subset A : |X| = k\}$. A member of $\mathcal{S}_k(A)$ is called a *committee* of size k, or a k-committee. We generally omit curly brackets when writing committees, *e.g.*, if $A = \{a, b, c, d\}$ then we note $\mathcal{S}_2(A) = \{ab, ac, ad, bc, bd, cd\}$ instead of $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.

$W(\mathcal{S}_k(A))$ is the set of weak orders (complete, reflexive and transitive relations) over $\mathcal{S}_k(A)$ and $\Pi(\mathcal{S}_k(A))$ is the set of strict partial orders (transitive and asymmetric relations) over $\mathcal{S}_k(A)$. The strict partial order associated with $\sqsupseteq_i \in W(\mathcal{S}_k(A))$ is defined as usual by $x \sqsupset_i y$ if $x \sqsupseteq_i y$ and not $y \sqsupseteq_i x$. If $\sqsupseteq_1, \ldots, \sqsupseteq_n$ are weak orders over $\mathcal{S}_k(A)$, $Q = (\sqsupseteq_1, \ldots, \sqsupseteq_n)$ is called a weak order profile over $\mathcal{S}_k(A)$. Given a strict partial order $\sqsupset_i$ over $\mathcal{S}_k(A)$, a *completion* of $\sqsupset_i$ is a complete weak order $\sqsupseteq_i$ over $\mathcal{S}_k(A)$ such that for all $x, y \in A$, if $x \sqsupset_i y$ then $x \sqsupseteq_i y$.

2.2 Multi-winner rules

A multiwinner rule is a function f that, given a profile P and an integer $k \leq m$, outputs a nonempty subset of $\mathcal{S}_k(A)$. We list here a few prominent multiwinner rules that have been well-studied in the literature. The most recent survey on multiwinner rules defined from ordinal profiles is Faliszewski et al. (2017).

2.2.1 Committee scoring rules (CSR)

Committee scoring rules were first defined in Elkind et al. (2017). Given a vote $\succ_i$ and a candidate c, we denote by $pos(c, \succ_i)$ the position of c in $\succ_i$ (the top-ranked candidate has position 1, the one ranked last has position m). Given $S \in \mathcal{S}_k(A)$, the position of S in $\succ_i$, denoted by $pos(S, \succ_i)$ is the sequence of positions of the members of S sorted increasingly. We denote by $[m]_k$ the set of all size-k increasing sequences of elements from $\{1, \ldots, m\}$. For $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ in $[m]_k$, we say that $I \succeq J$ if for each $t = 1, \ldots, k$ we have $i_t \leq j_t$. We write $I \succ J$ for $I \succeq J$ and not $J \succeq I$.

A *committee scoring function* $\gamma_{m,k} : [m]_k \to \mathbb{R}$ associates each committee position with a score and satisfies *monotonicity*: if $I \succeq J$ then $\gamma_{m,k}(I) \geq \gamma_{m,k}(J)$. Moreover, $\gamma_{m,k}$ is *strict* if $\gamma_{m,k}(I) > \gamma_{m,k}(J)$ whenever $I, J \in [m]_k$ are such that $I \succ J$.

Given $\gamma_{m,k}$ and profile P, the committee scoring rule (CSR) $f_{\gamma_{m,k}}$, which we will write simply f_γ by abuse of notation, outputs committees S maximising

$$score(S, P) = \sum_{i=1}^{n} \gamma_{m,k}(pos(S, \succ_i))$$

A CSR f_γ is strict if γ is strict. A few well-known particular CSRs are listed below:

- A committee scoring function γ is *additively decomposable* if

  $$\gamma_{m,k}(i_1, \ldots, i_k) = \sum_{j=1}^{k} \gamma_m(i_j)$$

  for some function $\gamma_m = \{1, \ldots, m\} \to \mathbb{R}$. If γ is additively decomposable then f_γ outputs the k candidates x maximizing $score(x, P) = \sum_{i=1}^{n} \gamma_m(pos_{\succ_i}(x))$. Such a CSR is the natural multiwinner extension of a single-winner positional scoring rule; it is called a *best-k* CSR. Well-known examples of best-k CSR are Single Non-Transferable Vote (SNTV), defined by $\gamma_m(1) = 1$ and $\gamma_m(j) = 0$ for each $j > 1$, and k-Borda, defined by $\gamma_m(j) = m - j$ for all $j = 1, \ldots, m$.
- if $\gamma_{m,k}(i_1, \ldots, i_k) = |\{j : i_j \leq k\}|$ then f_γ is the *Bloc* rule. In words, the Bloc rule outputs the candidates listed most often in the top k candidates of the votes. Note that, although $\gamma_{m,k}$ is additively decomposable, Bloc is not a best-k rule, because $\gamma_{m,k}$ depends on k.
- if $\gamma_{m,k}(1, \ldots, k) = 1$ and $\gamma_{m,k}(I) = 0$ for all $I \neq (1, \ldots, k)$ then f_γ is called the *perfectionist rule* (Faliszewski et al. 2018).
- let $s = (s_1, \ldots, s_m)$ with $s_1 \geq \ldots \geq s_m$ and $s_1 > s_m$. The Chamberlin-Courant k-multiwinner rule associated with scoring vector s, denoted by s-CC, is the CSR defined by $\gamma_{m,k}(i_1, \ldots, i_k) = s_{i_1}$. If s is the Borda vector, defined by $s_i = m - i + 1$ for every i then f_γ is the *Borda-Chamberlin-Courant* rule (β-CC for short).
- A family of rules, which contains both Chamberlin-Courant and best-k rules, is obtained by using an ordered weighted average (OWA) to compute the satisfaction of an agent: the score of her jth best candidate in the selection is weighted by w_j (Skowron et al. 2016). For a reason that will become clear in Sect. 5, we consider a specific rule in this family, a lexicographic refinement of β-CC, which we denote by β-CC*: let $\varepsilon < \frac{1}{nm}$, then

  $$\gamma_{m,k}(i_1, \ldots, i_k) = \sum_{j=1}^{k} \varepsilon^{j-1}(m - i_j).$$

  It can be checked easily that the winning committees do not depend on ε and that β-CC* is a refinement of β-CC.

Example 2.1 Let $A = \{a, b, c, d, e, f\}$, $n = 10$, $k = 2$, and $P = (4 \times fedbca, 3 \times abcdef, 2 \times bcaedf, 1 \times dcabef)$.

- $SNTV(P) = \{af\}$: f and a, in this order, are the two candidates ranked first in the largest number of votes.
- $2-\text{Borda}(P) = \{bc, bd\}$: b has the highest Borda score, followed by c and d (tied).
- $Bloc(P) = \{be, bf\}$: b is ranked in the top 2 positions in 5 votes; e and f, in 4 votes (and other candidates, in at most 3 votes).
- β-CC$(P) = \{af, bf\}$: 7 votes have a or f in first position, and for the other 3, the better candidate among a and f is in third position: the β-CC score of $\{a, f\}$ is $7 \times 5 + 3 \times 3 = 44$. Next, 6 votes have b or f in first position, 3 in second position, in one on fourth position: the β-CC score of $\{b, f\}$ is $6 \times 5 + 3 \times 4 + 2 = 44$. It can be checked that all other committees of size 2 have a smaller β-CC score.
- β-CC$^*(P) = \{bf\}$: the tie between the tied winning committees for β-CC is resolved by looking at the position of the second best (that is: worst!) committee member in all votes. For $\{a, f\}$, this second best candidate appears in the last position in all votes, while for $\{b, f\}$, it appears in the last position in 6 votes and in position 4 in 4 votes.
- the perfectionist rule applied to P outputs $\{ef\}$: 4 votes have $\{e, f\}$ as their top two elements, and no set of two candidates does better.

2.2.2 Condorcetian rules

Two ways of extending the Condorcet criterion from single winners to candidates are discussed in Aziz et al. (2017): Gehrlein stability (a committee S is Gehrlein stable if every $x \in S$ majority defeats every $y \in A \setminus S$) (Gehrlein 1985; Kamwa 2017), and local stability for quota q (S is locally stable for quota q if for any $y \in A \setminus S$, at least qn voters prefer some candidate in X to y (Elkind et al. 2015)).

A rule is Gehrlein-consistent if it elects the (unique) Gehrlein stable committee whenever there exists one. Two specific Gehrlein-consistent rules are NED (for "number of external defeats") and SEO (for "size of external opposition") (Coelho 2004), that can be seen as the respective multiwinner counterparts of the Copeland and maximin single-winner rules. The NED rule outputs committees S that maximize the number of pairs $(x, y) \in S \times A \setminus S$ such that x majority-beats y in P. The SEO rule outputs committees S that maximise $\min_{x \in S, y \in A \setminus S} |\{i : x \succ_i y\}|$. We give only one locally stable rule: the maximal θ-winning sets rule (Elkind et al. 2015), also called the *locally stable extension of maximin* (LSE-maximin) in Aziz et al. (2017): it outputs the sets S that are locally stable for the maximal possible quota q.

Continuing Example 2.1:

- there is no Gehrlein-stable committee; there is however a unique *weak* Gehrlein stable committee: each of b and c defeats a, e and f, and weakly defeats d. This leads to NED(P) = SEO$(P) = \{bc\}$:
- LSE-maximin$(P) = \{af\}$: $\{a, f\}$ is locally stable for $q = \frac{7}{10}$, because 8 voters prefer either a or f to b, 7 voters prefer either a or f to c, 9 voters prefer either a or f to d, and all voters prefer either a or f to e; and no committee does better or equally good.

2.2.3 Compromise rules

Let $\alpha \in [0, 1)$. For a given profile P, for each alternative x, the *compromise index of x with respect to α and P*, $\lambda(\alpha, P, x)$, is the smallest integer j such that x appears in the first j positions in more than αn votes: that is, x appears in the first $\lambda(\alpha, P, x)$ positions in more than αn votes but in the first $\lambda(\alpha, P, x) - 1$ positions in at most αn votes.

The compromise rule MC_k^α (Sertel 1986; Yilmaz and Sertel 1999) identifies the smallest integer j such that there exist at least k alternatives with $\lambda(\alpha, P, x) \leq j$, and then outputs the k alternatives with the smallest values of $\lambda(\alpha, P, x)$; in case of a tie, meaning that there are more alternatives with $\lambda(\alpha, P, x) = j$ than necessary, the tie is broken according to the number of voters who rank them in the first j positions.

Note that $MC_k^{1/2}$, called *majoritarian compromise*, is a multiwinner version of the *Bucklin* rule.

Continuing Example 2.1: let us first take $\alpha = 1/3$. We have

$$\begin{array}{lll} \lambda(1/3, P, a) = 3 & \lambda(1/3, P, b) = 2 & \lambda(1/3, P, c) = 3 \\ \lambda(1/3, P, d) = 3 & \lambda(1/3, P, e) = 2 & \lambda(1/3, P, f) = 1 \end{array}$$

We have $j = 2$. There is a tie between b and e, resolved in favour of b, since b and e are ranked in the first 3 positions by respectively 5 and 4 voters. Therefore, $MC_k^{1/3}(P) = \{bf\}$.

Let us now take $\alpha = 1/2$. We have $j = 3$ and $MC_k^{1/2}(P) = \{ac\}$ (no tie-breaking is needed).

2.2.4 Sequential rules

There are several variants of the multiwinner version of *single transferable vote* (STV). We present the most common one: let $q = \lceil \frac{n}{k} \rceil$ (quota). If some candidate x has a plurality score $S(x) \geq q$, then x is elected, and each of the votes for x becomes a fractional vote with weight $1 - \frac{q}{S(x)}$, with x removed; otherwise the candidate with the lowest plurality score is eliminated from all votes (using tie-breaking if necessary). This operation is repeated until k candidates have been elected.

Sequential plurality elects first the plurality winner (using tie-breaking if necessary), removes it from the list of candidates, then elects the plurality winner from the obtained profile, and so on until k candidates have been elected.

