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Abstract

This paper shows how local optimum may ensure constant differential approximation ratio for the *Maximum H_0-free Partial Subgraph* and the *Minimum H_0-cover (Partial) Subgraph problems* on graphs with maximum degree bounded by B. We first prove that 1-locally optimal solutions achieve differential approximation ratio $\delta_0/(B+1)$ for both *$MaxH_0$-freePartialSG-B* and *$MinH_0$-coverPartialSG-B*, where δ_0 refers to the minimum degree on H_0; next, we show that this ratio rises up to $2/B$ when H_0 is a 3-clique. We finally demonstrate that, when increasing slightly the size of the neighborhood, 3-local optima constitute a $(\delta_0+1)/(B+2+\nu_0)$-differential approximation for *Maximum H_0-free Subgraph problem* and *$MinH_0$-coverPartialSG-B* for any graph H_0, with $\nu_0 = (|V(H_0)|+1)/\delta_0$. Finally, we show that, when H_0 is a 3-clique, 3-local optimum achieve ratio $3/(B+1)$.

1 Introduction

The *Maximum H_0-free Partial Subgraph problem* or *$MaxH_0$-freePartialSG* in short (*resp., Maximum H_0-free Subgraph problem, $MaxH_0$-freeSG* in short) can be described as follows:

Given a graph $G = (V, E)$, we look for a maximum size subset $V' \subseteq V$ so that the induced graph from V' does not contain any partial subgraph (*resp.,* induced subgraph) isomorphic to H_0, where $H_0 = (V_0, E_0)$ is a connected graph. The value of an optimal solution on G will be denoted by $\alpha_{H_0}(G)$ (*resp., $\alpha'_{H_0}(G)$*); furthermore, we will denote by *$MaxH_0$-freePartialSG-B* (*resp., $MaxH_0$-freeSG-B*) the problem restriction to graphs with degrees bounded by B.

$MaxH_0$-freePartialSG and *$MaxH_0$-freeSG* are part of a more general problem family called *Maximum Induced Subgraph with property P problem*, or more commonly *Maximum Subgraph problem*; for a specific graph property P, the maximum induced subgraph problem with respect to P consists in finding, in a given graph $G = (V, E)$, a largest subset of vertices V' so that the graph V' induces satisfies P. The graph property P must be hereditary, which means that any time P is satisfied by a graph G, then it as well is satisfied by every G induced subgraph. Such a graph

property P can then be characterized by the forbidden set $\mathcal{H}_P$ made up of the minimal graphs (with respect to inclusion) that do not satisfy P: then a graph satisfies P if and only if it does not contain any graph from $\mathcal{H}_P$. Therefore, *MaxH$_0$-freeSG* and *MaxH$_0$-freePartialSG* are special cases of the maximum subgraph problem where, for *MaxH$_0$-freeSG*, the forbidden set $\mathcal{H}_P$ is restricted to the lonely graph H_0, whereas, for *MaxH$_0$-freePartialSG*, the forbidden set $\mathcal{H}_P$ is made up of supergraphs that contain H_0 (i.e., $\mathcal{H}_P = \{G' = (V_0, E') : E_0 \subseteq E'\}$); one can easily note that *MaxH$_0$-freeSG* and *MaxH$_0$-freePartialSG* obviously match each other perfectly when H_0 is a clique.

The maximum subgraph problem has already been dealt with in the literature at the beginning of the 80's, notably by Lewis [7] and Yannakakis [11] who independently made the evidence of its **NP-hardness**, as soon as H_0 contains at least 2 vertices. The authors have then generalized their results in Lewis and Yannakakis [8], giving the proof of the Maximum Subgraph problem **NP-hardness** for any hereditary property P. Ten years later, Lund and Yannakakis [9] proved that, on the one hand, MaxH$_0$-freePartialSG is not standard approximable within $1/|V|^\varepsilon$ for any $\varepsilon > 0$ unless **P** $=$ **NP** and, on the other hand, that Maximum Subgraph problem is not standard approximable within $1/2^{\log^{1/2-\varepsilon}|V|}$ for any $\varepsilon > 0$ unless **NP** $\subset$ **QP** if P is a non-trivial hereditary graph property[1]. Halldórsson and Lau have proved in [4] that Maximum Subgraph problem is approximable within $3/(B+1)$, while in [3] it has been proved that this problem is standard approximable within $O(\log(|V|)/|V|)$. Obviously, the same bounds hold for *MaxH$_0$-freePartialSG* problem.

