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Conjoint measurement tools for MCDM. A brief introduction

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Conjoint measurement tools for MCDM
A brief introduction[1]

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Abstract

This paper offers a brief and nontechnical introduction to the use of conjoint measurement in multiple criteria decision making. The emphasis is on the, central, additive value function model. We outline its axiomatic foundations and present various possible assessment techniques to implement it. Some extensions of this model, e.g. nonadditive models or models tolerating intransitive preferences are then briefly reviewed.

Keywords: Conjoint Measurement, Additive Value Function, Preference Modelling.

1 Introduction and motivation

Conjoint measurement is a set of tools and results first developed in Economics [41] and Psychology [127] in the beginning of the '60s. Its, ambitious, aim is to provide measurement techniques that would be adapted to the needs of the Social Sciences in which, most often, multiple dimensions have to be taken into account.

Soon after its development, people working in decision analysis realized that the techniques of conjoint measurement could also be used as tools to structure preferences [46, 148]. This is the subject of this paper which offers a brief and nontechnical introduction to conjoint measurement models and their use in multiple criteria decision making. More detailed treatments may be found in [57, 72, 108, 121, 191]. Advanced references include [52, 115, 193].

1.1 Conjoint measurement models in Decision Theory

The starting point of most works in Decision Theory is a binary relation $\succsim$ on a set A of objects. This binary relation is usually interpreted as an "at least as good as" relation between alternative courses of action gathered in A.

Manipulating a binary relation can be quite cumbersome as soon as the set of objects is large. Therefore, it is not surprising that many works have looked for a *numerical representation* of the binary relation $\succsim$. The most obvious numerical representation amounts to associate a real number $V(a)$ to each object $a \in A$ in such a way that the comparison between these numbers faithfully reflects the original relation $\succsim$. This leads to defining a real-valued function V on A, such that:

$$a \succsim b \Leftrightarrow V(a) \geq V(b), \tag{1}$$

for all $a, b \in A$. When such a numerical representation is possible, one can use V in lieu of $\succsim$ and, e.g. apply classical optimization techniques to find the most preferred elements in A given $\succsim$. We shall call such a function V a *value function*.

It should be clear that not all binary relations $\succsim$ may be represented by a value function. Condition (1) imposes that $\succsim$ is complete (i.e. $a \succsim b$ or $b \succsim a$, for all $a, b \in A$) and transitive (i.e. $a \succsim b$ and $b \succsim c$ imply $a \succsim c$, for all $a, b, c \in A$). When A is finite or countably infinite, it is well-known [52, 115] that these two conditions are, in fact, not only necessary but also sufficient to build a value function satisfying (1).

Remark 1
The general case is more complex since (1) implies, for instance, that there must be "enough" real numbers to distinguish objects that have to be distinguished. The necessary and sufficient conditions for (1) can be found in [52, 115]. An advanced treatment is [13]. Sufficient conditions that are well-adapted to cases frequently encountered in Economics can be found in [39]. •

It is vital to note that, when a value function satisfying (1) exists, it is by no means unique. Taking any increasing function ϕ on $\mathbb{R}$, it is clear that $\phi \circ V$ gives another acceptable value function. A moment of reflection will convince the reader that only such transformations are acceptable and that if V and U are two real-valued function on A satisfying (1), they must be related by an increasing transformation. In other words, a value function in the sense of (1) defines an *ordinal scale*.

Ordinal scales, although useful, do not allow the use of sophisticated assessment procedures, i.e. of procedures that allow an analyst to assess the relation $\succsim$ through a structured dialogue with the decision-maker. This is because the knowledge that $V(a) \geq V(b)$ is strictly equivalent to the knowledge of $a \succsim b$ and no inference can be drawn from this assertion besides the use of transitivity.

It is therefore not surprising that much attention has been devoted to numerical representations leading to more constrained scales. Many possible avenues have been explored to do so. Among the most well-know, let us mention:

- the possibility to compare *probability distributions* on the set A [52, 189]. If it is required that, not only (1) holds but that the numbers attached to the objects should be such that their expected values reflects the comparison of probability distributions on the set of objects, a much more constrained numerical representation clearly obtains,

- the introduction of "preference difference" comparisons of the type: the difference between a and b is larger than the difference between c and d, see [41, 115, 163, 181]. If it is required that, not only (1) holds, but that the differences between numbers also reflect the comparisons of preference differences, a more constrained numerical representation obtains.

When objects are evaluated according to several dimensions, i.e. when $\succsim$ is defined on a product set, new possibilities emerge to obtain numerical representations that would specialize (1). The purpose of conjoint measurement is to study such kinds of models.

There are many situations in decision theory which call for the study of binary relations defined on product sets. Among them let us mention:

- *Multiple criteria decision making* using a preference relation comparing alternatives evaluated on several attributes [15, 108, 145, 156, 191],
- *Decision under uncertainty* using a preference relation comparing alternatives evaluated on several states of nature [62, 95, 160, 167, 192, 193],
- *Micro Economics* manipulating preference relations for bundles of several goods [40],
- *Dynamic decision making* using a preference relation between alternatives evaluated at several moments in time [108, 111, 112],
- *Social choice* comparing distribution of wealth across several individuals [5, 16, 17, 198].

The purpose of this paper is to give an introduction to the main models of conjoint measurement useful in multiple criteria decision making. The results and concepts that are presented may however be of interest in all of the afore-mentioned areas of research.

Remark 2
Restricting ourselves to applications in multiple criteria decision making will not allow us to cover every aspects of conjoint measurement. Among the most important topics left aside, let us mention: the introduction of statistical elements in conjoint measurement models [49, 96] and the test of conjoint measurement models in experiments [121]. •

Given a binary relation $\succsim$ on a product set $X = X_1 \times X_2 \times \cdots \times X_n$, the theory of conjoint measurement consists in finding conditions under which it is possible to build a convenient numerical representation of $\succsim$ and to study the uniqueness of this representation. The central model is the *additive value function* model in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i) \qquad (2)$$

where v_i are real-valued functions, called *partial value functions*, on the sets X_i and it is understood that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Clearly if $\succsim$ has a representation in model (2), taking any common increasing transformation of the v_i will *not* lead to another representation in model (2).

Specializations of this model in the above-mentioned areas give several central models in decision theory:

- The Subjective Expected Utility model, in the case of decision-making under uncertainty,
- The discounted utility model for dynamic decision making,
- Inequality measures *à la* Atkinson/Sen in the area of social welfare.

The axiomatic analysis of this model is now quite firmly established [41, 115, 193]; this model forms the basis of many decision analysis techniques [72, 108, 191, 193]. This is studied in sections 3 and 4 after we introduce our main notation and definitions in section 2.

Remark 3
One possible objection to the study of model (2) is that the choice of an *additive* model seems arbitrary and restrictive. It should be observed here that the functions v_i will precisely be assessed so that additivity holds.

It is also useful to notice that this model can be reformulated so as to make addition disappear. Indeed if there are partial value functions v_i such that (2) holds, it is clear that $V = \sum_{i=1}^{n} v_i$ is a value function satisfying (1). Now, since V defines an ordinal scale, taking the exponential of V leads to another valid value function W. Clearly W has now a multiplicative form:

$$x \succsim y \Leftrightarrow W(x) = \prod_{i=1}^{n} w_i(x_i) \geq W(y) = \prod_{i=1}^{n} w_i(y_i).$$

where $w_i(x_i) = \mathrm{e}^{v_i(x_i)}$.

The reader is referred to chapter XXX (Chapter 6, Dyer) for the study of situations in which V defines a scale that is more constrained than an ordinal scale, e.g. because it is supposed to reflect preference differences or because it allows to compute expected utilities. In such cases, the additive form (2) is no more equivalent to the multiplicative form envisaged above. •

In section 5 we envisage a number of extensions of this model going from nonadditive representations of transitive relations to model tolerating intransitive indifference and, finally, nonadditive representations of nontransitive relations.