Given a scoring vector $s = (s_1, \ldots, s_m)$ with $s_1 \geq \ldots \geq s_m$ and $s_1 > s_m$, and a subset of candidates T with $|T| \leq k$, let

$$score^s_{CC}(T, \succ) = \sum_{i=1}^{n} \max_{y \in T} s_{pos_{\succ_i}}(y).$$

If $|T| < k$ and $x \notin T$, $score^s_{CC}(x|T, \succ) = score^s_{CC}(x \cup T, \succ) - score^s_{CC}(T, \succ)$ is the marginal score of x with respect to T and $\succ$. *Greedy s-Chamberlin-Courant* (*s*-

GCC) elects the k winning candidates in sequence, including at each step the candidate with the largest marginal score with respect to the candidates already included.

- it first elects y_1 maximising $score^s_{CC}(\{y_1\}, \succ)$
- then, for each $j \in \{2, \ldots, k\}$, it elects y_j maximising

$$score^s_{CC}(\{y_j | \{y_1, \ldots, y_{j-1}\}, \succ).$$

Note that sequential plurality coincides with s-GCC for $s = (1, 0, \ldots, 0)$.

For all these sequential rules, if ties occur, then all possibilities for resolving them are taken into account (which is sometimes called the "parallel universe" assumption).

Continuing Example 2.1:

- $STV(P) = \{af\}$: the quota is 5, no candidate reaches it; c and e are eliminated, still no candidate reaches the quota; d is eliminated, then b.
- SeqPlu$(P) = \{ef\}$: f is elected first, then e.
- β−GCC$(P) = \{be\}$: the Borda winner b is selected first, and e gives the highest marginal score given that b has been selected.

3 Preference extensions

3.1 Extension principles

An *extension principle* is a function $E : \mathcal{L}(A) \to \Pi(\mathcal{S}_k(A))$; informally, E maps a linear order over candidates to a strict order over committees of size k. We note $\succ_i^E$ for $E(\succ_i)$. The implicit assumption is that i's actual preference $\sqsupset_i$ is compatible with $\succ_i^E$, or in other words, that it is one of its completions. We write, for each $\succ_i \in \mathcal{L}(A)$,

$$\kappa^E(\succ_i) = \{\sqsupset_i \in \ W(\mathcal{S}_k(A)) \ : \sqsupset_i \ \text{ is a completion of } \ \succ_i^E\}$$

and, by a slight abuse of notation, for each profile $P = (\succ_1, \ldots, \succ_n)$,

$$\kappa^E(P) = \kappa^E(\succ_1) \times \ldots \times \kappa^E(\succ_n).$$

Remark 1 Let E be an extension principle. For every distinct $X, Y \in \mathcal{S}_k(A)$ and every $\succ_i \in \mathcal{L}(A)$, the following three statements are equivalent:

1. $X \succ_i^E Y$;
2. $X \sqsupset_i Y$ for all $\sqsupset_i \in \kappa^E(\succ_i)$;
3. $X \sqsupseteq_i Y$ for all $\sqsupseteq_i \in \kappa^E(\succ_i)$.

Proof (1) $\implies$ (2) and (2) $\implies$ (3) can be directly observed. To see (3) $\implies$ (1), suppose $X \succ_i^E Y$ fails for some $X, Y \in \mathcal{S}_k(A)$ and some $\succ_i \in \mathcal{L}(A)$. By definition of κ^E, there exists some $\sqsupseteq_i \in \kappa^E(\succ_i)$ with $Y \sqsupset_i X$, establishing the failure of (3). □

The choice of an extension principle depends before all of the interpretation of the set of objects. See (Barbera et al. 2004) for an extensive discussion on this topic.

Multi-winner elections, whose output is typically a committee, or a set of projects built for the community, correspond to the conjunctive interpretation (*sets of objects as final outcomes*, cf. Section 5 of Barbera et al. (2004)). Under this interpretation, two prominent extension principles are the *responsive* and the *lexicographic* principles, which we define below.

3.2 The responsive extension principle

The responsive extension principle, which we denote by ρ, was introduced in [6]. It says that for any subset A of candidates containing x and not containing y, if B is obtained from A by replacing x by y, then B is preferred to A if and only if y is preferred to x.

Formally, given any $X, Y \in \mathcal{S}_k(A)$ and any $\succ_i \in \mathcal{L}(A)$, we say that Y is an elementary improvement for X at $\succ_i$ if and only if $Y = (X \setminus \{x\}) \cup \{y\}$ for some $x \in X$ and $y \in A \setminus X$ with $y \succ_i x$.

The responsive extension ρ of $\succ_i$ is then defined as the inclusion-wise smallest transitive relation satisfying $Y \succ_i^\rho X$ whenever Y is an elementary improvement for X at $\succ_i$. Equivalently, $Y \succ_i^\rho X$ if and only if there is a sequence of sets X_0,..., X_t in $\mathcal{S}_k(A)$ where $X_0 = X$, $X_t = Y$, and X_{s+1} is an elementary improvement for X_s at $\succ_i$ for each $s \in \{0, ..., t-1\}$. We say that Y is an improvement for X at $\succ_i$.

The responsive extension $\succ_i^\rho$ can be characterised equivalently by stochastic dominance. For any $h \in \{1, ..., |X|\}$, we write $r_h(X; \succ_i) \in X$ for the hth ranked alternative in $X \subseteq A$ at $\succ_i \in \mathcal{L}(A)$. At each $\succ_i \in \mathcal{L}(A)$ and for any distinct $X, Y \in \mathcal{S}_k(A)$, we define the stochastic dominance relation $\sigma^k(\succ_i)$ over $\mathcal{S}_k(A)$ as $X\ \sigma^k(\succ_i)\ Y$ iff $r_h(X; \succ_i) = r_h(Y; \succ_i)$ or $r_h(X; \succ_i) \succ_i r_h(Y; \succ_i)$ for all $h \in \{1, ..., k\}$.

Lemma 3.1 *For every distinct $X, Y \in \mathcal{S}_k(A)$ and every $\succ_i \in \mathcal{L}(A)$, the following five statements are equivalent:*

(i) $X\ \sigma^k(\succ_i)\ Y$;
(ii) *X is an improvement for Y at $\succ_i$;*
(iii) $X \sqsupset_i Y$ *for all* $\sqsupseteq_i \in \kappa^\rho(\succ_i)$;
(iv) $X \sqsupseteq_i Y$ *for all* $\sqsupseteq_i \in \kappa^\rho(\succ_i)$.

Proof The equivalence of (i) and (ii) is shown in Aziz et al. (2015) (Theorem 1). The equivalence between (ii), (iii) and (iv) is a consequence of Remark 1. □

Example 3.1 Let $A = \{a, b, c, d\}$, $k = 2$, and $a \succ_1 b \succ_1 c \succ_1 d$. The responsive extension $\kappa^\rho(\succ_1)$ is the partial order on $S_2(A)$ depicted on Fig. 1.

Note that $\kappa^\rho(\succ_1)$ has three completions: one where $\{a, d\} \sqsupset \{b, c\}$, one where $\{b, c\} \sqsupset \{a, d\}$, and one where $\{a, d\} \sqsupseteq \{b, c\}$ and $\{b, c\} \sqsupseteq \{a, d\}$.

3.3 The lexicographic extension principle

The lexicographic extension principle lex associates with every $\succ_i \in \mathcal{L}(A)$ the linear order $lex(\succ_i) = \succ_i^{lex} \in \mathcal{L}(\mathcal{S}_k(A))$ defined by: for all $X \in S_k(A)$ and $Y \in S_k(A)$, $X \succ_i^{lex} Y$ if and only if for some $h^* \in \{1, ..., k\}$, the following two conditions hold:

$$\begin{tikzcd}
 & & \{b,c\} \arrow[dr] & & \\
\{a,b\} \arrow[r] & \{a,c\} \arrow[ur] \arrow[dr] & & \{b,d\} \arrow[r] & \{c,d\} \\
 & & \{a,d\} \arrow[ur] & &
\end{tikzcd}$$

Fig. 1 Responsive extension of $a \succ_1 b \succ_1 c \succ_1 d$

$$\{a,b\} \longrightarrow \{a,c\} \longrightarrow \{a,d\} \longrightarrow \{b,c\} \longrightarrow \{b,d\} \longrightarrow \{c,d\}$$

Fig. 2 Lexicographic extension of $a \succ_1 b \succ_1 c \succ_1 d$

- $r_{h^*}(X; \succ_i) < r_{h^*}(Y; \succ_i)$, and
- $r_h(X; \succ_i) = r_h(Y; \succ_i)$ for all $h \in \{1, ..., h^* - 1\}$.

Unlike the responsive extension, the lexicographic extension outputs a total order on $\mathcal{S}_k(A)$, which is one of the completions of the responsive extension. Therefore:

Remark 2 $X \succ_i^{lex} Y$ implies $X \succ_i^{\rho} Y$.

Example 3.2 Let again $A = \{a, b, c, d\}$, $k = 2$, and $a \succ_1 b \succ_1 c \succ_1 d$. The lexicographic extension $\succ_i^{lex}$ of $\succ_1$ is depicted on Fig. 2.

4 Pareto-optimality and Pareto-efficiency

As already mentioned, Pareto-optimality is a property conditional to the choice of an extension principle.

4.1 Lexicographic Pareto-optimality and Pareto-efficiency

As the lexicographic extension principle generates a linear order over $\mathcal{S}_k(A)$, Pareto-optimality is defined in a natural way.

Definition 1 Given a profile $P = (\succ_i)_{i \in N} \in \mathcal{L}(A)^n$, and two committees $X, Y \in S_k(A)$, Y lexicographically Pareto-dominates X at P if $Y \succ_i^{lex} X$ holds for every $i \in N$, and X is lexicographically Pareto optimal at P if it is not lexicographically Pareto-dominated by any other committee in $S_k(A)$.

Note that this formulation is equivalent to "$Y \succeq_i^{lex} X$ for all i and $Y \succ_i^{lex} X$ for some i" because $Y \succ_i^{lex} X$ is asymmetric.

Example 4.1 Let $P = (efbdca, abcdef)$.

- $\{a, b\}$ is lexicographically Pareto optimal at P, because it is the most preferred committee for the second voter.
- $\{a, e\}$ is lexicographically Pareto optimal at P: the first voter lexicographically prefers $\{a, e\}$ to any committee that does not contain e, and the second voter lexicographically prefers $\{a, e\}$ to any committee that does not contain a.

- $\{b, e\}$ is lexicographically Pareto optimal at P: the only committee that the first voter lexicographically prefers to $\{b, e\}$ is $\{e, f\}$, but the second voter lexicographically prefers $\{b, e\}$ to $\{e, f\}$.
- $\{c, e\}$ is not lexicographically Pareto optimal at P, as it is lexicographically Pareto-dominated by $\{b, e\}$.
- $\{b, f\}$ and $\{b, d\}$ are not lexicographically Pareto optimal at P, as they are lexicographically Pareto-dominated by $\{a, e\}$.

4.2 Pareto-optimality and Pareto-efficiency under the responsive extension

As the responsive extension generates only a partial order over committees, we cannot directly apply Pareto-optimality. One classical way of extending to collections of partial orders a notion that usually applies to collections of total orders consists in quantifying over completions. Possible and necessary Pareto-efficiency correspond respectively to existential and universal quantification (see (Brams et al. 2003; Bouveret et al. 2010; Aziz et al. 2015, 2019)).

We start by defining Pareto-dominance. We recall that ρ is the responsive extension principle.

Definition 2 Let $X, Y \in \mathcal{S}_k(A)$.