The *minimum H$_0$-cover Partial subgraph problem, MinH$_0$-CoverPartialSG* in short, (*resp., minimum H$_0$-cover subgraph problem, MinH$_0$-CoverSG* in short) has been less widely studied in the literature (see [9] for a short description) and consists in finding a minimum vertex subset that intersects any subgraph H (with $|V_0|$ vertices) of G containing a partial graph isomorphic to H_0. More formally, for a special connected graph $H_0 = (V_0, E_0)$ the problem can be defined as follows: given a graph $G = (V, E)$, we look for a minimum size subset $V' \subseteq V$ so that, for any subgraph H of G isomorphic to one in the set $\{G' = (V_0, E') : E_0 \subseteq E'\}$ (*resp.*, H_0), there exists a vertex $v \in V'$ that belongs to H.

As previously, these problems are special cases of a more general problem called *Minimum vertex deletion to obtain subgraph with property P*; for this latter, we look for a minimum size subset $V' \subseteq V$ so that the subgraph $V - V'$ induces satisfies P. One can easily see that when the forbidden set $\mathcal{H}_P$ characterizing the hereditary property P is $\{G' = (V_0, E') : E_0 \subseteq E'\}$ (*resp.*, H_0), then *MinH$_0$-CoverPartialSG* (*resp., MinH$_0$-CoverSG*) and Minimum vertex deletion problem are identical.

In this paper we study the approximability of *MaxH$_0$-freeSG, MaxH$_0$-freePartialSG, MinH$_0$-CoverPartialSG* and *MinH$_0$-CoverSG* using the so-called differential approximation ratio. This ratio, for an instance I of a combinatorial problem Π and an approximation algorithm **A** for Π is defined as $|\omega_\Pi(I) - \lambda_\mathbf{A}(I)|/|\omega_\Pi(I) - \beta_\Pi(I)|$, where $\omega_\Pi(I)$ is the value of a worst feasible solution for I, $\beta_\Pi(I)$ is the value of an optimal one and $\lambda_\mathbf{A}(I)$ is the value computed by **A** on I. The differential ratio is still less popular than the standard one, defined by $\lambda_\mathbf{A}(I)/\beta_\Pi(I)$, but it has some interesting properties that can be used in uniformly analyzing approximation properties of classes of

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[1]A non-trivial hereditary graph property is a hereditary property satisfied for infinitely many graphs and not satisfied by infinitely many.

maximization and minimization problems. For instance, it is stable under affine transformations of the objective function of a problem.

Minimum H_0-cover partial subgraph and minimum H_0-cover subgraph problems are standard-approximable within $|V_0|$, by a kind of greedy algorithm which generalizes the *matching algorithm* for *minimum vertex cover problem* (where H_0 is an edge) to any connected graph H_0. This family of algorithms can be described as: starting from $V' = \emptyset$, while there exists in G a subgraph $H = (V(H), E(H))$ isomorphic to a graph of size $|V_0|$ containing H_0 (*resp.*, isomorphic to H_0), add $V(H)$ to V' and delete H from G.

Furthermore, depending on the heridetary property P, the more general Minimum vertex deletion problem to obtain subgraph with property P) may be standard approximable within some constant: this is notably true when P describes a finite number of minimal forbidden subgraphs (it is, for instance, the case of line and interval graphs) as proved in [9], and also when P can be expressed through a universal first order formula over the graph edge subsets (see Kolaitis and Thakur [6]).

Whereas the computation of an optimum solution for $MinH_0$-$CoverPartialSG$ (*resp.*, $MinH_0$-$CoverSG$) is obviously no harder than for $MaxH_0$-$freePartialSG$ (*resp.*, $MaxH_0$-$freeSG$) (in fact, both computations are of equivalent hardness since a solution value (a posteriori the optimal one) is given by the number of deleted vertices for the former and of remaining vertices for the latter), these two problems become strikingly different in terms of their standard approximation: in fact, as we previously said, $MaxH_0$-$freePartialSG$ and $MaxH_0$-$freeSG$ cannot be standard approximable within any constant, while $MinH_0$-$Cover$ partial and induced subgraph problems are constant approximable.

On the other hand, since differential ratio is stable under affine transformation (see Demange and Paschos [2]), $MinH_0$-$CoverPartialSG$ (*resp.*, $MinH_0$-$CoverSG$) and $MaxH_0$-$freePartialSG$ (*resp.*, $MaxH_0$-$freeSG$) are equivalent regarding to their differential approximation. Finally, let us note that standard and differential approximation ratios coincide for $MaxH_0$-freePartialSG as well as for $MaxH_0$-freeSG, since, for both problems, the value of the worst solution of any instance is 0.