Remark 4
We shall limit our attention in this paper to the case in which alternatives may be evaluated on the various attributes without risk or uncertainty. Excellent overviews of these cases may be found in [108, 191]. •

Before starting our study of conjoint measurement oriented towards MCDM, it is worth recalling that conjoint measurement aims at establishing measurement models in the Social Sciences. To many, the very notion of "measurement in the Social Sciences" may appear contradictory. It may therefore be useful to briefly envisage how the notion of measurement can be modelled in the realm of Physics and to explain how a "measurement model" may indeed be useful in order to structure preferences.

1.2 An aside: measuring length

Physicists usually take measurement for granted and are not particularly concerned with the technical and philosophical issues it raises (at least when they work within the realm of Newtonian Physics). However, for a Social Scientist, these question are of utmost importance. It may thus help to have an idea of how things appear to work in Physics before tackling more delicate cases.

Suppose that you are on a desert island and that you want to "measure" the length of a collection of rigid straight rods. Note that we do not discuss here the "pre-theoretical" intuition that "length" is a property of these rods that can be measured, as opposed, say, to their rigidity, their softness or their beauty.

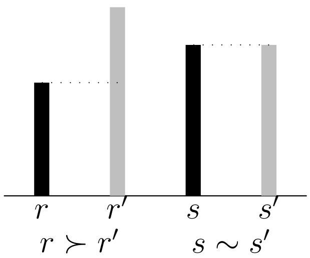


Figure 1: Comparing the length of two rods.

A first simple step in the construction of a measure of length is to place the two rods side by side in such a way that one of their extremities is at the same level (see Figure 1). Two things may happen: either the upper extremities of the two rods coincide or not. This seems to be the simplest way to devise an experimental procedure leading to the discovery of which rod "has more length" than the other. Technically, this leads to defining two binary relations $\succ$ and $\sim$ on the set of rods in the following way:

$$r \succ r' \text{ when the extremity of } r \text{ is higher than the extremity of } r',$$
$$r \sim r' \text{ when the extremities of } r \text{ and } r' \text{ are at the same level,}$$

Clearly, if length is a quality of the rods that can be measured, it is expected that these pairwise comparisons are somehow consistent, e.g.,

- if $r \succ r'$ and $r' \succ r''$, it should follow that $r \succ r''$,
- if $r \sim r'$ and $r' \sim r''$, it should follow that $r \sim r''$,
- if $r \sim r'$ and $r' \succ r''$, it should follow that $r \succ r''$.

Although quite obvious, these consistency requirements are stringent. For instance, the second and the third conditions are likely to be violated if the experimental procedure involves some imprecision, e.g if two rods that slightly differ in length are nevertheless judged "equally long". They represent a form of *idealization* of what could be a perfect experimental procedure.

With the binary relations $\succ$ and $\sim$ at hand, we are still rather far from a full-blown measure of length. It is nevertheless possible to assign numbers to each of the rods in such a way that the comparison of these numbers reflects what has been obtained experimentally. When the consistency requirements mentioned above are satisfied, it is indeed generally possible to build a real-valued function Φ on the set of rods that would satisfy:

$$r \succ r' \Leftrightarrow \Phi(r) > \Phi(r') \text{ and}$$
$$r \sim r' \Leftrightarrow \Phi(r) = \Phi(r').$$

If the experiment is costly or difficult to perform, such a numerical assignment may indeed be useful because it summarizes, once for all, what has been obtained in experiments. Clearly there are many possible ways to assign numbers to rods in this way. Up to this point, they are equally good for our purposes. The reader will easily check that defining $\succsim$ as $\succ$ or $\sim$, the function Φ is noting else than a "value function" for length: any increasing transformation may therefore be applied to Φ.

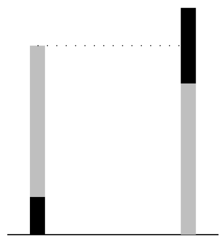


Figure 2: Comparing the length of composite rods.

The next major step towards the construction of a measure of length is the realization that it is possible to form new rods by simply placing two or more rods "in a row", i.e. you may *concatenate* rods. From the point of view of length, it seems obvious to expect this concatenation operation $\circ$ to be "commutative" ($r \circ s$ has the same length as $s \circ r$) and associative ($(r \circ s) \circ t$ has the same length as $r \circ (s \circ t)$).

You clearly want to be able to measure the length of these composite objects and you can always include them in in our experimental procedure outlined above (see Figure 2). Ideally, you would like your numerical assignment Φ to be somehow compatible with the concatenation operation: knowing the numbers assigned to two rods, you want to be able to deduce the number assigned to their concatenation. The most obvious way to achieve that is to require that the numerical assignment of a composite object can be deduced by addition from the numerical assignments of the objects composing it, i.e. that

$$\Phi(r \circ r') = \Phi(r) + \Phi(r').$$

This clearly places many additional constraints on the results of your experiment. One obvious one is that $\succ$ and $\sim$ should somehow be compatible with the concatenation operation $\circ$, e.g.

$$r \succ r' \text{ and } t \sim t' \text{ should lead to } r \circ t \succ r' \circ t'.$$

These new constraints may or not be satisfied. When they are, the usefulness of the numerical assignment Φ is even more apparent: a simple arithmetic operation will allow to infer the result of an experiment involving composite objects.

Let us take a simple example. Suppose that you have 5 rods $r_1, r_2, \ldots, r_5$ and that, because space is limited, you can only concatenate at most two rods and that not all concatenations are possible. Let us suppose, for the moment, that you do not have much technology available so that you may only experiment using *different* rods. You may well collect the following information, using obvious notation exploiting the transitivity of $\succ$ which holds in this experiment,

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1.$$

Your problem is then to find a numerical assignment Φ to rods such that using an addition operation, you can infer the numerical assignment of composite objects consistently with your observations. Let us envisage the following three assignments:

	Φ	Φ'	Φ''
r_1	14	10	14
r_2	15	91	16
r_3	20	92	17
r_4	21	93	18
r_5	28	100	29

These three assignments are equally valid to reflect the comparison of single rods. Only the first and the third allow to capture the comparison of composite objects. Note that, going from Φ to Φ'' does not involve just changing the "unit of measurement": since $\Phi(r_1) = \Phi''(r_1)$ this would imply that $\Phi = \Phi''$, which is clearly false. This implies that such numerical assignments have limited usefulness. Indeed, it is tempting to use them to predict the result of comparisons that we have not been able to perform. But this turns out to be quite disappointing: using Φ you would conclude that $r_2 \circ r_3 \sim r_1 \circ r_4$ since $\Phi(r_2)+\Phi(r_3) = 15+20 = 35 = \Phi(r_1)+\Phi(r_4)$, but, using Φ'', you would conclude that $r_2 \circ r_3 \succ r_1 \circ r_4$ since $\Phi''(r_2) + \Phi''(r_3) = 16 + 17 = 33$ while $\Phi''(r_1) + \Phi''(r_4) = 14 + 18 = 32$.

Intuitively, "measuring" calls for some kind of a *standard* (e.g. the "Mètre-étalon" that can be found in the Bureau International des Poids et Mesures in Sèvres, near Paris). This implies choosing an appropriate "standard" rod *and* being able to prepare perfect copies of this standard rod (we say here "appropriate" because the choice of a standard should be made in accordance with the lengths of the objects to be measured: a tiny or a huge standard will not facilitate experiments). Let us call s_0 the standard rod. Let us suppose that you have been able to prepare a large number of perfect copies s_1, $s_2, \dots$ of s_0. We therefore have:

$$s_0 \sim s_1, s_0 \sim s_2, s_0 \sim s_3, \dots$$

Let us also agree that the length of s_0 is 1. This is your, arbitrary, unit of length. How can you use s_0 and its perfect copies so as to determine unambiguously the length of any other (simple or composite) object? Quite simply, you may prepare a "standard sequence of length n", $S(n) = s_1 \circ s_2 \circ \dots \circ s_{n-1} \circ s_n$, i.e. a composite object that is made by concatenating n perfect copies of our standard rod s_0. The length of a standard sequence of length n is exactly n since we have concatenated n objects that are perfect copies of the standard rod of length 1. Take any rod r and let us now compare r with several standard sequences of increasing length: $S(1)$, $S(2), \dots$

Two cases may arise. There may be a standard sequence $S(k)$ such that $r \sim S(k)$. In that case, we know that the number $\Phi(r)$ assigned to r must be exactly k. This is unlikely however. The most common situation is that we will find two consecutive standard sequences $S(k-1)$ and $S(k)$ such that $r \succ S(k-1)$ and $S(k) \succ r$ (see Figure 3). This means that $\Phi(r)$ must be such that $k-1 < \Phi(r) < k$. We seem to be in trouble here since, as before, $\Phi(r)$ is not exactly determined. How can you proceed? This depends on your technology for preparing perfect copies.