- Y *Pareto-dominates* X at $Q = (\sqsupseteq_1, \ldots, \sqsupseteq_n) \in W(\mathcal{S}_k(A))^N$ if $Y \sqsupseteq_j X$ for all $j \in N$ and $Y \sqsupset_i X$ for some $j \in N$.
- Y *possibly Pareto-dominates* X at $P \in \mathcal{L}(A)^N$ for ρ if Y Pareto-dominates X at Q for some $Q \in \rho(P)$.
- Y *necessarily Pareto-dominates* X at $P \in \mathcal{L}(A)^N$ for ρ if Y Pareto-dominates X at Q for every $Q \in \rho(P)$.

We now define Pareto optimality.

Definition 3 Let $X \in \mathcal{S}_k(A)$.

- X is *Pareto optimal* at $Q \in W(\mathcal{S}_k(A))^N$ if and only if there is no $Y \in \mathcal{S}_k(A)$ such that Y *Pareto-dominates* X at Q.
- X is *necessarily Pareto optimal* at $P \in \mathcal{L}(A)^N$ for ρ if and only if X is Pareto optimal at every $Q \in \rho(P)$.
- X is *possibly Pareto optimal* at $P \in \mathcal{L}(A)^N$ for ρ if and only if X is Pareto optimal at some $Q \in \rho(P)$.

Remark 2 implies the following:

Proposition 1

1. *if X is necessarily Pareto at $P \in \mathcal{L}(A)^N$ for ρ then X is lexicographically Pareto optimal at P.*
2. *if X is lexicographically Pareto optimal at P then X is possibly Pareto at $P \in \mathcal{L}(A)^N$ for ρ.*

The following characterizations will be particularly useful.

Theorem 2 *Let $X \in \mathcal{S}_k(A)$ and $P \in \mathcal{L}(A)^N$. The following statements are equivalent:*

1. *X is necessarily Pareto optimal at P for ρ.*
2. *There is no $Y \in \mathcal{S}_k(A)$ that possibly Pareto-dominates X at P for ρ.*
3. *For every $Y \in \mathcal{S}_k(A) \setminus \{X\}$, there exists $i \in N$ such that $X \succ_i^\rho Y$ holds.*

Proof Let $X \in \mathcal{S}_k(A)$ and $P \in \mathcal{L}(A)^N$.

- We first show that 1 implies 2. Assume there exists $Y \in \mathcal{S}_k(A)$ which possibly Pareto-dominates X at P for ρ. By definition, this means that there exists $\widetilde{Q} = (\widetilde{\sqsupseteq}_1, \ldots, \widetilde{\sqsupseteq}_n) \in \kappa^\rho(P)$ at which Y Pareto-dominates X. Hence, X is not Pareto optimal at every $Q \in \kappa^\rho(P)$, which shows that X is not necessarily Pareto optimal at P for ρ.
- We show that 2 implies 3. Assume there exists $Y \in \mathcal{S}_k(A)\backslash\{X\}$ such that $X \succ_i^\rho Y$ fails for all $i \in N$. This implies the existence of $Q = (\sqsupseteq_1, \ldots, \sqsupseteq_n) \in \kappa^\rho(P)$ such that $Y \sqsupseteq_i X$ for all $i \in N$. By Remark 3.1, this is equivalent to saying that there exists $Q = (\sqsupset_1, \ldots, \sqsupset_n) \in \kappa^\rho(P)$ such that $Y \sqsupset_i X$ for all i. Thus, Y Pareto-dominates X at some $Q \in \kappa^\rho(P)$, hence Y possibly Pareto-dominates X.
- Finally, we show that 3 implies 1. Assume that for every $Y \in \mathcal{S}_k(A)\backslash\{X\}$ there exists $i(Y) \in N$ such that $X \succ_{i(Y)}^\rho Y$. By Remark 3.1, for every $Y \in \mathcal{S}_k(A)\backslash\{X\}$ there exists $i(Y) \in N$ such that $X \sqsupset_{i(Y)} Y$ for every $\sqsupseteq_{i(Y)} \in \kappa^\rho(\succ_{i(Y)})$. Pick any $Y \in \mathcal{S}_k(A)\backslash\{X\}$ and any $Q = (\sqsupseteq_1, \ldots, \sqsupseteq_n) \in \kappa^\rho(P)$. Since $X \sqsupset_{i(Y)} Y$ for some $i(Y) \in N$, Y does not Pareto-dominates X at Q. As this holds for any $Y \in \mathcal{S}_k(A)\backslash\{X\}$, X is Pareto optimal at Q for ρ. Finally, as the argument applies to any $Q \in \kappa^\rho(P)$, X is necessarily Pareto optimal at P for ρ.

□

Theorem 3 *Let $X \in \mathcal{S}_k(A)$ and $P \in \mathcal{L}(A)^N$. The following statements are equivalent:*

1. *X is possibly Pareto optimal at P for ρ.*
2. *There is no $Y \in \mathcal{S}_k(A)$ that necessarily Pareto-dominates X at P for ρ.*
3. *For every $Y \in \mathcal{S}_k(A)$, there exists $i \in N$ such that $Y \succ_i^\rho X$ fails.*

Before proving Theorem 3, we introduce a definition and several simple lemmas. Given $\succ_i \in \mathcal{L}(A)$, $X \in \mathcal{S}_k(A)$, and $\sqsupseteq_i \in \kappa(\succ_i^\rho))$, we say that $\sqsupseteq_i^{X+}$ is an X-best completion of $\succ_i^\rho$ if for any $Y \in \mathcal{S}_k(A)$, if $Y \succ_i^\rho X$ does not hold then $X \sqsupseteq_i^{X+} Y$. Moreover, we say that $Q^{X+} = (\sqsupseteq_1^{X+}, \ldots, \sqsupseteq_n^{X+})$ is an X-best completion of $P^\rho = (\succ_1^\rho, \ldots, \succ_n^\rho)$ if for every i, $\sqsupseteq_i^{X+}$ is an X-best completion of $\succ_i^\rho$.

Lemma 4.1 *There exists an X-best completion of $\rho(P)$.*

Proof For each i, consider the relation $\rhd_i = \succ_i^\rho \cup \{(X, Y) | Y \succ_i^\rho X \text{ does not hold }\}$. $\rhd_i$ is acyclic: if it had a cycle, since $\succ_i^\rho$ is acyclic, the cycle would be contain a pair (X, Y) such that $Y \succ_i^\rho X$ does not hold, and thus would contain a path from X to Y in $\succ_i^\rho$; because $\succ_i^\rho$ is transitive, this would contradict the fact that $Y \succ_i^\rho X$ does not hold. Therefore the transitive closure $\rhd_i$ is a strict partial order, and any of its completions is a X-best completion of $\succ_i^\rho$. This being true for every i, there exists an X-best completion of $\rho(P)$. □

Lemma 4.2 *For any $i \in N$ and $X, Y \in \mathcal{S}_k(A)$, $Y \sqsupseteq_i^{X+} X$ holds for some X-best completion of $\succ_i^{\rho}$ if and only if $Y \sqsupseteq_i X$ for all completions $\sqsupseteq_i$ of $\succ_i^{\rho}$.*

Proof The right-to-left direction is trivial. From left to right, assume that $Y \sqsupseteq_i X$ fails in some completion $\sqsupseteq_i$ of $\succ_i^{\rho}$. Then $X \sqsupset_i Y$, which implies that $Y \succ_i^{\rho} X$ does not hold; by definition of an X-best completion $\sqsupseteq_i^{X+}$, we have $X \sqsupset_i^{X+} Y$, and therefore $Y \sqsupseteq_i^{X+} X$ does not hold. □

Lemma 4.3 *For any $i \in N$ and $X, Y \in \mathcal{S}_k(A)$, $X \sqsupseteq_i Y$ holds for some completion of $\succ_i^{\rho}$ if and only if $X \sqsupseteq_i^{X+} Y$ holds for some X-best completion of $\succ_i^{\rho}$.*

Proof The right-to-left direction is trivial. From left to right, assume that for some X-best completion of $\succ_i^{\rho}$, $X \sqsupseteq_i^{X+} Y$ does not hold. Then, by definition of an X-best completion, we have $Y \succ_i^{\rho} X$, therefore $X \sqsupseteq_i Y$ does not hold in any completion of $\succ_i^{\rho}$. □

Lemma 4.4 *$X \in \mathcal{S}_k(A)$ is Pareto-dominated at some X-best completion Q^{X+} of $\rho(P)$ if and only if X is not possibly Pareto-optimal.*

Proof The right-to-left direction is a direct consequence of Lemma 4.1. From left to right, assume $X \in \mathcal{S}_k(A)$ is Pareto-dominated at some X-best completion $Q = (\sqsupseteq_1^{X+}, \ldots, \sqsupseteq_n^{X+})$ of $\rho(P)$. Then there is an $Y \in \mathcal{S}_k(A)$ such that $Y \sqsupseteq_i^{X+} X$ for all i, and $Y \sqsupset_i^{X+} X$ for some i. By Lemma 4.2, $Y \sqsupseteq_i^{X+} X$ implies $Y \sqsupseteq_i X$ for all completions $\sqsupseteq_i$ of $\succ_i^{\rho}$. $Y \sqsupset_i^{X+} X$ implies that $X \sqsupseteq_i^{X+} Y$ does not hold, and by Lemma 4.3, $X \sqsupseteq_i Y$ holds for no completion of $\succ_i^{\rho}$, therefore $Y \sqsupseteq_i X$ holds for any completion of $\succ_i^{\rho}$. This allows us to conclude that if X is Pareto-dominated at all $Q \in \kappa^{\rho}(P)$, which is equivalent to saying that is not possibly Pareto-optimal. □

Now we are ready to prove Theorem 3.

Proof
- We show by contradiction that 1 implies 2. Assume there is an $Y \in \mathcal{S}_k(A)$ that necessarily Pareto-dominates X at P for ρ. Then for any completion Q of $\rho(P)$, Y Pareto-dominates X at Q, which implies that X is not Pareto-optimal at Q. This being true for all Q, X is not possibly Pareto-optimal at P.
- We show by contradiction that 2 implies 3. Assume there is $Y \in \mathcal{S}_k(A)$ such that $Y \succ_i^{\rho} X$ holds for all i: then for all i, and for any extension $\sqsupseteq_i$ of $\succ_i^{\rho}$, we have $Y \sqsupseteq_i X$. By Remark 1, this is equivalent to saying that for all i, and for any extension $\sqsupseteq_i$ of $\succ_i^{\rho}$, we have $Y \sqsupset_i X$. Therefore X is necessarily Pareto-dominated by Y.
- We show by contradiction that 3 implies 1. Assume X is not possibly Pareto-optimal: then by Lemma 4.4, X is Pareto-dominated by some $Y \in \mathcal{S}_k(A)$ at some X-best completion $Q^{X+} = (\sqsupseteq_1^{X+}, \ldots, \sqsupseteq_n^{X+})$ of P^{ρ}, which implies that for all i we have $Y \sqsupseteq_i^{X+} X$. By Lemmas 4.3 and 4.4, for all i we have $Y \sqsupseteq_i X$ and all completions $\sqsupseteq_i$ of $\succ_i^{\rho}$, which implies that $Y \succ_i^{\rho} X$ holds.

□

Example 4.2 Consider again profile $P = (efbdca, abcdef)$.

- $\{a, b\}$ is necessarily Pareto optimal at P, because it is the most preferred committee for the second voter.
- $\{e, b\}$ is necessarily Pareto optimal at P: because of the first voter, the only committee that can possibly Pareto-dominate it is $\{e, f\}$; but the second voter necessarily prefers $\{e, b\}$ to $\{e, f\}$;
- $\{a, e\}$ is not necessarily Pareto optimal at P, as it is possibly Pareto-dominated by $\{b, c\}$. As it is lexicographically Pareto optimal it is *a fortiori* possibly Pareto optimal.
- $\{b, d\}$ is not lexicographically Pareto optimal at P, because it is lexicographically Pareto-dominated by $\{a, e\}$. Therefore it is not necessarily Pareto optimal. It is possibly Pareto optimal: if it was necessarily Pareto-dominated by another committee, this would be $\{b, e\}$, $\{b, f\}$, $\{d, e\}$, $\{d, f\}$ or $\{e, f\}$ because of the first voter; but the second voter necessarily prefers $\{b, d\}$ to all of these.
- $\{d, f\}$ is not possibly Pareto optimal at P, as it is necessarily Pareto-dominated by $\{b, e\}$.