In what follows, we analyze the differential approximation ratio achieved by local search approximation algorithms for $MaxH_0$-freePartialSG and $MaxH_0$-freeSG. The results obtained hold, as it has already been mentioned just above, for standard approximation and improve the ones of [4] for any H_0 of $\delta_0 > 3$. Moreover, thanks to the stability of differential ratio regarding to affine transformations, these results identically apply to the cases of $MinH_0$-Cover partial or induced subgraph.

2 The 1-OPT algorithm

In this section we study the behavior of maximal solutions (corresponding to 1-local optimum) with respect to both standard and differential measures. Such solutions can be easily computed: starting from a worst solution and in an iterative manner, just add (or delete according to the problem goal) vertices, as long as the current solution satisfies a given property. Concerning $MaxH_0$-$freePartialSG$ (*resp.*, $MinH_0$-$CoverPartialSG$), this algorithm, $1 - OPT$, works as follows:

1. Start with $V' = \emptyset$ (*resp.*, $V' = V$);

2. While there exists $v \notin V'$ (*resp.*, $v \in V'$) such that the subgraph induced by $V' \cup \{v\}$ (*resp.*, $V' \setminus \{v\}$) does not contain any partial subgraph isomorphic to H_0, do $V' := V' \cup \{v\}$ (*resp.*, $V' := V' \setminus \{v\}$);

3. Output V'.

The time-complexity of this algorithm is bounded by $O(n^{|V_0|+2})$ where V_0 is the vertex set of H_0.

2.1 1-local optima for any H_0

We will respectively denote by δ_0 and B the minimum degree on H_0 and the maximum degree on G; furthermore, $V(G')$ and $E(G')$ will represent for any graph G' its vertex and edge sets; finally, we denote by $\langle X, Y \rangle_{G'}$ the cut set between two vertex subsets X and Y on G'. However, when dealing with these sets, we will omit to mention G' as soon as no confusion is possible.

Theorem 2.1 *The 1-OPT algorithm is a $\delta_0/(B+1)$-differential approximation for $\mathrm{Max}H_0$-freePartialSG-B and $\mathrm{Min}H_0$-CoverPartialSG $- B$.*

Proof. Let U^* be an optimal solution for the graph $G = (V, E)$ and U be the solution provided by the 1-OPT algorithm; U is a 1-local optimum if and only if, on the one hand, the subgraph induced by U does not contain any partial subgraph isomorphic to H_0 and, on the other hand, the addition of any vertex v from $V \setminus U$ into U induces a partial subgraph isomorphic to H_0.

We set $U_+^* = U^* \setminus U$ and U' at the set of vertices from U that are in a graph H_0 when adding to U a vertex from U_+^*; formally, if we denote for any $v \in U_+^*$ by $\mathcal{G}_v$ the set of H_0 graphs on $U \cup \{v\}$, then U' is defined by:

$$U' = \{w \in U : \ \exists v \in U_+^*, \exists H_0 \in \mathcal{G}_v \text{such that } w \in V(H_0) \}$$

The local optimality of U indicates that a partial subgraph induced by $U' \cup \{v\}$ contains H_0 for any vertex v from U_+^*, that is to say: $\forall v \in U_+^*, \ \mathcal{G}_v \neq \emptyset$. We can therefore deduce from the local optimality of U the following properties on the bipartite graph $BP = (U', U_+^* ; \langle U', U_+^* \rangle)$:

1. $\forall v \in U_+^*, \ d_{BP}(v) \geq \delta_0$

2. $\forall v \in U', \ d_{BP}(v) \leq B - \delta_0 + 1$

For property 1: for any vertex v from U_+^*, there exists a H_0 graph in $\mathcal{G}_v$ that contains v and whose vertices but v are in U'; v is therefore adjacent to at least δ_0 vertices from U'. For property 2: any vertex u from U' is also on a H_0 graph whose vertices (but v) are in $U' \cup \{v\}$, for some vertex v from U_+^*. Thus, u is adjacent to at least δ_0 vertices from $U' \cup \{v\}$ and then, to at least $\delta_0 - 1$ vertices from U'; since U' and U_+^* are separate sets and because the degrees on G are upper bounded by B, we deduce that u may not be connected to more than $B - (\delta_0 - 1)$ vertices from U_+^*.