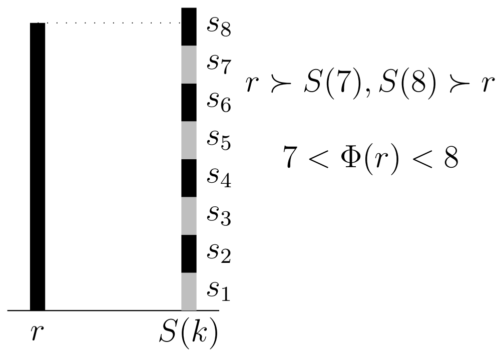


Figure 3: Using standard sequences.

Imagine that you are able to prepare perfect copies not only of the standard rod but also of any object. You may then prepare several copies (r_1, $r_2, \ldots$) of the rod r. You can now compare a composite object made out of two perfect copies of r with your standard sequences $S(1), S(2), \ldots$ As before, you shall eventually arrive at locating $\Phi(r_1 \circ r_2) = 2\Phi(r)$ within an interval of width 1. This means that the interval of imprecision surrounding $\Phi(r)$ has been divided by two. Continuing this process, considering longer and longer sequences of perfect copies of r, you will keep on reducing the width of the interval containing $\Phi(r)$. This means that you can approximate $\Phi(r)$ with any given level of precision. Mathematically a unique value for $\Phi(r)$ will be obtained using a simple limiting argument.

Supposing that you are in position to prepare perfect copies of any object is a strong technological requirement. When this is not possible, there still exists a way out. Instead of preparing a perfect copy of r you may also try to increase the granularity of your standard sequence. This means building an object t that you would be able to replicate perfectly and such that concatenating t with one of its perfect replicas gives an object that has exactly the length of the standard object s_0, i.e. $\Phi(t) = 1/2$. Now considering standard sequences based on t, you will be able to increase by a factor 2 the precision with which we measure the length of r. Repeating the process, i.e. subdividing t, will lead, as before, to a unique limiting value for $\Phi(r)$.

The mathematical machinery underlying the measurement process informally described above (called "extensive measurement") rests on the theory of ordered groups. It is beautifully described and illustrated in [115]. Although the underlying principles are simple, we may expect complications to occur e.g. when not all concatenations are feasible, when there is some level (say the velocity of light if we were to measure speed) that cannot be exceeded or when it comes to relate different measures. See [115, 126, 151] for a detailed treatment.

Clearly, this was an overly detailed and unnecessary complicated description of how length could be measured. Since our aim is to eventually deal with "measurement" in the Social Sciences, it may however be useful to keep the above process in mind. Its basic ingredients are the following:

- well-behaved relations $\succ$ and $\sim$ allowing to compare objects,
- a concatenation operation $\circ$ allowing to consider composite objects,
- consistency requirements linking $\succ$, $\sim$ and $\circ$,
- the ability to prepare perfect copies of some objects in order to build standard sequences.

Basically, conjoint measurement is a quite ingenious way to perform related measurement operations when no concatenation operation is available. This will however require that objects can be evaluated along several dimensions. Before explaining how this might work, it is worth explaining the context in which such measurement might prove useful.

Remark 5
It is often asserted that "measurement is impossible in the Social Sciences" precisely because the Social Scientist has no way to define a concatenation operation. Indeed, it would seem hazardous to try to concatenate the intelligence of two subjects or the pain of two patients. Even, when there seems to be a concatenation operation readily available, it does not always fit the purposes of extensive measurement. Consider for instance an individual expressing preferences for the quantity of the 2 goods he consumes. The objects therefore take the well structured form of points in the positive orthant of $\mathbb{R}^2$. There seems to be an obvious concatenation operation here: $(x, y) \circ (z, w)$ might simply be taken to be $(x + y, z + w)$. However a fairly rational person, consuming pants and jackets, may indeed prefer $(3, 0)$ (3 pants and no jacket) to $(0, 3)$ (no pants and 3 jackets) but at the same time prefer $(3, 3)$ to $(6, 0)$. This implies that these preferences cannot be explained by a measure that would be additive with respect to the concatenation operation consisting in adding the quantities of the two goods consumed. Indeed $(3, 0) \succ (0, 3)$ implies $\Phi(3, 0) > \Phi(0, 3)$, which implies

$\Phi(3,0) + \Phi(3,0) > \Phi(0,3) + \Phi(3,0)$. Additivity with respect to concatenation should then imply that $(3,0) \circ (3,0) \succ (0,3) \circ (3,0)$ that is $(6,0) \succ (3,3)$.

The power of conjoint measurement will precisely be to provide a means to bypass this absence of readily available concatenation operation as soon as the objects are evaluated on several dimensions. •

1.3 An example: Even swaps

The even swaps technique described and advocated in [107, 108, 148] is a simple way to deal with decision problems involving several attributes that do not have recourse to a formal representation of preferences, which will be the subject of conjoint measurement. Because this technique is simple and may be quite useful, we describe it below using the same example as in [107]. This will also allow to exemplify the type of problems that are dealt with in decision analysis applications of conjoint measurement.

Example 6 (Even swaps technique)
A consultant considers renting a new office. Five different locations have been identified after a careful consideration of many possibilities, rejecting all those that do not meet a number of requirements.

His feeling is that five distinct characteristics, we shall say five attributes, of the possible locations should enter into his decision: his daily commute time (expressed in minutes), the ease of access for his clients (expressed as the percentage of his present clients living close to the office), the level of services offered by the new office (expressed on an ad hoc scale with three levels: A (all facilities available), B (telephone and fax), C (no facilities)), the size of the office expressed in square feet, and the monthly cost expressed in dollars.

The evaluation of the five offices is given in Table 1 The consultant has

	a	b	c	d	e
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

Table 1: Evaluation of the 5 offices on the 5 attributes.

well-defined preferences on each of these attributes. His preference increases with the level of access for his clients, the level of services of the office and its size. It decreases with commute time and cost. This gives a first easy way to compare alternatives through the use of *dominance*.

An alternative y is dominated by an alternative x if x is at least as good as y on *all* attributes while being strictly better for at least one attribute. Clearly dominated alternatives are not candidate for the final choice and may, thus, be dropped from consideration. The reader will easily check that, on this example, alternative b dominates alternative e: e and b have similar size but b is less expensive, involves a shorter commute time, an easier access to clients and a better level of services. We may therefore forget about alternative e. This is the only case of "pure dominance" in our table. It is however easy to see that d is "close" to dominating a, the only difference in favor of a being on the cost attribute (50 $ per month). This is felt more than compensated by the differences in favor of d on all other attributes: commute time (20 minutes), client access (35 %) and size (150 sq. feet).

Dropping now all alternatives that are not candidate for choice, this initial investigation allows to reduce the problem to:

	b	c	d
Commute	25	20	25
Clients	80	70	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

A natural way to proceed is then to assess trade-offs. Observe that all alternatives but b have a common evaluation on commute time. We may therefore ask the consultant, starting with office c, what gain on client access would compensate a loss of 5 minutes on commute time. We are looking for an alternative c' that would be evaluated as follows:

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

and judged indifferent to c. Although this is not an easy question, it is clearly crucial in order to structure preferences.