We make the following useful observation. We denote by $top(k, \succ_i)$ the set of candidates ranked in position $1, \dots, k$ in $\succ_i$.

Proposition 4 *Given any $P \in \mathcal{L}(A)^N$ and any $i \in N$:*

1. *$top(k, \succ_i)$ is necessarily Pareto optimal at P;*
2. *let $X = top(k+1, \succ_i) \setminus \{x\}$ for some $x = \{r_t(A; \succ_i)\}$ with $t \in \{1, \dots, k\}$. Then X is necessarily Pareto optimal at P if and only if $y \succ_j x$ for some $j \in N$ and $y \in top(k+1, \succ_i) \setminus top(t, \succ_i)$.*

Proof For 1, observe that $top(k, \succ_i)$ is the most preferred committee by i: there is no $Y \in \mathcal{S}_k(A)$ such that $Y \neq X$ and $Y \sqsupseteq_i X$: $top(k, \succ_i)$ cannot be possibly Pareto-dominated, and is therefore necessarily Pareto optimal at P.

For 2, let $X = top(k+1, \succ_i) \setminus \{x\}$ for some $x = \{r_t(A; \succ_i)\}$ with $t \in \{1, \dots, k\}$. Assume X is necessarily Pareto optimal at P: in particular, for any i, it is not possibly Pareto-dominated by $top(k, \succ_i)$, which means that there is a j such that $X \sqsupseteq_j top(k, \succ_i)$. Since X is obtained from $top(k, \succ_i)$ by replacing $\{r_t(A; \succ_i)\}$ by $\{r_{k+1}(A; \succ_i)\}$, this means that $\{r_t(A; \succ_i)\} \succ_j x$ for some $t \in \{1, \dots, k\}$. □

The sets described by Proposition 4 need not be the only ones that are necessarily Pareto-optimal. To see this, let $k = 2$ and take the profile $(abcdef, dcbaef, efadbc)$. The committee ad is necessarily Pareto-optimal: the committees which the first voter possibly prefers to ad are ab, ac and bc; however the second voter necessarily prefers ad to ab and ac, and the third voter necessarily prefers ad to bc.

We have defined so far possible and necessary Pareto optimality of a *committee* with respect to some extension principle. For multiwinner rules, we have to take irresoluteness into account. We define the following five levels of efficiency:[4]

[4] We do not define a weak version of necessary Pareto-efficiency, which would have little interest: we do not know any interesting rule that is guaranteed to output *some* necessarily Pareto-optimal committee, but that fails necessary Pareto-efficiency. (Of course, such rules exist: for instance, the rule that outputs all possible committees).

$$\begin{tikzcd}
& \text{necessary Pareto-efficiency (NPE)} \arrow[d] & \\
& \text{lexicographic Pareto-efficiency (LPE)} \arrow[dl] \arrow[dr] & \\
\text{possible Pareto-efficiency (PPE)} \arrow[dr] & & \text{weak lexicographic Pareto-efficiency (WLPE)} \arrow[dl] \\
& \text{weak possible Pareto-efficiency (WPPE)} &
\end{tikzcd}$$

Fig. 3 Relations between the five levels of Pareto-efficiency

Definition 4 Given an extension principle E, and a multiwinner voting rule f, we say that

- f is *necessarily Pareto-efficient* for E if for any profile P over A, every $S \in f(P)$ is necessarily Pareto-optimal for E.
- f is *possibly Pareto-efficient* for E if for any profile P, every $S \in f(P)$ is possibly Pareto-optimal for E.
- f is *weakly possibly Pareto-efficient* for E if for any profile P, some $S \in f(P)$ is possibly Pareto-optimal for E.
- f is *lexicographically Pareto-efficient* if for any profile P, every $S \in f(P)$ is lexicographically Pareto-optimal.
- f is *weakly lexicographically Pareto-efficient* if for any profile P, some $S \in f(P)$ is lexicographically Pareto-optimal.

Based on the implication relationships between the notions of Pareto-optimality as stated in Proposition 1, Fig. 3 shows the logical relations between the five levels of Pareto-efficiency.

As we see further, for some rules, weak possible Pareto-efficiency serves to guarantee possible Pareto-efficiency even for pathological profiles.

When $k = 1$, f becomes a single-winner rule. Then possible and necessary Pareto-efficiency reduce to standard Pareto-efficiency: if f is an irresolute single-winner voting rule f with ordinal input, f is *Pareto-efficient* if for every profile P, every $x \in f(P)$ is Pareto-optimal (there is no y such that $y \succ_i x$ for all i). Let us say that f is *weakly Pareto-efficient* if for every profile P, some $x \in f(P)$ is Pareto-optimal.

As far as we can tell, all irresolute single-winner rules that have received some attention in the literature satisfy at least weak Pareto-efficiency. Most of them satisfy the stronger Pareto-efficiency property; a few exceptions are some positional scoring rules with a scoring vector that is not strictly decreasing (such as k-approval for $k \geq 2$),[5] as well as maximin, and tournament solutions such as the Top Cycle and the Banks set (Laslier 1997).

[5] For an exact characterization of Pareto-efficient positional scoring rules see (Llamazares and Peña 2015).

5 Lexicographic Pareto-efficiency

We now have all the elements that we need so as to proceed with Pareto-efficiency of various multiwinner rules. We start by lexicographic Pareto-efficiency. For a reason that will become clear soon, we consider sequential rules first.

5.1 Sequential rules

Given any $P_i \in \mathcal{L}(A)$ together with $B \subseteq A$, define $1(\succ_i, B)$ as the top candidate in B for $\succ_i$. We introduce below a property of multiwinner rules which has its own interest: *top-sequentiality* expresses that candidates are selected in a sequence, and at each step, the selected candidate is the most preferred candidate, among those that have not been selected yet, for at least one voter. For instance, sequential dictatorship — where a voter picks her preferred candidate, then a second voter (who can be the same one) picks her preferred candidate among those who remain, etc. — is obviously top-sequential. As we see below, many more interesting rules are top-sequential as well.

Definition 5 A multiwinner rule f is top-sequential if for all $P \in \mathcal{L}(A)^n$, $S = \{a_1, ..., a_k\} \in f(P)$, one can order candidates $a_1, ..., a_k$ so that $\forall h \in \{1, ..., k\}$, $a_h = 1(\succ_{i_h}, A \backslash \{a_1, ..., a_{h-1}\})$ for some $i_h \in N$.

Lemma 5.1

1. *Sequential plurality is top-sequential.*
2. *STV is top-sequential.*

Proof 1. at each step, the selected candidate maximizes the plurality score among the remaining candidates; therefore it is ranked first by at least one voter.

2. at each step where a candidate is selected, it reaches the quota, therefore it is ranked first by at least one voter.

□

Proposition 5 *Every top-sequential rule is LPE.*

Proof Let f be top-sequential and pick $P \in \mathcal{L}(A)^n$ and $X \in f(P)$. Writing $S = \{a_1, ..., a_k)$, and assuming elements of S are selected w.r.t. order $a_1 > ... > a_k$, the definition of top-sequentiality implies that for all $h \in \{1, ..., k\}$, $a_h = 1(\succ_{i_h}, A \backslash \{a_1, ..., a_{h-1}\})$ for some $i_h \in N$. If $S' \succ_i^{lex} S$ for all $i \in N$, one must have $a_1 \in S'$ (otherwise, by definition of ρ^{lex}, $S \succ_{i_1}^{lex} S'$). Replicating this argument for $a_2, ..., a_k$ shows that $S = S'$, which is impossible. Hence, S is lex-Pareto optimal at P, which shows that f satisfies LPE. □

As an immediate consequence of Proposition 5 and Lemma 5.1 we have

Proposition 6 *Sequential plurality and STV are LPE.*

Proposition 5 can also be used to prove that sequential dictatorships are LPE, which can also be obtained as a by-product of Theorem 2 in Aziz and Monnot (2020).

The last remaining sequential rule is greedy β-CC, for which we have a negative result.

Proposition 7 *Greedy β-CC is not WLPE.*

Proof Let $k = 2$ and $P = (axztyub, ayutxzb, bxztyua, byztxua)$. At first step, β-CC selects x or y, both with optimal Borda scores. Suppose it selects x (respectively y), then at step 2 it selects y (respectively x) with maximal marginal contribution. Therefore, the output is $\{x, y\}$, which is lexicographically Pareto-dominated by $\{a, b\}$. □

5.2 Committee scoring rules

We use Proposition 5 to prove that SNTV is "almost" LPE, in the sense that it is LPE when being restricted to profiles for which at least k alternatives are ranked first by some voter. When n is large enough compared to m, the fraction of profiles that satisfies this condition is close to 1.

Proposition 8 *SNTV is WLPE but not LPE. Its restriction to profiles where at least k alternatives are ranked first by some voter is LPE.*

Proof For any profile P let $Top(P)$ be the set of candidates that are ranked on top by at least one voter. Let $q = |Top(P)|$. If $q \geq k$, then SNTV is equivalent to the top-sequential rule that selects the candidates with the highest k plurality scores, and therefore, by Proposition 5, the restriction of SNTV to such profiles is LPE. If $q < k$, consider the following top-sequential rule: all q candidates with strictly positive plurality score are selected, and then the remaining $k-q$ candidates are voters 1's top $k-q$ candidates among those remaining. This rule is top-sequential by definition, and its outcome belongs to $SNTV(P)$, therefore SNTV is WLPE. SNTV is however not LPE, because of pathological profiles with less than k alternatives ranked first by some voter. For instance, if $k = 2$ and $P = (abcd, acdb)$, then SNTV outputs $\{ab, ac, ad\}$; while ab and ac are lexicographically Pareto-optimal, ad is lexicographically Pareto-dominated. □

Characterizing LPE and WLPE committee scoring rules appears to be difficult in the general case. As we already observed in Sect. 4, when $k = 1$, LPE coincides with standard Pareto-efficiency, and characterizing Pareto-efficient single-winner positional scoring rules is already not trivial (Llamazares and Peña 2015). Therefore, we should not expect to obtain an easy characterization of lexicographic Pareto-efficiency of multiwinner CSRs in the general case. But somewhat surprisingly, obtaining such a characterization is difficult even for the simple case of $k = 2$ and "best-k" committee scoring rules. To give an idea of the difficulty, let us restrict to additive CSRs and consider the case $k = 2$, $m = 4$.

An additive CSR for $k = 2$ is associated with an additive scoring function $\gamma_{m,2}$: there exists a non-increasing scoring vector $(s_1, \ldots, s_m)$, with $s_1 > s_m$, such that $\gamma_{m,2}(i, j) = s_i + s_j$. Without loss of generality, we assume $s_m = 0$. Given profile $\succ = (\succ_1, \ldots, \succ_n)$, recall that $Score(x, \succ) = \sum_{i=1}^{n} s_{pos(x,\succ_i)}$. We denote by f_2^s the corresponding additive CSR.

Proposition 9 *If $m = 4$ and $k = 2$, then f_2^s is LPE if and only if $s_1 > s_2 + s_3$ and $s_2 > s_3$.*

Proof If $s_1 \leq s_2+s_3$, take $\succ = (abcd, dcba)$. We have $Score(a, \succ) = Score(d, \succ) = s_1 \leq Score(b, \succ) = Score(c, \succ) = s_2 + s_3$, therefore $\{b, c\}$ is a winning committee, which is lexicographically Pareto-dominated by $\{a, d\}$. If $s_2 = s_3$, then the single-voter profile $(abcd)$ has $\{a, c\}$ as a winning committee although it is dominated by $\{a, b\}$. This shows the necessary part.