From properties 1 and 2, we deduce $\delta_0|U_+^*| \leq \sum_{w \in U_+^*} d_{BP}(w) = \sum_{v \in U'} d_{BP}(v) \leq (B - \delta_0 + 1)|U'|$ and finally get:

$$\begin{aligned} \delta_0 \alpha_{H_0}(G) &= \delta_0(|U^* \cap U| + |U_+^*|) & \leq & \ \delta_0|U^* \cap U| + (B - \delta_0 + 1)|U'| \\ & \leq \ \delta_0|U| + (B - \delta_0 + 1)|U| & \leq & \ (B+1)|U| \end{aligned}$$

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which achieves the proof. $\qquad\square$

Algorithm 1-OPT reaches ratio at least δ_0/B; in some special H_0 topologies, for instance when H_0 has a unique vertex of degree δ_0 (one only has to notice that property 1 above turns to: $\forall v \in U_+$, $d_{BP}(v) \leq B - \delta_0$). Furthermore, this ratio is also reached when $H_0 = K_2$ or $H_0 = K_3$; the former case amounts to the independent set problem (then consider the complete bipartite graph $K_{1,B}$), while the latter one amounts to the search of triangle-free maximum subgraphs. We conjecture that 1-local optima do reaches ratio δ_0/B as soon as H_0 is a clique; We show in section 2.2 the truthfulness of this conjecture in the case of triangles.

2.2 The Maximum Partial subgraph K_3-free

Recall that, when dealing with a clique K_n for H_0, the maximum H_0-free partial subgraph (*resp.*, minimum H_0-cover partial subgraph) problem is equivalent to the maximum H_0-free subgraph (*resp.*, minimum H_0-cover subgraph) problem. In our case, $H_0 = K_3$ (and thus, $\delta_0 = 2$).

Proposition 2.2 *1-OPT algorithm is a $2/B$-differential approximation for* $\mathrm{Max}K_3\text{-freeSG} - B$ *and* $\mathrm{Min}K_3\text{-CoverSG} - B$, *and this ratio is tight.*

Proof. We will refine the previous proof by studying a new graph H: still considering the family of graphs $\mathcal{G}_v$ when v goes over U_+^*. Vertex and edge sets of H will be respectively defined by $V(H) = U' \cup U_+^*$ and $E(H) = \cup_{v \in U_+^*} E(\mathcal{G}_v)$.

Notice that H contains the bipartite graph BP discussed in the previous proof, whereas there is no longer reason that it is bipartite by itself. Moreover, let us denote by U_{com} the intersection of the global and the local optimum U and U^*, by U'_{com} its restriction to U' and by U'' the adjacent set of U'_{com} in U'; in other words, we set $U_{com} = U^* \cap U$, $U'_{com} = U^* \cap U'$ and $U'' = U' \cap \Gamma_H(U'_{com})$. We can notice that U'_{com} and U'' are independent from each other, and also that any vertex v of U' is in U'' if and only if it is on a triangle $\{u, v, w\}$ with v in U_+^* and w in U'_{com}; finally, u does not belong to U'_{com}, otherwise there exists a triangle on U^*. This new description of G leads to the following inequalities for the edges of H:

$$
\begin{array}{rclcl}
\sum_{v \in U'} d_H(v) & = & |\langle U', U_+^* \rangle_H| + 2|\langle U', U' \rangle_H| & \geq & |\langle U', U_+^* \rangle_H| + 2|\langle U'_{com}, U'' \rangle_H| \\
\Rightarrow \quad |\langle U', U_+^* \rangle_H| & \leq & \sum_{v \in U'} d_G(v) - 2|\langle U'_{com}, U'' \rangle_H| & \leq & B|U'| - 2|\langle U'_{com}, U'' \rangle_H|
\end{array}
$$

From the above inequalities we deduce the properties:

1. $\forall v \in U_+^* \ d_H(v) \geq 2$

2. $|\langle U'_{com}, U'' \rangle_H| \geq Max\{|U'_{com}|; |U''|\}$

The first property holds according to the arguments similar to the ones for property 1 in the proof of theorem 2.1, with $\delta_0 = 2$. Let us then show the second one: each vertex from U'' is by definition related in H to at least one vertex from U'_{com} and conversely, each vertex from U'_{com} is related in H to at least one vertex from U' (by way of a triangle from $\mathcal{G}_v$ for a special vertex v from U_+^*), which ensures this vertex to be in U'' (or U'_{com} and U'' are not separate sets).