Remark 7

We do not envisage in this paper the possibility of lexicographic preferences, in which such tradeoffs do not occur, see [53, 54, 143]. Lexicographic preferences may also be combined with the possibility of "local" tradeoffs, see [20, 58, 122]. •

Suppose that the answer is that for $\delta = 8$, it is reasonable to assume that c and c' would be indifferent. This means that the decision table can now be reformulated as follows:

	b	c'	d
Commute	25	25	25
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

It is then apparent that all alternatives have a similar evaluation on the first attribute which, therefore, is not useful to discriminate between alternatives and may be forgotten. The reduced decision table is now as follows:

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

There is no case of dominance in this reduced table. Therefore further simplification calls for the assessment of new tradeoffs. Using cost as the reference attribute, we then proceed to "neutralize" the service attribute. Starting with office c', this means asking for the increase in monthly cost that the consultant would just be prepared to pay to go from level "C" of service to level "B". Suppose that this increase is roughly 250 \$. This defines alternative c''. Similarly, starting with office d we ask for the reduction of cost that would exactly compensate a reduction of services from "A" to "B". Suppose that the answer is 100 \$ a month, which defines alternative d'. The decision table is now reshaped as:

	b	c''	d'
Clients	80	78	85
Services	B	**B**	**B**
Size	700	500	950
Cost	1700	**1750**	**1800**

We may now forget about the second attribute which does not discriminate any more between alternatives. When this is done, it is apparent that c'' is now dominated by b and can be suppressed. Therefore, the decision table at this stage looks like the following:

	b	d'
Clients	80	85
Size	700	950
Cost	1700	1800

Unfortunately, this table reveals no case of dominance. New tradeoffs have to be assessed. We may now ask, starting with office b, what additional cost the consultant would be ready to incur to increase its size by 250 square feet. Suppose that the rough answer is 250 $ a month, which defines b'. We are now facing the following table:

	b'	d'
Clients	80	85
Size	**950**	950
Cost	**1950**	1800

Attribute size may now be dropped from consideration. But, when this is done, it is clear that d' dominates b'. Hence it seems obvious to recommend office d as the final choice. ◇

The above process is simple and looks quite obvious. If this works, why be interested at all in "measurement" if the idea is to help someone to come up with a decision?

First observe that in the above example, the set of alternatives was relatively small. In many practical situations, the set of objects to compare is much larger than the set of alternatives in our example. Using the even swaps technique could then require a considerable number of difficult tradeoff questions. Furthermore, as the output of the technique is not a preference model but just the recommendation of an alternative in a given set, the appearance of new alternatives (e.g. because a new office is for rent) would require starting a new round of questions. This is likely to be highly frustrating. Finally, the informal even swaps technique may not be well adapted to the, many, situations, in which the decision under study takes place in a complex organizational environment. In such situations, having a formal model to be able to communicate and to convince is an invaluable asset. Such a model will furthermore allow to conduct extensive sensitivity analysis and, hence, to deal with imprecision both in the evaluations of the objects to compare and in the answers to difficult questions concerning tradeoffs.

This clearly leaves room for a more formal approach to structure preferences. But where can "measurement" be involved in the process? It should be observed that, beyond surface, there are many analogies between the even swap process and the measurement of length envisaged above.

First, note that, in both cases, objects are compared using binary relations. In the measurement of length, the binary relation $\succ$ reads "is longer than". Here it reads "is preferred to". Similarly, the relation $\sim$ reading before "has equal length" now reads "is indifferent to". We supposed in the measurement of length process that $\succ$ and $\sim$ would nicely combine in experiments: if $r \succ r'$ and $r' \sim r''$ then we should observe that $r \succ r''$. Implicitly, a similar hypothesis was made in the even swaps technique. To realize that this is the case, it is worth summarizing the main steps of the argument.

We started with the following decision table 1. Our overall recommendation was to rent office d. This means that we have reason to believe that d is preferred to all other potential locations, i.e. $d \succ a$, $d \succ b$, $d \succ c$, and $d \succ e$. How did we arrive logically at such a conclusion?

Based on the initial table, using dominance and quasi-dominance, we concluded that b was preferable to e and that d was preferable to a. Using symbols, we have $b \succ e$ and $d \succ a$. After assessing some tradeoffs, we concluded, using dominance, that $b \succ c''$. But remember, c'' was built so as to be indifferent to c' and, in turn, c' was built so as to be indifferent to c. That is, we have $c'' \sim c'$ and $c' \sim c$. Later, we built an alternative d' that is indifferent to d ($d \sim d'$) and an alternative b' that is indifferent to b ($b \sim b'$). We then concluded, using dominance, that d' was preferable to b' ($d' \succ b'$). Hence, we know that:

$$d \succ a, b \succ e,$$
$$c'' \sim c', c' \sim c, b \succ c'',$$
$$d \sim d', b \sim b', d' \succ b'.$$

Using the consistency rules linking $\succ$ and $\sim$ that we envisaged for the measurement of length, it is easy to see that the last line implies $d \succ b$. Since $b \succ e$, this implies $d \succ e$. It remains to show that $d \succ c$. But the second line leads to, combining $\succ$ and $\sim$, $b \succ c$. Therefore $d \succ b$ leads to $d \succ c$ and we are home. Hence, we have used the same properties for preference and indifference as the properties of "is longer than" and "has equal length" that we hypothesized in the measurement of length.

Second it should be observed that expressing tradeoffs leads, indirectly, to equating the "length" of "preference intervals" on different attributes. Indeed, remember how c' was constructed above: saying that c and c' are indifferent more or less amounts to saying that the interval $[25, 20]$ on commute time has exactly the same "length" as the interval $[70, 78]$ on client access. Consider now an alternative f that would be identical to c except that it has a client access at 78%. We may again ask which increase in client access would compensate a loss of 5 minutes on commute time. In a tabular form we are now comparing the following two alternatives:

	f	f'
Commute	20	25
Clients	78	$78 + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

Suppose that the answer is that for $\delta = 10$, f and f' would be indifferent. This means that the interval $[25, 20]$ on commute time has exactly the same length as the interval $[78, 88]$ on client access. Now, we know that the preference intervals $[70, 78]$ and $[78, 88]$ have the same "length". Hence, tradeoffs provide a means to equate two preference intervals on the same attribute. This brings us quite close to the construction of standard sequences. This, we shall shortly do.

How does this information about the "length" of preference intervals relate to judgements of preference or indifference? Exactly as in the even swaps technique. You can use this measure of "length" modifying alternatives in such a way that they only differ on a single attribute and then use a simple dominance argument.

Conjoint measurement techniques may roughly be seen as a formalization of the even swaps technique that leads to building a numerical model of preferences much in the same way that we built a numerical model for length. This will require assessment procedures that will rest on the same principles as the standard sequence technique used for length. This process of "measuring preferences" is not an easy one. It will however lead to a numerical model of preference that will not only allow us to make a choice within a limited number of alternatives but that can serve as an input of computerized optimization algorithms that will be able to deal with much more complex cases.

2 Definitions and notation

Before entering into the details of how conjoint measurement may work, a few definitions and notation will be needed.

2.1 Binary relations

A *binary relation* $\succsim$ on a set A is a subset of $A \times A$. We write $a \succsim b$ instead of $(a, b) \in \succsim$. A binary relation $\succsim$ on A is said to be:

- *reflexive* if $[a \succsim a]$,
- *complete* if $[a \succsim b$ or $b \succsim a]$,
- *symmetric* if $[a \succsim b] \Rightarrow [b \succsim a]$,
- *asymmetric* if $[a \succsim b] \Rightarrow [Not[b \succsim a]]$,
- *transitive* if $[a \succsim b$ and $b \succsim c] \Rightarrow [a \succsim c]$,
- *negatively transitive* if $[\,Not[\,a \succsim b\,]$ and $Not[\,b \succsim c\,]\,] \Rightarrow Not[\,a \succsim c\,]$,

for all $a, b, c \in A$.