For the sufficiency part, assume that $s_1 > s_2 + s_3$ and $s_2 > s_3$, and let a profile $\succ$ such that $\{y_1, y_2\}$ is lexicographically Pareto-dominated by $\{x_1, x_2\}$.

If $\{y_1, y_2\} \cap \{x_1, x_2\} \neq \emptyset$ then without loss of generality, $\{y_1, y_2\} = \{x_1, y_2\}$ with $y_2 \neq x_1$. Since $\{x_1, y_2\}$ is lexicographically Pareto-dominated by $\{x_1, x_2\}$, y_2 is Pareto-dominated by x_2. If x_2 is ranked at least once in position 1 or 2, then $s_1 > s_2 + s_3$ and $s_2 > s_3$ imply that $Score(x_2, \succ) > Score(y_2, \succ)$. If x_2 is always ranked in position 3, then y_2 is always ranked in position 4 and the single winning committee consists of the other two candidates. In both cases, $\{y_1, y_2\}$ cannot be a winning committee.

Now, assume $\{y_1, y_2\} \cap \{x_1, x_2\} = \emptyset$. Because $\{y_1, y_2\}$ is lexicographically Pareto-dominated by $\{x_1, x_2\}$, each vote $\succ_i$ has the form $xyy'x'$ or $xyx'y'$ or $xx'yy'$, where $\{x, x'\} = \{x_1, x_2\}$ and $\{y, y'\} = \{y_1, y_2\}$. Then, $s_1 > s_2 + s_3$ implies that $Score(x_1, \succ) + Score(x_2, \succ) > Score(y_1, \succ) + Score(y_2, \succ)$. Thus, $\{y_1, y_2\}$ cannot be a winning committee. □

Replacing strict inequalities by weak inequalities in the proof of Proposition 9 leads to the characterization of WLPE best-k rules for $m = 4$ and $k = 2$: if $m = 4$, then f_2^s is WLPE if and only if $s_1 \geq s_2 + s_3$ and $s_2 \geq s_3$.

When m becomes larger, generalizing such a characterization becomes difficult.

As a corollary of Proposition 9, k-Borda is not LPE. It is WLPE for $m = 4$ and $k = 2$, but no longer if $m \geq 5$, as witnessed by the profile $(abcde-, ebcda-)$, for which the unique winning committee for $k = 2$ is $\{b, c\}$, which is lexicographically Pareto-dominated by $\{a, e\}$.

On the other hand, k-Harmonic, defined by the scoring vector $(1-\frac{1}{4}, \frac{1}{2}-\frac{1}{4}, \frac{1}{3}-\frac{1}{4}, 0)$, is LPE for $k = 2$ and $m = 4$. But it is not WLPE in the general case: let $k = 3$, $m = 14$, and $P = (atuv \dots bc, buvt \dots ca, ctuv \dots ab)$: the winning committee $\{u, u, v\}$ is lexicographically Pareto-dominated by $\{a, b, c\}$.

Moving away from best-k rules:

Proposition 10 *Bloc is not WLPE.*

Proof Let $k = 3$, $m = 6$, and $P = (axyzbc, byzxca, czxyab)$. The winning committee $\{x, y, z\}$ is lexicographically Pareto-dominated by $\{a, b, c\}$. □

Proposition 11 *β-CC is WLPE but not LPE.*

Proof If $S \in \beta\text{-CC}(P)$ is lexicographically Pareto-dominated, then some S' lexicographically dominates S: then, for each voter i, her best candidate in S' is at least as good as her best candidate in S, therefore the β-CC score of S' is no smaller than the β-CC score of S. We iterate this process until we reach a lexicographically optimal committee S^*, which is also in $\beta\text{-CC}(P)$. This implies that β-CC is WLPE.

Let $k = 2$, $m = 3$, and let P be the one-voter profile (abc). The winning committees for β-CC are $\{a, b\}$ and $\{a, c\}$; the latter is lexicographically Pareto-dominated . Therefore β-CC is WLPE but not LPE. $\square$

Proposition 12 *β-CC* is LPE.*

Proof Immediate from the fact that if S' lexicographically dominates S then it has a strictly larger β-CC* score. $\square$

5.3 Compromise rules

Proposition 13 *For any $\alpha \in (0, 1)$ and $k \geq 2$, MC_k^α is not WLPE.*

Proof Assume $\frac{1}{2} \leq \alpha < 1$. Let P be the following two-voter profile:

$$\begin{array}{l} 1 : z_1 \ldots z_{k-2}\ a\ u\ v\ b\ \ldots \\ 1 : z_1 \ldots z_{k-2}\ b\ v\ u\ a\ \ldots \end{array}$$

We have $\lambda(z_i, \alpha, P) = i$ for all $i = 1, \ldots, k-2$; $\lambda(u, \alpha, P) = \lambda(v, \alpha, P) = k+1$; and $\lambda(a, \alpha, P) = \lambda(b, \alpha, P) = k + 2$.

So $MC_k^\alpha(P) = \{z_1 \ldots z_{k-2}uv\}$ although $z_1 \ldots z_{k-2}ab$ lexicographically dominates $z_1 \ldots z_{k-2}uv$.

Now assume $0 < \alpha < \frac{1}{2}$. Let n be the smallest integer such that $n \geq \frac{1}{\alpha}$. (For instance, if $\frac{1}{3} \leq \alpha < \frac{1}{2}$ then $n = 3$.) From $\alpha < \frac{1}{2}$ we have $\frac{2}{\alpha} - \frac{1}{\alpha} = \frac{1}{\alpha} > 2$, therefore $n < \frac{2}{\alpha}$, so that $\frac{1}{n} < \alpha < \frac{2}{n}$. Let P be the following n-voter profile:

$$\begin{array}{rl} 1 : & z_1 \ldots z_{k-2}\ a\ \ u\ \ v\ \ \times\ \ldots \\ 1 : & z_1 \ldots z_{k-2}\ b\ \ v\ \ u\ \ \times\ \ldots \\ n-2 : & z_1 \ldots z_{k-2}\ \times\ \times\ \times\ a\ \ \ldots \end{array}$$

such that

1. in each of the last $n-2$ votes, none of the candidates ranked between $z_1, \ldots, z_{k-2}$ and a is b, u or v
2. no candidate appears more than once above a in the last $n - 2$ votes (note that for this to be possible we must have at least $3(n - 2) + 4$ candidates).

We have $\lambda(z_i, \alpha, P) = i$ for all $i = 1, \ldots, k-2$; $\lambda(u, \alpha, P) = \lambda(v, \alpha, P) = k+1$; $\lambda(a, \alpha, P) = k + 2$; and $\lambda(b, \alpha, P) > k + 2$.

So $MC_k^\alpha(P) = \{z_1, \ldots, z_{k-2}, u, v\}$, although $\{z_1, \ldots, z_{k-2}, a, b\}$ lexicographically dominates $\{z_1, \ldots, z_{k-2}, u, v\}$. $\square$

5.4 Condorcetian rules

The failure of LPE, and even WLPE, for Condorcetian rule, is a consequence of their failing possible Pareto-efficiency, which will prove in Section 6. However we give here a stronger result:

Proposition 14 *When $k \geq 3$, no Gehrlein-consistent rule is weakly lexicographically Pareto-efficient.*

Proof Let $k \geq 3$ and $P = (axyzbc-, byzxca-, czxyab-)$. This profile has a Gehrlein-stable committee $\{x, y, z\}$, which however is lexicographically Pareto-dominated by $\{a, b, c\}$. □

For $k = 2$, however, lexicographically Pareto-efficiency and Gehrlein-consistency are compatible. Let P be a profile, for which $\{x, y\}$ is Gehrlein stable. Assume that $\{x, y\}$ is lexicographically Pareto-dominated by $\{a, b\}$. If $\{x, y\} \cap \{a, b\} \neq \emptyset$ then without loss of generality, $\{a, b\} = \{a, y\}$; but then b is ranked above y in all votes, which contradicts the assumption that $\{x, y\}$ is Gehrlein stable. If $\{x, y\} \cap \{a, b\} = \emptyset$, then the restriction of P to $\{a, b, x, y\}$ must be such that every voter ranks a or b on top, therefore one of a an b is ranked on top in at least half of the votes, which contradicts the assumption that $\{x, y\}$ is Gehrlein stable. So we know that a Gehrlein stable committee cannot be Pareto-dominated. Now, any rule that outputs the Gehrlein-stable committee whenever there is one, and otherwise some lexicographically-optimal committee, is both Gehrlein stable and lexicographically Pareto-efficient.

6 Possible Pareto-efficiency under the responsive extension

We already know that every LPE (respectively, WLPE) rule is PPE (respectively, WPPE). Hence, in particular, sequential plurality, STC and β-CC* are PPE, while β-CC and SNTV are WPPE.

6.1 Committee scoring rules

Proposition 15 *Every CSR f_γ is WPPE. If γ is strict then f_γ is PPE.*

Proof The monotonicity of γ implies that if S necessarily Pareto-dominates S' for some profile P, then $score(S', P) \geq score(S, P)$, therefore f_γ is WPPE.

Assume γ is strict. Let $S \in f_\gamma(P)$ and let $S' \in \mathcal{S}_k(A)$ such that S' necessarily Pareto-dominates S. Then $pos(S, \succ_i) \succ pos(S', \succ_i)$ for each i, which implies $score(S', P) > score(S, P)$, contradicting $S \in f_\gamma(P)$. □

Going further, we can see easily that there are nonstrict CSRs that are PPE, and some that are not. This is true already for $k = 1$ (Llamazares and Peña 2015). To show intuitively why the question of identifying nonstrict CSRs that are PPE is nontrivial, even for best-k rules, we take $m = 5$ and $k = 2$, and consider a few examples:

1. the best-2 rule associated with scoring vector $(4, 4, 2, 1, 0)$ is PPE: suppose (1) $\{z, t\}$ necessarily dominates $\{x, y\}$. Suppose $\{z, t\} \cap \{x, y\} = \emptyset$. Without loss of generality, consider a vote $\succ_i = zxty$. Then $S_i(z) \geq S_i(x) > S_i(t) > S_i(y)$, therefore $S_i(\{z, t\}) > S_i(\{x, y\})$. This being true for each i, $\{x, y\}$ cannot be a winning committee. Now suppose $t = x$ and suppose (2) $\{x, y\}$ is a winning committee. From (2) we have $S(x) \geq S(y) \geq S(u)$ for all $u \neq x, y$. From

(1), we have $z \succ_i y$ for each i, and $S(z) + S(t) \geq S(x) + S(y)$. Therefore, $S(x) = S(y) = S(z)$, and for each vote i, (3) $S_i(y) = S_i(z)$. Now, if y and z are not ranked on the first two positions in vote i then $S_i(z) > S_i(y)$, contradicting (3). But if they are ranked in the first positions of all votes, then $\{z, y\}$ is the only winning committee, contradicting (2).

2. the best-2 rule associated with scoring vector $(4, 2, 2, 1, 0)$ is not PPE: consider the profile containing a single vote $xzyuv$; $\{x, y\}$ is winning but necessarily Pareto-dominated by $\{x, z\}$.
3. the best-2 rule associated with scoring vector $(4, 3, 1, 1, 0)$ is PPE. Suppose (1) $\{z, t\}$ necessarily dominates $\{x, y\}$. Then by a similar line of reasoning as for item 1 above, we cannot have $\{z, t\} \cap \{x, y\} = \emptyset$. Suppose $t = x$ and (2) $\{x, y\}$ is a winning committee. Then we must have z and y ranked in positions 3 and 4 in all votes, which imply $S(z) = S(y) = n$. Now, $S(x) \leq 4n$ and $S(u) + S(v) = 9n - S(x) - S(y) - S(y) \geq 9n - 4n - n - n = 3n$, contradicting (2).
4. the best-2 rules $(1 \ldots, 1, 0, \ldots, 0)$ with q 1's $(1 \leq q \leq 4)$ are not PPE: for $q = 1$ (SNTV) and $q \in \{3, 4\}$, consider a profile containing a single vote. For $q = 2$ (Bloc), consider the profile $P = (zyxuv, zyxuv, xuzyv, xvzyu)$: $\{x, y\}$ is winning but necessarily Pareto-dominated by $\{x, z\}$.