From properties 1, 2 and from the inequality on $|\langle U', U_+^* \rangle_H|$, we now conclude:

$$
\begin{aligned}
2\alpha_{K_3}(G) &= 2(|U_+^*| + |U'_{com}| + |U_{com} \setminus U'_{com}|) \leq |\langle U', U_+^* \rangle_H| + 2(|U'_{com}| + |U_{com} \setminus U'_{com}|) \\
&\leq B|U'| - 2|\langle U'_{com}, U'' \rangle| + 2(|U'_{com}| + |U_{com} \setminus U'_{com}|) \\
&\leq B|U| + 2(|U'_{com}| - Max\{|U'_{com}|; |U''|\}) \leq B|U|
\end{aligned}
$$

For tightness of the ratio derived, consider the graph $G = (V, E)$ with $V = \{x_1, x_2, y_1, y_2, \dots, y_{B-1}\}$ and $E = \{(x_1, x_2)\} \cup \{(x_1, y_i), (x_2, y_i) : 1 \leq i \leq B-1\}$; on such a graph, the solution $U^* = \{x_1, y_1, y_2, \dots, y_{B-1}\}$ is an optimum of value $\alpha_{K_3}(G) = B$ while the solution $U = \{x_1, x_2\}$ is a local optimum of value 2. $\qquad \square$

3   The 3-OPT algorithm

In this section we show that when allowing three moves from a solution to a neighboring solution, then the 3-local optimum of $MaxH_0$-$freePartialSG$ and $MinH_0$-$CoverPartialSG - B$ problems guarantee differential ratio $Max\{(\delta_0 + 1)/(B + 2 + \nu_0); (\delta_0)/(B + 1)\}$ with $\nu_0 = (|V(H_0)| - 1)/\delta_0$. For instance, when H_0 is a $(\delta_0 + 1)$ clique (i.e., $H_0 = K_{\delta_0+1}$), a 3-local optimum reaches the ratio $(\delta_0 + 1)/(B + 3)$, which is strictly better than $\delta_0/(B + 1)$ as soon as $B \geq 2\delta_0 - 1$, and better than δ_0/B when $B \geq 3\delta_0$. The 3-local neighborhood aims at improving a given solution U by removing one vertex from it and by adding two vertices from $V \setminus U$; thus, determining a 3-local optimum can be described for $MaxH_0$-$freePartialSG$ (resp., $MinH_0$-$CoverPartialSG$) as follows: starting from a maximal (resp., minimal) (i.e. 1-OPT algorithm) solution, remove (resp., add) one vertex from this solution and add (resp., remove) 2 other ones if this is possible, make the solution maximal (resp., minimal), and so on.

Proposition 3.1 *The 3-OPT algorithm is a $(\delta_0 + 1)/(B + 2 + \nu_0)$-differential approximation for* $MaxH_0$-freePartial$SG-B$ *and* $MinH_0$-CoverPartial$SG-B$ *problems, where $\nu_0 = (|V(H_0)| - 1)/(\delta_0)$.*

Proof. If U is a 3-local optimum, then every set $\{v_1, v_2\}$ of vertices in U_+^* of degree δ_0 in BP is on a H_0 graph with vertices in $U \setminus \{u\} \cup \{v_1, v_2\}$, for a special vertex u from U; for each set $\{v_1, v_2\}$, we will refer to the set of such H_0 graphs as $\mathcal{G}_{v_1, v_2}$. We introduce a new set V' made on vertices from U that are on a H_0 graph from $\cup_{v_1 \neq v_2 \in U_+^*} \mathcal{G}_{v_1, v_2}$, and will denote by W' its union with U': $W' = U' \cup V'$.

We then focus on a new bipartite graph $BP = (W', U_+^* ; \langle W', U_+^* \rangle)$. U_1^* will denote the subset of vertices from U_+^* which are related in BP to exactly δ_0 vertices from W', U_1' will refer to the set of vertices connected to U_1^* in BP, while BP_1 will describe the restriction of BP to these two sets: $U_1^* = \{v \in U_+^* : d_{BP}(v) = \delta_0\}$, $W_1' = \Gamma_{BP}(U_1^*)$ and $BP_1 = (W_1', U_1^* ; \langle W_1', U_1^* \rangle)$. Note that W_1' is a subset of U'; so,

$$|W_1'| \leq |U'| \tag{3.1}$$

Properties 1 and 2 in the proof of theorem 2.1 remain true and, more precisely, we get: $\forall v \in W' \setminus U' = V'$, $d_{BP}(v) \leq B - \delta_0 + 2$ and: $\forall v \in U_+^* \setminus U_1^*$, $d_{BP}(v) \geq \delta_0 + 1$. Moreover, the following inequality holds:

$$|U_1^*| \leq \nu_0 |W_1'| \tag{3.2}$$

In order to prove (3.2), we will show the inequality: $\forall u \in W_1'$, $d_{BP_1}(u) \le |V(H_0)| - 1$, which leads by summing on vertices of W_1' to the relation $\delta_0|U_1^*| = \sum_{v \in U_1^*} d_{BP_1}(v) = \sum_{u \in W_1'} d_{BP_1}(u) \le (|V(H_0)| - 1)|W_1'|$ and thus, to (3.2).