The *asymmetric* (resp. *symmetric*) part of $\succsim$ is the binary relation $\succ$ (resp. $\sim$) on A defined letting, for all $a, b \in A$, $a \succ b \Leftrightarrow [a \succsim b$ and $Not(b \succsim a)]$ (resp. $a \sim b \Leftrightarrow [a \succsim b$ and $b \succsim a]$). A similar convention will hold when $\succsim$ is subscripted and/or superscripted.

A *weak order* (resp. an *equivalence relation*) is a complete and transitive (resp. reflexive, symmetric and transitive) binary relation. For a detailed analysis of the use of binary relation as tools for preference modelling we refer to [4, 52, 60, 144, 150, 152]. The weak order model underlies the examples that were presented in introduction. Indeed, the reader will easily prove the following.

Proposition 8
Let $\succsim$ be a weak order on A. Then:

- $\succ$ *is transitive,*
- $\succ$ *is negatively transitive,*
- $\sim$ *is transitive,*
- $[a \succ b$ *and* $b \sim c] \Rightarrow a \succ c$,
- $[a \sim b$ *and* $b \succ c] \Rightarrow a \succ c$.

2.2 Binary relations on product sets

In the sequel, we consider a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements $x, y, z, \ldots$ of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes. A typical binary relation on X is still denoted as $\succsim$, interpreted as an "at least as good as" preference relation between multi-attributed alternatives with $\sim$ interpreted as indifference and $\succ$ as strict preference.

For any non empty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}).

Remark 9
Throughout this paper, we shall work with a binary relation defined on a product set. This setup conceals the important work that has to be done in practice to make it useful:

- the structuring of objectives [3, 14, 15, 104, 105, 106, 141, 146],
- the definition of adequate attributes to measure the attainment of objectives [73, 85, 103, 109, 156, 190, 197],
- the definition of an adequate family of attributes [22, 108, 156, 157, 191],
- the modelling of uncertainty, imprecision and inaccurate determination [21, 25, 108, 154].

The importance of this "preliminary" work should not be forgotten in what follows. •

2.3 Independence and marginal preferences

In conjoint measurement, one starts with a preference relation $\succsim$ on X. It is then of vital importance to investigate how this information makes it possible to define preference relations on attributes or subsets of attributes.

Let $J \subseteq N$ be a nonempty set of attributes. We define the *marginal relation* $\succsim_J$ induced on X_J by $\succsim$ letting, for all $x_J, y_J \in X_J$:

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with asymmetric (resp. symmetric) part $\succ_J$ (resp. $\sim_J$). When $J = \{i\}$, we often abuse notation and write $\succsim_i$ instead of $\succsim_{\{i\}}$. Note that if $\succsim$ is reflexive (resp. transitive), the same will be true for $\succsim_J$. This is clearly not true for completeness however.

Definition 10 (Independence)
Consider a binary relation $\succsim$ on a set $X = \prod_{i=1}^{n} X_i$ and let $J \subseteq N$ be a nonempty subset of attributes. We say that $\succsim$ is independent for J if, for all $x_J, y_J \in X_J$,

$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J.$$

If $\succsim$ is independent for all non empty subsets of N, we say that $\succsim$ is independent. *If $\succsim$ is independent for all subsets containing a single attribute, we say that $\succsim$ is* weakly independent.

In view of (2), it is clear that the additive value model will require that $\succsim$ is independent. This crucial condition says that common evaluations on some attributes do not influence preference. Whereas independence implies weak independence, it is well-know that the converse is not true [193].

Remark 11
Independence, or at least weak independence, is an almost universally accepted hypothesis in multiple criteria decision making. It cannot be overemphasized that it is easy to find examples in which it is inadequate.

If a meal is described by the two attributes, main course and wine, it is highly likely that most gourmets will violate independence, preferring red wine with beef and white wine with fish. Similarly, in a dynamic decision problem, a preference for variety will often lead to violating independence: you may prefer Pizza to Steak, but your preference for meals today (first attribute) and tomorrow (second attribute) may well be such that (Pizza, Steak) preferred to (Pizza, Pizza), while (Steak, Pizza) is preferred to (Steak, Steak).

Many authors [106, 156, 191] have argued that such failures of independence were almost always due to a poor structuring of attributes (in our choice of meal, preference for variety should be explicitly modelled). •

When $\succsim$ is a weakly independent weak order, marginal preferences are well-behaved and combine so as to give meaning to the idea of dominance that we already encountered. The easy proof of the following is left as an easy exercise.

Proposition 12
Let $\succsim$ be a weakly independent weak order on $X = \prod_{i=1}^{n} X_i$. Then:

- *$\succsim_i$ is a weak order on X_i,*
- *$[x_i \succsim_i y_i$, for all $i \in N] \Rightarrow x \succsim y$,*
- *$[x_i \succsim_i y_i$, for all $i \in N$ and $x_j \succ_j y_j$ for some $j \in N] \Rightarrow x \succ y$,*

3 The additive value model in the "rich" case

The purpose of this section and the following is to present the conditions under which a preference relation on a product set may be represented by the additive value function model (2) and how such a model can be assessed. We begin here with the case that most closely resembles the measurement of length envisaged in section 1.2.

3.1 Outline of theory

When the structure of X is supposed to be "adequately rich", conjoint measurement is a quite clever adaptation of the process that we described in section 1.2 for the measurement of length. What will be measured here are the "length" of preference intervals on an attribute using a preference interval on another attribute as a standard.

3.1.1 The case of two attributes

Consider first the two attribute case. Hence the relation $\succsim$ is defined on a set $X = X_1 \times X_2$. Clearly, in view of (2), we need to suppose that $\succsim$ is an *independent weak order*. Consider two levels $x_1^0, x_1^1 \in X_1$ on the first attribute such that $x_1^1 \succ_1 x_1^0$, i.e. x_1^1 is preferable to x_1^0. This makes sense because, we supposed that $\succsim$ is *independent*. Note also that we shall have to exclude the case in which all levels on the first attribute would be indifferent in order to be able to find such levels.

Now choose any $x_2^0 \in X_2$. The, arbitrarily chosen, element $(x_1^0, x_2^0) \in X$ will be our "reference point". The basic idea is to use this reference point and the "unit" on the first attribute given by the reference preference interval $[x_1^0, x_1^1]$ to build a standard sequence on the preference intervals on the second attribute. Hence, we are now looking for an element $x_2^1 \in X_2$ that would be such that:

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0). \tag{3}$$

Clearly this will require the structure of X_2 to be adequately "rich" so as to find the level $x_2^1 \in X_2$ such that the reference preference interval on the first attribute $[x_1^0, x_1^1]$ is exactly matched by a preference interval of the same "length" on the second attribute $[x_2^0, x_2^1]$. Technically, this calls for a solvability assumption or, more restrictively, for the supposition that X_2 has a (topological) structure that is close to that of an interval of $\mathbb{R}$ and that $\succsim$ is "somehow" continuous.

If such a level x_2^1 can be found, model (2) implies:

$$v_1(x_1^0) + v_2(x_2^1) = v_1(x_1^1) + v_2(x_2^0) \text{ so that}$$
$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0). \quad (4)$$

Let us now fix the origin of measurement letting:

$$v_1(x_1^0) = v_2(x_2^0) = 0,$$

and our unit of measurement letting:

$$v_1(x_1^1) = 1 \text{ so that } v_1(x_1^1) - v_1(x_1^0) = 1.$$

Using (4), we therefore obtain $v_2(x_2^1) = 1$. We have therefore found an interval between levels on the second attribute ($[x_2^0, x_2^1]$) that exactly matches our reference interval on the first attribute ($[x_1^0, x_1^1]$). We may now proceed to build our standard sequence on the second attribute (see Figure 4) asking for levels $x_2^2, x_2^3, \ldots$ such that:

$$\begin{array}{c}
(x_1^0, x_2^2) \sim (x_1^1, x_2^1), \\
(x_1^0, x_2^3) \sim (x_1^1, x_2^2), \\
\cdots \\
(x_1^0, x_2^k) \sim (x_1^1, x_2^{k-1}).
\end{array}$$

As above, using (2) leads to:

$$\begin{array}{c}
v_2(x_2^2) - v_2(x_2^1) = v_1(x_1^1) - v_1(x_1^0), \\
v_2(x_2^3) - v_2(x_2^2) = v_1(x_1^1) - v_1(x_1^0), \\
\cdots \\
v_2(x_2^k) - v_2(x_2^{k-1}) = v_1(x_1^1) - v_1(x_1^0),
\end{array}$$

so that:

$$v_2(x_2^2) = 2, v_2(x_2^3) = 3, \ldots, v_2(x_2^k) = k.$$

This process of building a standard sequence of the second attribute therefore leads to defining v_2 on a number of, carefully, selected elements of X_2.