Two (remarkable) examples of nonstrict CSR that do not satisfy possible Pareto-efficiency are SNTV and Bloc:

Proposition 16 *SNTV and Bloc are WPPE but not PPE.*

Proof They are WPPE because they are CSRs (Proposition 15). To show that they fail PPE:

- For SNTV, it suffices to consider a profile containing a single vote; the output of SNTV contains all committees containing the top candidate of the vote, and all but one are necessarily Pareto-dominated.
- For *Bloc*, take the two-voter profile $(x_1 \ldots x_{k-1}a-, abx_1 \ldots x_{k-2}-)$: the two winning committees are $\{a, x_1, \ldots, x_{k-1}\}$ and $\{b, x_1, \ldots, x_{k-1}\}$; the former necessarily Pareto-dominates the latter.

□

Giving a full characterization of possibly Pareto-efficient CSRs, or even a full characterization of possibly Pareto-efficient best-k CSRs, seems very complicated and is likely to yield non-appealing conditions, therefore not worth the effort.

Intuitively, a best-k rule is not PPE as soon as there are $k + 1$ alternatives $x_1, \ldots, x_{k-1}, x^*, y^*$ such that

1. $S(x_1) \geq \ldots \geq S(x_{k-1}) \geq S(x^*) = S(y^*) \geq S(z)$ for each $z \neq x_1, \ldots, x_{k-1}, x^*, y^*$, and
2. x^*, y^* are ranked only in positions associated with identical scores: in that case, $\{x_1, \ldots, x_{k-1}, x^*\}$ can be made a winning committee whereas y^* Pareto-dominates x^*. For this we must ensure that the equal scores in the vector are large enough so that two alternatives ranked only in positions associated with identical scores can have one of the top $k + 1$ global scores, and small enough so that two alternatives

ranked only in positions associated with identical scores can have one of the bottom $m - k + 1$ global scores.

For non-additive committee scoring rules a characterization seems even more out of reach. However, a specific case of interest is that of Chamberlin-Courant rules.

Proposition 17 *For any scoring vector s and any $k \geq 2$, s-CC is not PPE.*

Proof Take any profile such that the top alternative is the same (x^*) in all votes, and take two alternatives z, y such that $y \succ z$ in all votes. Any committee containing x^* and z, but not y, is a winning committee although it is necessarily Pareto-dominated. (This is trivially satisfied if $n = 1$). □

PPE is violated for s-CC rules, even if the scoring vector s is strictly decreasing. However, for such scoring vectors (such as Borda), the failure of PPE occurs only for pathological profiles for which the number of candidates appearing in top position in some votes is less than k.

6.2 Compromise rules

Proposition 18 *For any $\alpha \in [0, 1)$, MC_k^α is PPE.*

Proof Suppose not. Then there exists a profile P such that $MC_k^\alpha(P) = X \in \mathcal{S}_k(A)$ and a set $Y \in \mathcal{S}_k(A)$ such that $Y \succ_i^R X$ for all i (by Theorem 3.1). Let $X^* = X \setminus Y$ and $Y^* = Y \setminus X$. Note that $0 < |X^*| = |Y^*| = \bar{k} < k$. Now, for all $i \in N$ and $x \in X^*$, there is a $y_i \in Y^*$ such that $y_i \succ_i x$. Let $r_i(x)$ be the rank of x in $\succ_i$.

Let $s = \lfloor \alpha n + 1 \rfloor$. Let h^* be the smallest rank such that all $x \in X^*$ are ranked within the top h^* positions in at least s votes (that is, in more than αn votes). Let $S^{h^*}(x) = |\{i \in N | r_i(X) \leq h^*\}|$ for any $x \in X^*$. As all $x \in X^*$ are ranked within the top h^* positions in at least s votes, we have $\sum_{x \in X^*} S^{h^*}(x) \geq \bar{k}s$.

By definition of h^*, there is some x^* with $S^{h^*-1}(x) < s$ and $S^{h^*-1}(x) \geq s$.

By stochastic dominance, we have

$$\sum_{y \in Y^*} S^{h^*-1}(y) \geq \sum_{x \in X^*} S^{h^*-1}(y)$$

and as $|X^*| = |Y^*|$, there exists some $y^* \in Y^*$ such that

$$|\{i \in N | r_i(y^*) \leq h^* - 1\}| \geq s$$

contradicting $x^* \in MC_k^\alpha(P)$ and $y^* \notin MC_k^\alpha(P)$. □

6.3 Condorcetian rules

We start by SEO. Recall that when $k = 1$, SEO is the *maximin* rule, that is known to fail Pareto-efficiency, but to satisfy weak Pareto-efficiency. It turns out that for $k > 1$, SEO does not even satisfy weak possible Pareto-efficiency.

Proposition 19 *SEO is not WPPE.*

Proof We give a counterexample for $n = 20$, $k = 2$, $m = 12$, $C = \{b, d\} \cup A^* \cup C^*$ where $A^* = \{a_1, \ldots, a_5\}$ and $C^* = \{c_1, \ldots, c_5\}$; P is the following 20-voter profile where the 20 voters are grouped in four types;

$$\begin{array}{l} 5\ A^* \succ b \succ C^* \succ d \\ 5\ A^* \succ d \succ C^* \succ b \\ 5\ C^* \succ b \succ A^* \succ d \\ 5\ C^* \succ d \succ A^* \succ b \end{array}$$

The 5 voters of a given type have cyclic preferences over the clone sets A^* and C^*: for instance, those of type $A^* \succ b \succ C^* \succ d$ are

$$\begin{array}{l} a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5 \succ b \succ c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5 \succ d \\ a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_1 \succ b \succ c_2 \succ c_3 \succ c_4 \succ c_5 \succ c_1 \succ d \\ a_3 \succ a_4 \succ a_5 \succ a_1 \succ a_2 \succ b \succ c_3 \succ c_4 \succ c_5 \succ c_1 \succ c_2 \succ d \\ a_4 \succ a_5 \succ a_1 \succ a_2 \succ a_3 \succ b \succ c_4 \succ c_5 \succ c_1 \succ c_2 \succ c_3 \succ d \\ a_5 \succ a_1 \succ a_2 \succ a_3 \succ a_4 \succ b \succ c_5 \succ c_1 \succ c_2 \succ c_3 \succ c_4 \succ d \end{array}$$

For $S \in S_2(C)$, let $SEO(S) = \min_{x \in S, y \in C \setminus S} |\{i : x \succ_i y\}|$. For each $i \in \{1, \ldots, 5\}$, 16 voters out of 20 prefer a_i to $a_{i+1[5]}$, therefore, if $S \cap A^* \neq \emptyset$ then $SEO(S) \leq 4$. Similarly, if $S \cap C^* \neq \emptyset$ then $SEO(S) \leq 4$. Finally, for each i, 15 voters out of 20 prefer the a_i's to b, 15 prefer the a_i's to d, 15 prefer the c_i's to b, and 15 prefer the c_i's to b; therefore, $SEO(\{b, d\} = 5$. The only winning committee is $\{b, d\}$; however, it is necessarily Pareto-dominated by $\{a_1, c_1\}$. $\square$

We do not have any better news with NED. While, for $k = 1$, NED is the *Copeland* rule, which is Pareto-efficient, for $k > 1$ we do not even have weak possible Pareto-efficiency.

Proposition 20 *NED is not WPPE.*

Proof We give a counterexample with $n = 5$, $k = 2$, $m = 14$, and $C = \{a, b, c, d\} \cup E^* \cup F^*$ with $E^* = \{e_1, e_2, e_3, e_4, e_5\}$ and $F^* = \{f_1, f_2, f_3, f_4, f_5\}$. The five candidates $e_1, \ldots, e_5$ are clones and are ranked in the five votes in such a way that they form a cycle $e_1 \to e_2 \to e_3 \to e_4 \to e_5 \to e_1$ (see the proof of 19 for an explanation), and similarly for $f_1, \ldots, f_5$. P is a five-voter profile for whose votes are:

$$\begin{array}{l} a \succ b \succ F^* \succ c \succ d \succ E^* \\ c \succ b \succ E^* \succ a \succ d \succ F^* \\ E^* \succ a \succ d \succ F^* \succ c \succ b \\ a \succ b \succ c \succ d \succ E^* \succ F^* \\ F^* \succ c \succ d \succ E^* \succ a \succ b \end{array}$$

where $a \succ b \succ F^* \succ c \succ d \succ E^*$ means that a and b are preferred to all the f_i's, all of them being preferred to c, d and all the e_i's, and so on. The majority graph is as follows:

$$
\begin{aligned}
&a \to b, c, d, F^* \\
&b \to d, E^*, F^* \\
&c \to b, d, E^* \\
&d \to E^*, F^* \\
&E^* \to a, F^* \\
&F^* \to c
\end{aligned}
$$

b and d are the only candidates that beat the 10 candidates in $E \cup F$: the NED score of $\{b, d\}$ is 20 (as b and d both beat 10 candidates in $C \setminus \{b, d\}$). Any other candidate beats at most 8 candidates, therefore, any 2-committee different from $\{b, d\}$ has a NED score at most 11+8 = 19. Therefore the only winning committee is $\{b, d\}$, and it is necessarily Pareto-dominated by $\{a, c\}$. □

Given that NED ans SEO are Gehrlein-consistent but fail possible Pareto-efficiency, we may wonder whether Gehrlein-consistency and possible Pareto-efficiency are compatible. The answer is positive. We even have this more general result:

Proposition 21 *A Gehrlein stable committee is possibly Pareto-optimal.*

Proof For any profile P and $x, y \in C$, let

$$W(x, y, P) = |\{i : x \succ_i y\}.$$

For any committees $S, S' \in \mathcal{S}_k(A)$, define

$$G(S, S', P) = \sum_{x \in S} \sum_{y \in S'} W(x, y, P).$$

Assume $S' \in \mathcal{S}_k(A)$ necessarily Pareto-dominates $S \in \mathcal{S}_k(A)$. Then $S' \setminus S$ necessarily Pareto-dominates $S \setminus S'$. Let $|S' \setminus S| = |S \setminus S'| = r$. For every voter i, let $S' \setminus S = \{x_1^i, \dots, x_r^i\}$, with $x_1^i \succ_i x_2^i \succ_i \dots \succ_i x_r^i$. Then, for each $j = 1, \dots, r$, i prefers x_j^i to at most $r - j + 1$ candidates in $S \setminus S'$. This implies $G(S \setminus S', S' \setminus S, P) \leq n(1 + \dots + (r-1)) = \frac{r(r-1)}{2} n < r^2 \frac{n}{2}$.

Now, assume S is Gehrlein stable for P; then for each $x \in S \setminus S'$ and $y \in S' \setminus S$, $W(x, y, P) > \frac{n}{2}$, therefore $G(S \setminus S', S' \setminus S, P) > r^2 \frac{n}{2}$.

The contradiction between the two inequalities imply that S cannot be both Gehrlein stable and necessarily Pareto-dominated. □

It is then easy to construct a rule that is both Gehrlein-consistent and PPE. As an example: given any PPE rule f, the rule that outputs the unique Gehrlein stable k-committee if there is one, and the winner of f otherwise, is Gehrlein stable and PPE.

Proposition 22 *LSE-maximin fails possible Pareto-efficiency, but satisfies weak possible Pareto-efficiency.*

Proof For the failure of PPE, just consider the single-voter profile (abc): LSE-maximin outputs $\{a, b\}$ and $\{a, c\}$, the latter being necessarily Pareto-dominated.