Assume that a given vertex u from W_1' is connected in BP_1 to at least $n_0 = |V(H_0)|$ vertices from U_1^*, denote by $\{v_1, \ldots, v_{n_0}\}$ these vertices and arbitrarily pick two of them, say v_i and v_j. Since U is 3-optimal, the set $U \setminus \{u\} \cup \{v_i, v_j\}$ cannot be feasible, which means that it must contain one H_0 graph. Thus, v_i and v_j must be related to at least δ_0 vertices from $U \setminus \{u\} \cup \{v_i, v_j\}$, that implies, because they are related to precisely $\delta_0 - 1$ vertices from $U \setminus \{u\}$, that (v_i, v_j) is an edge on G. This being true for two arbitrary vertices $\{v_i, v_j\}$, we deduce that the whole set $\{v_1, \ldots, v_{n_0}\}$ forms a n_0-clicque on the optimal solution U^*, which contradicts its feasibility, H_0 being a K_{n_0} subgraph.

From the previous inequalities we get:

$$\delta_0|U_1^*| + (\delta_0 + 1)|U_+^* \setminus U_1^*| \le \sum_{v \in U_+^*} d_{BP}(v) = \sum_{u \in W'} d_{BP}(u) \le (B - \delta_0 + 1)|W'| + |W' \setminus U'|$$

which allows us to complete the proof:

$$
\begin{aligned}
(\delta_0 + 1)\alpha_{H_0}(G) &= (\delta_0 + 1)(|U_1^*| + |U_+^* \setminus U_1^*| + |U_{com}|) \\
&= |U_1^*| + (\delta_0|U_1^*| + (\delta_0 + 1)|U_+^* \setminus U_1^*|) + (\delta_0 + 1)|U_{com}| \\
&\le \nu_0|W_1'| + (B - \delta_0 + 1)|W'| + |W'| - |U'| + (\delta_0 + 1)|U_{com}| \\
&\le (\nu_0 - 1)|W_1'| + (B - \delta_0 + 1)|W'| + |W'| + (\delta_0 + 1)|U_{com}| \\
&\le (\nu_0 + B + 2)|U|
\end{aligned}
$$

As for 1-local optimum, we conjecture that 3-local optimum reach a differential ratio of at least $(\delta_0 + 1)/(B + 1)$, which would significantly improve the ratio $\delta_0/(B + 1)$ and even δ_0/B, the best ratio expected for 1-local optimum. The ratio conjectured is at least true for *MaxIS* (see [5]) and, as we are going to show, even in the case where forbidden graph is a triangle.

Proposition 3.2 *The 3-OPT algorithm is a $3/(B + 1)$-differential approximation for both $MaxK_3$-freeSG-B and $MinK_3$-CoverSG-B.*

Proof. In order to demonstrate it, we will refine proofs of propositions 2.2 and 3.1. We first build a graph H' which contains H (*cf., proposition 2.2*) and the triangles from the family $\mathcal{G}_{v_1, v_2}$: $H' = (V(H'), E(H'))$ with $V(H') = W' \cup U_+^*$ and $E(H') = \cup_{v \in U_+^*} E(\mathcal{G}_v) \cup_{v_1 \ne v_2 \in U_+^*} E(\mathcal{G}_{v_1, v_2})$. H is then a subgraph of H' whose vertex set $V(H)$ contains U', U'_{com} and U''. As well as we did in the proof of proposition 3.1, we also consider U_1^* the subset from U_+^* made on vertices which of degree 2 in $\langle U_+^*, W' \rangle_{H'}$ and denote by W_1' the subset of W' that are adjacent in H': $U_1^* = \{v \in U_+^* : |\langle \{v\}, W' \rangle_{H'}| = 2\}$ and $W_1' = \Gamma_{H'}(U_1^*)$. We then denote by H_1' the subgraph of H' induced by $W_1' \cup U_1^*$. W_1' is obviously a subset from U' since H_1' is an element of $\mathcal{G}_v$. We now partition U_1^* into two subsets U_2^* and $U_1^* \setminus U_2^*$ where U_2^* contains the vertices of U_1^* that are connected in H_1' to U_{com}: $U_2^* = \Gamma_{H_1'}(U_{com})$; $W_2' = \Gamma_{H_1'}(U_2^*)$; $W_3' = \Gamma_{H_1'}(U_1^* \setminus U_2^*)$ and, finally, H_2' and H_3' will respectively draw the subgraphs of H_1' induced by $U_2^* \cup W_2'$ and $(U_1^* \setminus U_2^*) \cup W_3'$. Note that these two last subgraphs are not necessarily disjoint, whereas we have $V(H_2') \cup V(H_3') = V(H_1')$. Note also that the two sets $W_2' \cap U_{com}$ and $W_2' \setminus U_{com}$ are respectively subsets of U'_{com} and U''.