Remember the standard sequence that we built for length in section 1.2. An implicit hypothesis was that the length of any rod could be exceeded by the length of a composite object obtained by concatenating a sufficient number of perfect copies of a standard rod. Such an hypothesis is called "Archimedean" since it mimics the property of the real numbers saying that

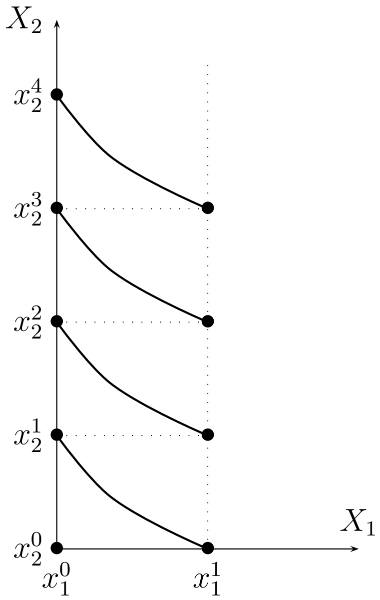


Figure 4: Building a standard sequence on X_2.

for any positive real numbers x, y it is true that $nx > y$ for some integer n, i.e. y, no matter how large, may always be exceeded by taking any x, no matter how small, and adding it with itself and repeating the operation a sufficient number of times. Clearly, we will need a similar hypothesis here. Failing it, there might exist a level $y_2 \in X_2$ that will never be "reached" by our standard sequence, i.e. such that $y_2 \succ_2 x_2^k$, for $k = 1, 2, \ldots$. For measurement models in which this Archimedean condition is omitted, see [139, 176].

Remark 13

At this point a good exercise for the reader is to figure out how we may extend the standard sequence to cover levels of X_2 that are "below" the reference level x_2^0. This should not be difficult. •

Now that a standard sequence is built on the second attribute, we may use any part of it to build a standard sequence on the first attribute. This will require finding levels $x_1^2, x_1^3, \ldots \in X_1$ such that (see Figure 5):

$$\begin{aligned}
(x_1^2, x_2^0) &\sim (x_1^1, x_2^1), \\
(x_1^3, x_2^0) &\sim (x_1^2, x_2^1), \\
&\cdots \\
(x_1^k, x_2^0) &\sim (x_1^{k-1}, x_2^1).
\end{aligned}$$

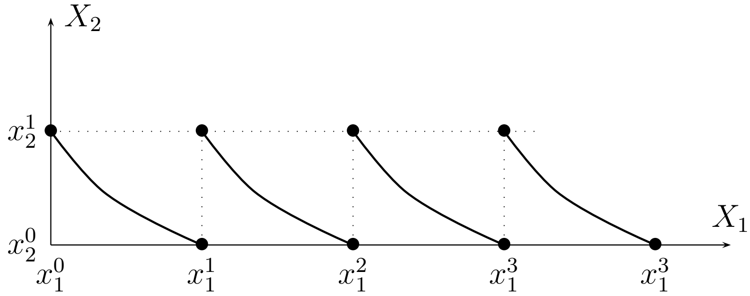


Figure 5: Building a standard sequence on X_1.

Using (2) leads to:

$$\begin{aligned} v_1(x_1^2) - v_1(x_1^1) &= v_2(x_2^1) - v_2(x_2^0), \\ v_1(x_1^3) - v_1(x_1^2) &= v_2(x_2^1) - v_2(x_2^0), \\ &\dots \\ v_1(x_1^k) - v_1(x_1^{k-1}) &= v_2(x_2^1) - v_2(x_2^0), \end{aligned}$$

so that:

$$v_1(x_1^2) = 2, v_1(x_1^3) = 3, \dots, v_1(x_2^k) = k.$$

As was the case for the second attribute, the construction of such a sequence will require the structure of X_1 to be adequately rich, which calls for a solvability assumption. An Archimedean condition will also be needed in order to be sure that all levels of X_1 can be reached by the sequence.

We have now defined a "grid" in X (see Figure 6) and we have $v_1(x_1^k) = k$ and $v_2(x_2^k) = k$ for all elements of this grid. Intuitively such numerical assignments seem to define an adequate additive value function on the grid. We have to prove that this intuition is correct. Let us first verify that:

$$\alpha + \beta = \gamma + \delta = \epsilon \Rightarrow (x_1^\alpha, x_2^\beta) \sim (x_1^\gamma, x_2^\delta). \tag{5}$$

When $\epsilon = 1$, (5) holds by construction because we have: $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$. When $\epsilon = 2$, we know that $(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$ and $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$ and the claim is proved using the transitivity of $\sim$.

Consider now the $\epsilon = 3$ case. We have $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$ and $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$. It remains to be shown that $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$ (see the dotted arc in Figure 6). This does not seem to follow from the previous conditions that we more or less explicitly used: transitivity, independence, "richness", Archimedean. Indeed, it does not. Hence, we have to suppose that: $(x_1^2, x_2^0) \sim (x_1^0, x_2^2)$

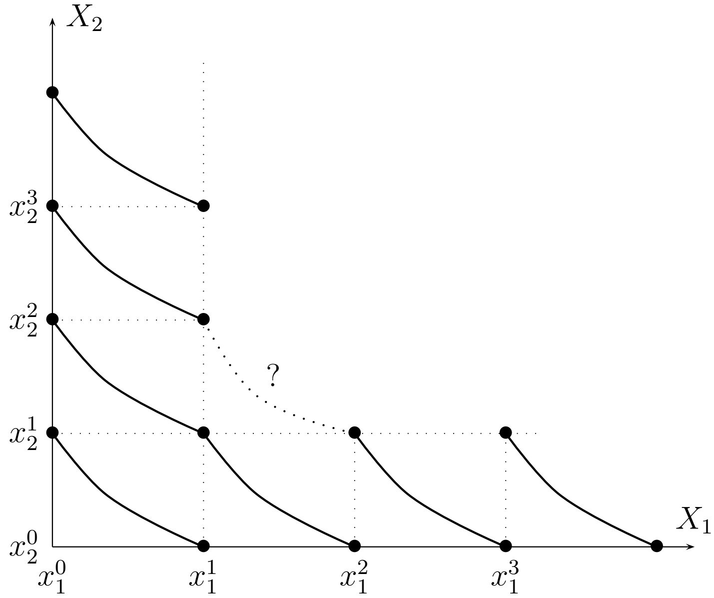


Figure 6: The grid.

and $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$ imply $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$. This condition, called the Thomsen condition, is clearly necessary for (2). The above reasoning now easily extends to all points on the grid, using weak ordering, independence and the Thomsen condition. Hence, (5) holds on the grid.

It remains to show that:

$$\epsilon = \alpha + \beta > \epsilon' = \gamma + \delta \Rightarrow (x_1^\alpha, x_2^\beta) \succ (x_1^\gamma, x_2^\delta). \tag{6}$$

Using transitivity, it is sufficient to show that (6) holds when $\epsilon = \epsilon' + 1$. By construction, we know that $(x_1^1, x_2^0) \succ (x_1^0, x_2^0)$. Using independence this implies that $(x_1^1, x_2^k) \succ (x_1^0, x_2^k)$. Using (5) we have $(x_1^1, x_2^k) \sim (x_1^{k+1}, x_2^0)$ and $(x_1^0, x_2^k) \sim (x_1^k, x_2^0)$. Therefore we have $(x_1^{k+1}, x_2^0) \succ (x_1^k, x_2^0)$, the desired conclusion.