For the satisfaction of WPPE, assume (1) S' necessarily dominates S and (2) S is locally stable for quota q. (1) implies that there is a bijection σ_i from S to S' such that for all $x \in S$, $\sigma_i(x) \succeq_i x$. (2) means that for all $y \in A \setminus S$, at least qn voters prefer some candidate in S to y. Let $I(S, y) \subseteq N$ be the set of voters who prefer some candidate $c(i, S, y) \in S$ to y.

We now show that S' is locally stable for quota q. Let $y \in A \setminus S'$; we have to show that at least qn voters prefer some candidate in S' to y.

Assume first that $y \in A \setminus S$. Let $i \in I(S, y)$. Therefore, $\sigma_i(c(i, S, y)) \succeq_i c(i, S, y) \succ_i y$, and so $i \in I(S', y)$. This implies that $I(S', y) \supseteq I(S, y)$ and so that $|I(S', y)| \geq qn$.

Assume now that $y \in S$. Because $y \notin S'$, (1) implies that $\sigma_i(x) \succ_i x$ for all $i \in N$. This implies that $I(S', y) = N$ and *a fortiori* that $|I(S', y)| \geq qn$. □

6.4 Sequential rules

As already mentioned, sequential plurality and STV are LPE, therefore PPE. As greedy β-CC is not WLPE, it remains to investigate whether it satisfies (W)PPE.

Proposition 23 *Greedy β-CC does not satisfy WPPE.*

Proof Consider the following profile with $n = 6$ and $m = 16$.

$$
\begin{array}{l}
a_1\ b_2\ z_1\ z_2\ z_3\ a_2\ b_1\ z_4\ z_5\ z_6\ z_7\ z_8\ z_9\ z_{10}\ a_3\ b_3\\
a_2\ b_2\ z_4\ z_5\ z_6\ a_3\ b_1\ z_7\ z_8\ z_9\ z_{10}\ z_1\ z_2\ z_3\ a_1\ b_3\\
a_3\ b_2\ z_7\ z_8\ z_9\ a_1\ b_1\ z_{10}\ z_1\ z_2\ z_3\ z_4\ z_5\ z_6\ a_2\ b_3\\
a_1\ b_3\ z_3\ z_2\ z_1\ a_2\ b_1\ z_{10}\ z_9\ z_8\ z_7\ z_6\ z_5\ z_4\ a_3\ b_2\\
a_2\ b_3\ z_6\ z_5\ z_4\ a_3\ b_1\ z_3\ z_2\ z_1\ z_{10}\ z_9\ z_8\ z_7\ a_1\ b_2\\
a_3\ b_3\ z_{10}\ z_9\ z_8\ a_1\ b_1\ z_7\ z_6\ z_5\ z_4\ z_3\ z_2\ z_1\ a_2\ b_3
\end{array}
$$

Let $k = 3$. Denote by $B(x|Y)$ the marginal β-CC-score of x with respect to Y. We have:

- $B(b_1) = 54$;
- $B(b_2) = B(B_3) = 42$;
- $B(a_1) = B(a_2) = B(a_3) = 52$;
- $B(z_i) \leq 43$ for all i.

Therefore the first selected candidate is b_1. Now:

- $B(b_2|b_1) = B(b_3|b_1) = 15$
- $B(a_1|b_1) = B(a_2|b_1) = B(a_3|b_1) = 14$
- $B(z_i|b_1) \leq 6$ for all i.

Therefore the first selected candidate is b_2. Finally;

- $B(b_3|b_1b_2) = 15$
- $B(a_1|b_1b_2) = B(a_2|b_1b_2) = B(a_3|b_1b_2) = 7$
- $B(z_i|b_1b_2) \leq 3$ for all i.

Therefore the first selected candidate is b_3. The unique winning committee is $b_1b_2b_3$; it is necessarily Pareto-dominated by $a_1a_2a_3$. □

7 Necessary Pareto-efficiency under the responsive extension

We already know that rules that fail LPE will also fail necessary Pareto-efficiency. Therefore we can restrict our study to rules that satisfy LPE.

7.1 Committee scoring rules

Again, obtaining a general (and simple) characterization of NPE committee scoring rules seems out of reach, even for best-k rules.

To get an idea of which CSR satisfy NPE, consider the specific case $m = 4$ and $k = 2$. In that case, a committee scoring rule is defined by $\gamma_{4,2}(i, j) = \mu_{ij}$ for $1 \leq i < j \leq 4$, with $\mu_{12} \geq \mu_{13} \geq \mu_{24} \geq \mu_{34}$, $\mu_{13} \geq \mu_{14} \geq \mu_{24}$, and $\mu_{13} \geq \mu_{23} \geq \mu_{24}$. Now, for $m = 4$ and $k = 2$, f_k^γ is necessarily Pareto-efficient if and only if

$$\max(\mu_{23}, \mu_{14}) < \frac{1}{4}(\mu_{12} + \mu_{13} + \mu_{24} + \mu_{34})$$

The following table shows some five examples of rules $f_2^1, \ldots, f_2^5$ that are necessarily Pareto-efficient.

	μ_{12}	μ_{13}	μ_{14}	μ_{23}	μ_{24}	μ_{34}
f_2^1	1	0	0	0	0	0
f_2^2	1	1	0	0	0	0
f_2^3	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0
f_2^4	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
f_2^5	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

Proposition 24 *The perfectionist rule is NPE.*

Proof For any committee S output by the perfectionist rule there is at least one voter i such that the set of i's top k candidates is S. Therefore i necessarily prefers S any other committee S', and S is necessarily Pareto-optimal. □

On the negative side:

Proposition 25 *For any $k \geq 2$ and $m \geq 2k + 1$, no best-k rule satisfies necessary Pareto-efficiency.*

Proof Let $k \geq 2$ and $m \geq 2k+1$. Consider the following m-candidate, k-voter profile:

$$\begin{array}{lcl} x_1 z_1 \ldots z_k & \text{——} & x_2 \ldots x_k \\ x_2 z_2 \ldots z_1 & \text{——} & x_3 \ldots x_2 \\ x_3 z_3 \ldots z_2 & \text{——} & x_4 \ldots x_3 \\ \ldots & & \\ x_k z_k \ldots z_{k-1} & \text{——} & x_1 \ldots x_{k-1} \end{array}$$

Let $X = \{x_1, \ldots, x_k\}$, $Z = \{z_1, \ldots, z_k\}$ and $Y = A \setminus (X \cup Z)$. Every candidate in Y is Pareto-dominated by every candidate in Z, therefore a necessarily Pareto optimal committee must be contained in $X \cup Z$.

Let $s = (s_1, \ldots, s_m)$ and consider the best-k rule f_s. The score of each $x \in X$ is $s_1 + s_{m-k+2} + \ldots + s_m$; the score of each $z \in Z$ is $s_2 + \ldots + s_{k+1}$. The score of every $y \in Y$ candidates is at most $s_{k+2} + \ldots + s_{2k+1}$. Therefore:

- If $s_1 + s_{m-k+2} + \ldots + s_m > s_2 + \ldots + s_{k+1}$ then X is the only winning committee. It is not necessarily Pareto optimal , as it is possibly Pareto-dominated by Z.
- If $s_2 + \ldots + s_{k+1} > s_1 + s_{m-k+2} + \ldots + s_m$ then Z is the only winning committee contained in $X \cup Z$ (recall that all other committees cannot be necessarily Pareto optimal). It is not necessarily Pareto optimal, as it is possibly Pareto-dominated by X.

The only remaining cases are when $s_1 + s_{m-k+2} + \ldots + s_m = s_2 + \ldots + s_{k+1}$, which we now assume.

- Assume $s_1 + s_{m-k+2} + \ldots + s_m = s_2 + \ldots + s_{k+1}$ and $s_{k+2} > 0$. Assume also $k \geq 3$. We consider the following profile:

$$\begin{array}{ll} x_1 z_1 \ldots z_k x_2 \ldots x_k & \text{———} \\ x_2 z_2 \ldots z_1 x_3 \ldots x_2 & \text{———} \\ x_3 z_3 \ldots z_2 x_4 \ldots x_3 & \text{———} \\ \ldots & \\ x_k z_k \ldots z_{k-1} x_1 \ldots x_{k-1} & \text{———} \end{array}$$

  The only winning committee is X (which is not necessarily Pareto optimal).
- Assume $k \geq 3$, $s_1 + s_{m-k+2} + \ldots + s_m = s_2 + \ldots + s_{k+1}$ and $s_{k+2} = 0$, which means that $s = (s_1, s_2, \ldots, s_{k+1}, 0, \ldots, 0)$ with $s_1 = s_2 + \ldots + s_{k+1}$. This also implies that $s_2 > 0$. Consider this profile:

$$\begin{array}{lll} x_1 x_2 z_1 \ldots z_{k-1} & z_k & \text{———} \\ x_2 x_3 z_2 \ldots z_k & z_1 & \text{———} \\ \ldots & & \\ x_k x_1 z_k \ldots z_{k-2} & z_{k-1} & \text{———} \end{array}$$

  The score of every x_i is $s_1 + s_2$ and that of every z_i is $s_3 + \ldots + s_{k+1}$. Since $s_2 > 0$, $s_1 + s_2 > s_3 + \ldots + s_{k+1}$, therefore the winning committee is X, which is not necessarily Pareto optimal because it is possible Pareto-dominated by Z.
- The only remaining case is $k = 2$ and $s = (s_1, s_2, s_3, 0, \ldots, 0)$ with $s_1 = s_2 + s_3$. Assume first $s_3 > 0$. Consider the profile

$$\begin{array}{l} x_1 z_1 z_3 \text{ ——— } x_2 \\ x_2 z_2 z_3 \text{ ——— } x_1 \end{array}$$

  The winning committee is $\{x_1, x_2\}$; it is not necessarily Pareto optimal. Finally, assume $s_3 = 0$, that is, $s = (s_1, s_1, 0, \ldots, 0)$. Consider the profile

$$\begin{array}{l} x_1 z_1 \text{ —— } x_2 \\ x_1 z_2 \text{ —— } x_2 \\ x_1 z_3 \text{ —— } x_2 \\ x_2 z_1 \text{ —— } x_1 \\ x_2 z_2 \text{ —— } x_1 \\ x_2 z_3 \text{ —— } x_1 \end{array}$$

The only winning committee is $\{x_1, x_2\}$; it is not necessarily Pareto optimal.

□

We note that in all of our profiles we needed at least $2k$ candidates, except in the last one where we needed at least $2k + 1$. Therefore, if $m \leq 2k$, some best-k rules may be NPE.[6]

7.2 Other rules

As compromise rules fail weak lexicographic Pareto-efficiency, *a fortiori* they fail necessary Pareto-efficiency.

Also, NED and SEO fail NPE because they fail PPE. More generally:

Proposition 26 *All Gehrlein-consistent rules fail necessary Pareto-efficiency.*

Proof Consider the profile $P = (axybc, bxyca, cxyab)$, and $k = 2$. $\{x, y\}$ is Gehrlein stable; it is however not necessarily Pareto optimal. □

This is no better for top-sequential rules:

Proposition 27 *STV and sequential plurality fail necessary Pareto-efficiency.*

Proof Consider the profile $P = (axyb, bxya)$, and $k = 2$. The winning STV, and sequential plurality, committee is $\{a, b\}$; it is not necessarily Pareto optimal. □

8 Discussion

Table 1 summarizes all results.

At one extreme, we have a class (Class 1) of rules that fail even the weakest notion of Pareto-efficiency (WLPE): two Gehrlein-consistent rules (SEO and NED), and Greedy β-CC. Failing such a weak property sends a negative signal about these rules: they should be selected with care, and for good reasons that counterbalance this failure.