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We now exploit local optimality of U according to 3-bounded neighborhoods, first through a preliminary remark from which various properties on H_1', H_2' et H_3' will follow.

$$\forall u \in W_1', \ \forall v_1 \neq v_2 \in U_1^*, \ u \in \Gamma_{H_1'}(v_1) \cap \Gamma_{H_1'}(v_2) \Rightarrow \{v_1, v_2\} \in E \tag{3.3}$$

Actually, if U is a local optimum, then there exists a triangle T on $U \setminus \{u\} \cup \{v_1, v_2\}$; since v_1 and v_2 are of degree 2 in W', the unique triangle of G on $v_1 \cup W'$ contains u and thus, T does contain edge $\{v_1, v_2\}$. Therefore, the following properties can be deduced:

1. $\forall u \in W_1', \ |\langle \{u\}, U_1^* \rangle_{H_1'}| \leq 2$

2. $\forall u \in W_2' \cap U_{com}, \ |\langle \{u\}, U_1^* \rangle_{H_1'}| = 1$

3. $\forall u \in W_2' \setminus U_{com}, \ |\langle \{u\}, U_1^* \rangle_{H_1'}| = 1$

For 1, we reason in the same way as we did in proposition 2.2: if a vertex from W_1' is adjacent to (at least) 3 vertices of U_1^*, then, according to the previous remark, these 3 vertices are all mutually connected, forming a triangle.

For 2, suppose that a vertex of $W_1' \cap U_{com}$ is adjacent to two vertices of U_+^*; then, these two vertices are also connected in G and, once more, we have exhibit a triangle in the optimum.

Finally, for 3, let us consider a vertex u of $W_2' \setminus U_{com}$ connected to two vertices v_1 and v_2 of U_1^* and denote by u_1 (*resp.*, u_2) the vertex of $W_2' \cap U_{com}$ lying on the unique triangle $\{v_1, u, u_1\}$ (*resp.*, $\{v_2, u, u_2\}$) derived from v_1 (*resp.*, v_2) in H'. Since u_1 and u_2 belong to $W_2' \cap U_{com}$, we deduce from the previous argument that these two vertices are distinct. U being optimal, there exists a triangle $\{v_1, v_2, w\}$ on $U \setminus \{u\} \cup \{v_1, v_2\}$; since u_1 and u_2 are distinct, we may suppose $w \neq u_1$; then, v_1 is connected to three vertices u, u_1 et w in H', in contradiction with the fact that its degree is 2.

From properties 1, 2 and 3 we deduce the following inequalities:

$$|U_1^* \setminus U_2^*| \ \leq \ |W_3'| \tag{3.4}$$
$$|U_2^*| \ = \ |W_2' \setminus U_{com}| \ = \ |W_2' \cap U_{com}| \tag{3.5}$$

For (3.4), in the same way as in proposition 3.1, but considering H_3' this time, we get:

$$2|U_1^* \setminus U_2^*| = \sum_{v \in U_1^* \setminus U_2^*} |\langle \{v\}, W_3' \rangle_{H_1'}| = |\langle U_1^* \setminus U_2^*, W_3' \rangle| = \sum_{u \in W_3'} |\langle \{u\}, U_1^* \setminus U_2^* \rangle_{H_1'}| \leq 2|W_3'|$$