Hence, we have built an additive value function of a suitably chosen grid (see Figure 7). The logic of the assessment procedure is then to assess more and more points somehow considering more finely grained standard sequences. The two techniques evoked for length may be used here depending on the underlying structure of X. A limiting process then unambiguously defines the functions v_1 and v_2. Clearly such v_1 and v_2 are intimately related. Once we have chosen an arbitrary reference point (x_1^0, x_2^0) and a level x_1^1 defining the unit of measurement, the process just described entirely defines v_1 and v_2. It follows that the only possible transformations that can be applied to v_1 and v_2 is to multiply both by the same positive number α and to add

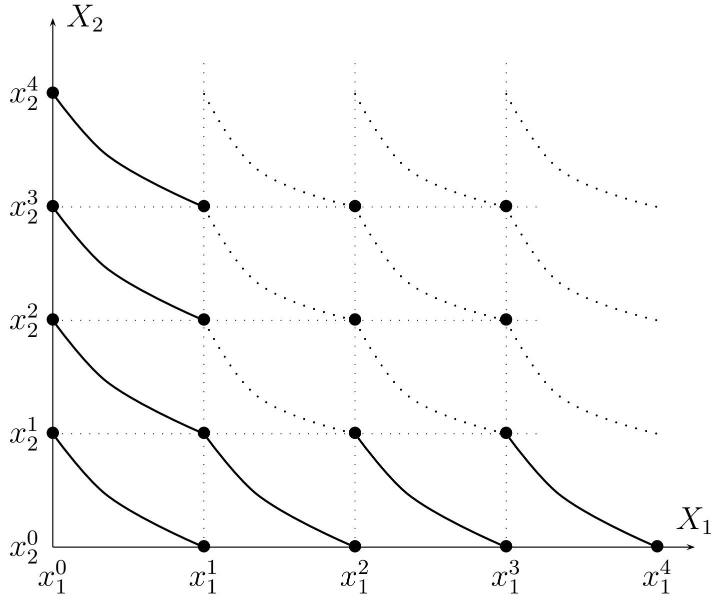


Figure 7: The entire grid.

to both a, possibly different, constant. This is usually summarized saying that v_1 and v_2 define interval scale with a common unit.

The above reasoning is a very rough sketch of the proof of the existence of an additive value function when $n = 2$, as well as a sketch of how it could be assessed. Careful readers will want to refer to [52, 115, 193].

Remark 14
As was already the case with the even swap technique, it is worth emphasizing that this assessment technique makes no use of the vague notion of the "importance" of the various attributes. The "importance" is captured here in the lengths of the preference intervals on the various attributes.

A common but critical mistake is to confuse the additive value function model (2) with a weighted average and to try to assess weights asking whether an attribute is "more important" than another. This makes no sense. •

Remark 15
The measurement of length through standard sequences envisaged above leads to a scale that is unique once the unit of measurement is chosen. At this point, a good exercise for the reader is to find an intuitive explanation to the fact that, when measuring the "length" of preference intervals, the origin of measurement becomes arbitrary. The analogy with the the measurement

of duration on the one hand and dates, as given in a calendar, on the other hand should help. •

3.1.2 The case of more than two attributes

The good news now is that the process is exactly the same when there are more than two attributes. With one surprise: the Thomsen condition is no more needed to prove that the standard sequences defined on each attribute lead to an adequate value function on the grid. A heuristic explanation of this strange result is that, when $n = 2$, there is no difference between independence and weak independence. This is no more true when $n \geq 3$ and assuming independence is much stronger than just assuming weak independence.

3.2 Statement of results

We use below the "algebraic approach" [113, 115, 127]. A more restrictive approach using a topological structure on X is given in [41, 193]. We formalize below the conditions informally introduced in the preceding section. The reader not interested in the precise statement of the results or, better, having already written down his own statement, may skip this section.

Definition 16 (Thomsen condition)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. It is said to satisfy the Thomsen condition if

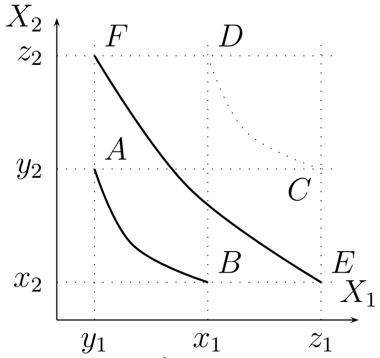
$$(x_1, x_2) \sim (y_1, y_2) \text{ and } (y_1, z_2) \sim (z_1, x_2) \Rightarrow (x_1, z_2) \sim (z_1, y_2),$$

for all $x_1, y_1, z_1 \in X_1$ and all $x_2, y_2, z_2 \in X_2$.

Figure 8 shows how the Thomsen condition uses two "indifference curves" to place a constraint on a third one. This was needed above to prove that an additive value function existed on our grid. Remember that the Thomsen condition is only needed when $n = 2$; hence, we only stated it in this case.

Definition 17 (Standard sequences)
A standard sequence on attribute $i \in N$ is a set $\{a_i^k : a_i^k \in X_i, k \in K\}$ where K is a set of consecutive integers (positive or negative, finite or infinite) such that there are $x_{-i}, y_{-i} \in X_{-i}$ satisfying $Not[x_{-i} \sim_{-i} y_{-i}]$ and $(a_i^k, x_{-i}) \sim (a_i^{k+1}, y_{-i})$, for all $k \in K$.

A standard sequence on attribute $i \in N$ is said to be *strictly bounded* if there are $b_i, c_i \in X_i$ such that $b_i \succ_i a_i^k \succ_i c_i$, for all $k \in K$. It is then clear



$$\left.\begin{array}{l} A \sim B \\ E \sim F \end{array}\right\} \Rightarrow C \sim D$$

Figure 8: The Thomsen condition.

that, when model (2) holds, any strictly bounded standard sequence must be finite.

Definition 18 (Archimedean)
For all $i \in N$, any strictly bounded standard sequence on $i \in N$ is finite.

The following condition rules out the case in which a standard sequence cannot be built because all levels are indifferent.

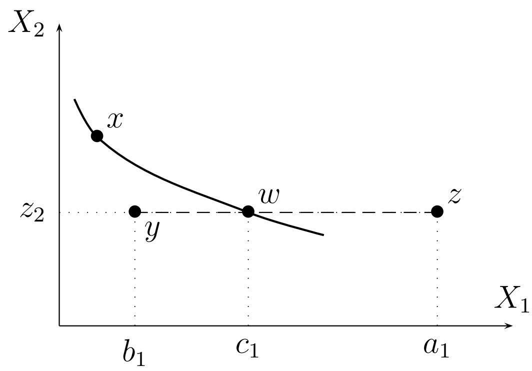
Definition 19 (Essentiality)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \cdots \times X_n$. Attribute $i \in N$ is said to be essential if $(x_i, a_{-i}) \succ (y_i, a_{-i})$, for some $x_i, y_i \in X_i$ and some $a_{-i} \in X_{-i}$.

Definition 20 (Restricted Solvability)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \cdots \times X_n$. Restricted solvability is said to hold with respect to attribute $i \in N$ if, for all $x \in X$, all $z_{-i} \in X_{-i}$ and all $a_i, b_i \in X_i$, $[(a_i, z_{-i}) \succsim x \succsim (b_i, z_{-i})] \Rightarrow [x \sim (c_i, z_{-i})$, for some $c_i \in X_i]$.

Remark 21
Restricted solvability is illustrated in Figure 9 in the case where $n = 2$. It says that, given any $x \in X$, if it is possible find two levels $a_i, b_i \in X_i$ such that when combined with a certain level $z_{-i} \in X_{-i}$ on the other attributes, (a_i, z_{-i}) is above x and (b_i, z_{-i}) is below x, it should be possible to find a level c_i, "in between" a_i and b_i, such that such that (c_i, z_{-i}) is exactly indifferent to x.