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[6] This is actually the case: take $k = 2$, $m = 3$, $A = \{x, y, z\}$, $s = (s_1, s_2, 0)$. Assume that for some profile P, $\{x, y\}$ is a winning committee but is not necessarily Pareto optimal; without loss of generality, assume $\{x, y\}$ is possibly Pareto-dominated by $\{x, z\}$. Because $\{x, y\}$ is possibly Pareto-dominated by $\{x, z\}$ P contains no vote where z is ranked last and no vote yzx. Therefore, P consists of α votes xzy, β votes zxy, and γ votes zyx. But now, the score of z is $(\beta + \gamma)s_1 + \alpha s_2$ and the score of y is γs_2, so $\{x, y\}$ cannot be a winning committee unless $(\beta + \gamma)s_1 + \alpha s_2 = \gamma s_2$, that is, $\alpha = \beta = 0$ and $s_1 = s_2$; but then P contains only votes zyx and the winning committee is $\{y, z\}$.

Table 1 Multiwinner rules and degrees of Pareto-efficiency

	WPPE	PPE	WLPE	LPE	NPE	Class
CSRs	+	Some	Some	Some	Some	2, 3, 4, 5, 6
Strict CSRs	+	+	Some	Some	Some	3, 5, 6
Best-k	+	Some	Some	Some	–	2,3,4,5
SNTV	+	–	+	–	–	4
k-Borda	+	+	–	–	–	3
β-CC	+	-	+	–	–	4
β-CC*	+	+	+	+	–	5
Perfectionist	+	+	+	+	+	6
Bloc	+	–	–	–	–	2
MC_k^{α}	+	+	–	–	–	3
STV	+	+	+	+	–	5
Sequential plurality	+	+	+	+	–	5
Greedy β-CC	–	–	–	–	–	1
LSE-maximin	+	–	–	–	–	2
Gehrlein-consistent rules	Some	Some	–	–	–	1,2,3
SEO, NED	–	–	–	–	–	1

Just above, we find Class 2, containing LSE-Maximin and Bloc, who satisfy WPPE but nothing above. LSE-Maximin, SEO and NED are the three Condorcetian rules (extending Condorcet-consistent rules) we considered: the message is that Gehrlein stability does not fit well with Pareto (which perhaps does not come as a surprise). As for Bloc, this is one more slightly negative signal, which should contribute to be cautious about using it.

Two classes of rules are above this class. Class 3 contains compromise rules, and some committee scoring rules, including k-Borda, that satisfy PPE, but fail WLPE. WLPE is rather strong, so we can consider that some safety test is passed as to what concerns Pareto-efficiency, in the sense that this should not be a reason to exclude these rules.

The other class above Class 2 is Class 4, which is incomparable with Class 3. It contains some committee scoring rules, including β-CC and SNTV: they satisfy WPPE and WLPE but fail PPE, but this is mostly because of pathological profiles, so again they should probably not be excluded on this ground.

Now we move towards classes of rules that behave very well regarding Pareto-efficiency. Class 5 contains rules that satisfy LPE (but fail NPE). It contains some CSRs, including the new rule β-CC*, as well as STV and sequential plurality. Satisfying LPE is a good arguments to choose one of these rules in any context where the lexicographic extension makes sense, that is, when an agent pays more attention to her preferred alternative in a committee than on the other ones.

Finally, Class 6 is composed of rules that satisfy the strongest property (NPE). We have identified only one known rule that satisfies it (the perfectionist rule); on the other hand, this rule has so many other drawbacks that it should be chosen with extreme care. These rather negative findong about NPE sends the signal that this property is too strong, rather than the signal that we should select rules that satisfy it.

Appendix

Notation

Notation	Meaning	Subsection
N	Set of voters	2.1
n	Number of voters	2.1
A	Set of alternatives	2.1
m	Number of alternatives	2.1
$\mathcal{L}(A)$	The set of linear orders (rankings) over A	2.1
$P = (\succ_1, \ldots, \succ_n)$	Profile over A	2.1
$\succ_i$	Vote of voter i Over A	2.1
k	Size of the committee	2.1
$\mathcal{S}_k(A)$	Set of all committees of size k	2.1
$W(\mathcal{S}_k(A))$	Set of weak orders over $\mathcal{S}_k(A)$	2.1
$\Pi(\mathcal{S}_k(A))$	Set of strict partial orders over $\mathcal{S}_k(A)$	2.1
$\sqsupseteq_i$	Weak order over $\mathcal{S}_k(A)$ of voter i	2.1
$\sqsupset_i$	Strict order over $\mathcal{S}_k(A)$ of voter i	2.1
$Q = (\sqsupseteq_1, \ldots, \sqsupseteq_n)$	Weak order profile over $\mathcal{S}_k(A)$	2.1
f	Multiwinner rule	2.2
$pos(c, \succ_i)$	Position of c in $\succ_i$	2.2.1
$pos(S, \succ_i)$	(multiset) position of S in $\succ_i$	2.2.1
$[m]_k$	Set of size-k increasing sequences	2.2.1
$I = (i_1, \ldots, i_k)$	Element of $[m]_k$	2.2.1
$\gamma_{m,k}$	Committee scoring function	2.2.1
$score(S, P)$	Score of committee S for profile P	2.2.1
$score^s_{CC}(T, \succ)$	s-Chamberlin-Courant score of x for $\succ$	2.2.4
$\lambda(\alpha, P, x)$	Compromise index of x with respect to α and P	2.2.3
$score^s_{CC}(x\|T, \succ)$	Marginal score of x with respect to T for $\succ$	2.2.4
E	Extension principle	3.1
$\succ_i^E$	Extension of $\succ_i$ with respect to E	3.1
$\kappa^E(\succ_i)$	Set of completions of $\succ_i$	3.1
$\kappa^E(P)$	Set of completions of P	3.1
ρ	Responsive extension principle	3.1
$r_h(X; \succ_i)$	h-th ranked alternative in $\succ_i$	3.1
$\sigma^k(\succ_i)$	Stochastic dominance	3.1
lex	lexicographic Extension principle	3.3
$\succ_i^{lex}$	Lexicographic extension of $\succ_i$	3.3
$top(k, \succ_i)$	Top k elements of $\succ_i$	4.2

Voting rules

Notation	Rule	Subsection
$f_{\gamma_{m,k}}$	Rule associated with $\gamma_{m,k}$	2.2.1
β-CC	Borda Chamberlin-Courant	2.2.1
β-CC*	Lexicographic refinement of β-CC	2.2.1
	Bloc	2.2.1
	Perfectionist	2.2.1
	SNTV	2.2.1
	k-Borda	2.2.1
NED	Number of External Defeats	2.2.2
SEO	Size of External Opposition	2.2.2
	LSE-maximin	2.2.2
MC_k^α	Compromise rule associated with α	2.2.3
	Majoritarian compromise	2.2.3
$SeqPlu$	Sequential plurality	2.2.4
STV	Single Transferable vote	2.2.4
s-GCC	Greedy s-Chamberlin-Courant	2.2.4

Further terminology

Terminology	Subsection
Completion	2.1
Gehrlein stability	2.2.2
Elementary improvement	3.1
Necessary Pareto-optimality	4.2
Possible Pareto-optimality	4.2
Necessary Pareto-efficiency (NPE)	4.2
Possible Pareto-efficiency (PPE)	4.2
Weak possible Pareto-efficiency (WPPE)	4.2
Lexicographic Pareto-efficiency (LPE)	4.2
Weak lexicographic Pareto-efficiency (WLPE)	4.2
Top sequentiality	4.2

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References

Aziz H, Biró P, Lang J, Lesca J, Monnot J (2019) Efficient reallocation under additive and responsive preferences. Theor. Comput. Sci. 790:1–15

Aziz H, Elkind E, Faliszewski P, Lackner M, Skowron P (2017) The Condorcet principle for multiwinner elections: From shortlisting to proportionality. In: Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017, pp. 84–90
Aziz H, Gaspers S, Mackenzie S, Walsh T (2015) Fair assignment of indivisible objects under ordinal preferences. Artif. Intell. 227:71–92
Aziz H, Lang J, Monnot J (2016) Computing Pareto optimal committees. In: Kambhampati, S. (ed.) Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016, pp. 60–66
Aziz H, Monnot J (2020) Computing and testing Pareto optimal committees. Auton. Agents Multi Agent Syst. 34(1):24
Barbera S, Bossert W, Pattanaik PK (2004) Ranking sets of objects. In: Handbook of Utility Theory, Volume II, pp. 893–977. Kluwer Academic Publishers, ??? . Chap. 17
Belahcène K, Mousseau V, Wilczynski A (2021) Combining fairness and optimality when selecting and allocating projects. In: IJCAI, pp. 38–44
Bossert W (1995) Preference extension rules for ranking sets of alternatives with a fixed cardinality. Theory and Decision 39:301–317
Bouveret S, Endriss U, Lang J (2010) Fair division under ordinal preferences: Computing envy-free allocations of indivisible goods. In: ECAI 2010 - 19th European Conference on Artificial Intelligence, Lisbon, Portugal, August 16-20, 2010, Proceedings, pp. 387–392
Brams S, Edelman PH, Fishburn PC (2003) Fair division of indivisible items. Theory and Decision 55:147–180
Coelho D (2004) Understanding, evaluating and selecting voting rules through games and axioms. PhD thesis
Elkind E, Faliszewski P, Skowron P, Slinko A (2017) Properties of multiwinner voting rules. Soc. Choice Welf. 48(3):599–632
Elkind E, Lang J, Saffidine A (2015) Condorcet winning sets. Soc. Choice Welf. 44(3):493–517
Faliszewski P, Skowron P, Slinko A, Talmon N (2017) Multiwinner voting: A new challenge for social choice theory. In: Endriss, U. (ed.) Trends in Computational Social Choice
Faliszewski P, Skowron P, Slinko A, Talmon N (2018) Multiwinner analogues of the plurality rule: axiomatic and algorithmic perspectives. Soc. Choice Welf. 51(3):513–550
Gehrlein W (1985) The Condorcet criterion and committee selection. Mathematical Social Sciences 10(3):199–209
Kamwa E (2017) Stable rules for electing committees and divergence on outcomes. Group Decision and Negotiation 26(3):547–564
Kerkmann AM, Lang J, Rey A, Rothe J, Schadrack H, Schend L (2020) Hedonic games with ordinal preferences and thresholds. J. Artif. Intell. Res. 67:705–756
Khare R, Roy S, Storcken T (2021) Stability in matching with couples having responsive preferences. Technical report
Klamler C, Pferschy U, Ruzika S (2012) Committee selection under weight constraints. Math. Soc. Sci. 64(1):48–56. https://doi.org/10.1016/j.mathsocsci.2011.11.006
Lackner M, Skowron P (2020) Utilitarian welfare and representation guarantees of approval-based multiwinner rules. Artif. Intell. 288:103366
Lang J, Mengin J, Xia L (2018) Voting on multi-issue domains with conditionally lexicographic preferences. Artif. Intell. 265, 18–44 https://doi.org/10.1016/j.artint.2018.05.004
Laslier J-F (1997) Tournament Solutions and Majority Voting. Springer, ???
Llamazares B, Peña T (2015) Scoring rules and social choice properties: some characterizations. Theory and Decision 78(3):429–450
Lucchetti R, Moretti S, Rea T (2022) Coalition formation games and social ranking solutions. In: AAMAS, pp. 862–870
Roth A, Sotomayor M (1990) Two-sided matching: A study in game-theoretic modeling and analysis
Roth AE. The college admissions problem is not equivalent to the marriage problem. Journal of economic Theory **36**(2), 277–288
Segal-Halevi E, Hassidim A, Aziz H (2020) Fair allocation with diminishing differences. J. Artif. Intell. Res. 67:471–507
Sertel MR (1986) Lecture Notes in Microeconomics

Skowron P, Faliszewski P, Lang J (2016) Finding a collective set of items: From proportional multirepresentation to group recommendation. Artif. Intell. 241:191–216

Yilmaz B, Sertel MR (1999) The majoritarian compromise is majoritarian-optimal and subgame-perfect implementable. Social Choice and Welfare 16(4):615–627