For (3.5), note that any vertex of U_2^* is connected in H' to a unique pair of vertices from $(W_2' \cap U_{com}) \times (W_2' \setminus U_{com})$; then,

$$
\begin{aligned}
|U_2^*| \ &= \ \sum_{v \in U_2^*} |\langle \{v\}, W_2' \cap U_{com} \rangle_{H_1'}| \ &= \ |\langle U_2^*, W_2' \cap U_{com} \rangle| \\
&= \ \sum_{u \in W_2' \cap U_{com}} |\langle \{u\}, U_2^* \rangle_{H_1'}| \ &= \ |W_2' \cap U_{com}| \\
&= \ \sum_{v \in U_2^*} |\langle \{v\}, W_2' \setminus U_{com} \rangle_{H_1'}| \ &= \ |\langle U_2^*, W_2' \setminus U_{com} \rangle| \\
&= \ \sum_{u \in W_2' \setminus U_{com}} |\langle \{u\}, U_2^* \rangle_{H_1'}| \ &= \ |W_2' \setminus U_{com}|
\end{aligned}
$$

Finally remark that, since each vertex u of W_3' is on a triangle $\{v, u, u'\}$ for a given $v \in U_1^*$ and for a given u' from W_3', the following relationship can be derived:

$$2|\langle W_3', W_3' \rangle_{H_3'}| \geq |W_3'| \tag{3.6}$$

From all these inequalities, given that $B \geq 3$ (otherwise we have the optimum), we get:

$$
\begin{aligned}
3\alpha_{K_3}(G) &= 3(|U_1^*| + |U_+^* \setminus U_1^*| + |U_{com}|) = |U_1^*| + (2|U_1^*| + 3|U_+^* \setminus U_1^*|) + 3|U_{com}| \\
&\leq |U_1^*| + |\langle U_+^*, W' \rangle_{H'}| + 3|U_{com}| \\
&\leq |U_1^*| + \sum_{u \in W'} d_{H'}(u) - 2|\langle W', W' \rangle_{H'}| + 3|U_{com}| \\
&\leq |U_1^*| + B|W'| - 2(|\langle U_{com}', U'' \rangle_H| + |\langle W_3', W_3' \rangle_{H_3'}|) + 3|U_{com}| \\
&\leq |U_1^*| + B(|U| - |U_{com} \setminus U_{com}'|) - 2Max\{|U_{com}'|, |U''|\} - |W_3'| + 3|U_{com}| \\
&\leq |U_1^*| + B|U| - 2Max\{|U_{com}'|, |U''|\} - |W_3'| + 3|U_{com}'| \\
&\leq |U_1^* \setminus U_2^*| + |U_2^*| + B|U| - |W_3'| + |U_{com}'| \\
&\leq |W_3'| + |W_2' \setminus U_{com}'| + B|U| - |W_3'| + |U_{com}'| \quad \leq \quad (B+1)|U| \ \square
\end{aligned}
$$

Note that the result of proposition 3.2 already slightly improves ratio $3/(B+2)$ obtained in [4].

4 Conclusion

We have shown that, in the case of $MaxH_0\text{-}freePartialSG$ and $MinH_0\text{-}CoverPartialSG$, 3-local optima are better solutions than 1-local optima; the following question can be therefore posed: is the approximation ratio provided by local optima strictly improved when increasing the size of the neighborhood (or, equivalently: are $(k+1)$-local optima strictly better solutions than k-local ones?). In fact, if Δ_k refers to the differential ratio provided by k-local optima, we wonder if the ratio Δ_k/Δ_3 becomes, for a certain k, greater than, or equal to, 1. If Δ_k/Δ_3 tends to 1, then local optima according to larger neighborhoods will never improve Δ_3 within better than an additive constant. On the other hand, if Δ_k/Δ_3 tends to something strictly greater than 1, then local optima of larger neighborhoods improve Δ_3 by multiplicative factors. We cannot answer this question yet; all we can say is that, in general, the quality of local optima is not necessarily correlated to the size of neighborhood there are defined from. For instance, it is proved in [1] that, in the special case of the *Minimum label spanning tree problem with bounded color classes*, 3-local optima provide a $(r+1)/2$-standard approximation, while the ratio provided by k-local optima will not exceed $(r/2) + \epsilon$ for any $k > 3$ and $\epsilon > 0$, where r bounds the number of occurrence of each color on the edges of the graph considered. Another illustration of this fact is provided by $Max2-CCSP$: it is shown in [10] that 3-local optima for $Max2-CCSP$ ensure a 1/3-standard approximation, but this ratio is tight for *any k-local optimal*, for a special neighborhood structure called *mirror k-bounded neighborhood*. Do there exist problems for which use of wider neighborhoods leads to strictly better solutions? This is an open question and matter for further research.

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