A much stronger hypothesis is unrestricted solvability asserting that for all $x \in X$ and all $z_{-i} \in X_{-i}$, $x \sim (c_i, z_{-i})$, for some $c_i \in X_i$. Its use leads however to much simpler proofs.

It is easy to imagine situations in which restricted solvability might hold while unrestricted solvability would fail. Suppose, e.g. that a firm has to choose between several investment projects, two attributes being the Net Present Value (NPV) of the projects and their impact on the image of the firm in the public. Consider a project consisting in investing in the software market. It has a reasonable NPV and no adverse consequences on the image of the firm. Consider now another project that could have dramatic consequences on the image of the firm, because it leads to investing the market of cocaine. Unrestricted solvability would require that by sufficiently increasing the NPV of the second project it would become indifferent to the more standard project of investing in the software market. This is not required by restricted solvability. •



$$\left.\begin{array}{l} z \succ x \\ x \succ y \end{array}\right\} \Rightarrow \text{ there is a } w \text{ such that } x \sim w$$

Figure 9: Restricted Solvability on X_1.

We are now in position to state the central results concerning model (2). Proofs may be found in [115, 194].

Theorem 22 (Additive utility when $n = 2$)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. If restricted solvability holds on all attributes and each attribute is essential then $\succsim$ has a representation in model (2) if and only if $\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions

Furthermore in this representation, v_1 and v_2 define an interval scale with a common unit, i.e. if v_1, v_2 and w_1, w_2 are two pairs of functions satisfying (2), there are real numbers α, β_1, β_2 with $\alpha > 0$ such that, for all $x_1 \in X_1$ and all $x_2 \in X_2$

$$v_1(x_1) = \alpha w_1(x_1) + \beta_1 \text{ and } v_2(x_2) = \alpha w_2(x_2) + \beta_2$$

When $n \geq 3$ and at least three attributes are essential, the above result simplifies in that the Thomsen condition can now be omitted.

Theorem 23 (Additive utility when $n \geq 3$)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \ldots X_n$ with $n \geq 3$. If restricted solvability holds on all attributes and at least 3 attributes are essential then $\succsim$ has a representation in model (2) if and only if $\succsim$ is an independent weak order satisfying the Archimedean condition.

Furthermore this representation defines an interval scale with a common unit.

Remark 24
As mentioned in introduction, the additive value model is central to several fields in decision theory. It is therefore not surprising that much energy has been devoted to analyze variants and refinements of the above results. Among the most significant ones, let us mention:

- the study of cases in which solvability holds only on some or none of the attributes [76, 77, 78, 138],
- the study of the relation between the "algebraic approach" introduced above and the topological one used in [41], see e.g. [102, 110, 193, 194].

The above results are only valid when X is the entire Cartesian product of the sets X_i. Results in which X is a subset of the whole Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ are not easy to obtain, see [34, 164] (the situation is "easier" in the special case of homogeneous product sets, see [195, 196]). •

3.3 Implementation: Standard sequences and beyond

We have already shown above how additive value functions can be assessed using the standard sequence technique. It is worth recalling here some of the characteristics of this assessment procedure:

- It requires the set X_i to be *rich* so that it is possible to find a preference interval on X_i that will exactly match a preference interval on another attribute. This excludes using such an assessment procedure when some of the sets X_i are discrete.

- It relies on *indifference* judgements which, a priori, are less firmly established than preference judgements.

- It relies on judgements concerning fictitious alternatives which, a priori, are harder to conceive than judgements concerning real alternatives.

- The various assessments are thoroughly intertwined and, e.g., an imprecision on the assessment of x_2^1, i.e. the endpoint of the first interval in the standard sequence on X_2 (see Figure 4) will propagate to many assessed values.

The assessment procedure based on standard sequences is therefore rather demanding; this should be no surprise given the proximity between this form of measurement and extensive measurement illustrated above on the case of length. Hence, the assessment procedure based on standard sequences seems to be seldom used in the practice of decision analysis [108, 191].

Many other simplified assessment procedures have been proposed that are less firmly grounded in theory. In many of them, the assessment of the partial value functions v_i relies on *direct* comparison of preference differences without recourse to an interval on another attribute used as a "meter stick". We refer to [45] for a theoretical analysis of these techniques. They are also studied in detail in Chapter XX of this volume (Chapter 6, Dyer).

These procedures include:

- *direct rating* techniques in which values of v_i are directly assessed with reference to two arbitrarily chosen points [47, 48],

- procedures based on *bisection*, the decision-maker being asked to assess a point that is "half way" in terms of preference two reference points,

- procedures trying to build *standard sequences* on each attribute in terms of "preference differences" [115, ch. 4].

An excellent overview of these techniques may be found in [191].

4 The additive value model in the "finite" case

4.1 Outline of Theory

We suppose in this section that $\succsim$ is a binary relation on a finite set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$ (contrary to the preceding section, dealing with subsets of product sets will raise no difficulty here). The finiteness hypothesis clearly invalidates the standard sequence mechanism used till now. On each attribute there will only be finitely many "preference intervals" and exact matches between preference intervals will only happen exceptionally.

Clearly, independence remains a necessary condition for model (2) as before. Given the absence of structure of the set X, it is unlikely that this condition is sufficient to ensure (2). The following example shows that this intuition is indeed correct.

Example 25
Let $X = X_1 \times X_2$ with $X_1 = \{a, b, c\}$ and $X_2 = \{d, e, f\}$. Consider the weak order on X such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ af \succ be \succ cd \succ ce \succ bf \succ cf.$$

It is easy to check that $\succsim$ is independent. Indeed, we may for instance check that:

$$\begin{gathered} ad \succ bd \text{ and } ae \succ be \text{ and } af \succ bf, \\ ad \succ ae \text{ and } bd \succ be \text{ and } cd \succ ce. \end{gathered}$$

This relation cannot however be represented in model (2) since:

$$\begin{aligned} af \succ be &\Rightarrow v_1(a) + v_2(f) > v_1(b) + v_2(e), \\ be \succ cd &\Rightarrow v_1(b) + v_2(e) > v_1(c) + v_2(d), \\ ce \succ bf &\Rightarrow v_1(c) + v_2(e) > v_1(b) + v_2(f), \\ bd \succ ae &\Rightarrow v_1(b) + v_2(d) > v_1(a) + v_2(e). \end{aligned}$$

Summing the first two inequalities leads to:

$$v_1(a) + v_2(f) > v_1(c) + v_2(d).$$

Summing the last two inequalities leads to:

$$v_1(c) + v_2(d) > v_1(a) + v_2(f),$$

a contradiction.

Note that, since no indifference is involved, the Thomsen condition is trivially satisfied. Although it is clearly necessary for model (2), adding it to independence will therefore not solve the problem. ◇

The conditions allowing to build an additive utility model in the finite case were investigated in [1, 2, 162]. Although the resulting conditions turn out to be complex, the underlying idea is quite simple. It amounts to finding conditions under which a system of linear inequalities has a solution.

Suppose that $x \succ y$. If model (2) holds, this implies that:

$$\sum_{i=1}^{n} v_i(x_i) > \sum_{i=1}^{n} v_i(y_i). \tag{7}$$

Similarly if $x \sim y$, we obtain:

$$\sum_{i=1}^{n} v_i(x_i) = \sum_{i=1}^{n} v_i(y_i). \tag{8}$$

The problem is then to find condition on $\succsim$ such that the system of finitely many equalities and inequalities (7-8) has a solution. This is a classical problem in Linear Algebra [74].

Definition 26 (Relation E^m)
Let $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$. We say that

$$(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m)$$

if, for all $i \in N$, $(x_i^1, x_i^2, \ldots, x_i^m)$ is a permutation of $(y_i^1, y_i^2, \ldots, y_i^m)$.

Suppose that $(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m)$ then model (2) implies that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} v_i(x_i^j) = \sum_{j=1}^{m} \sum_{i=1}^{n} v_i(y_i^j).$$

Therefore if $x^j \succsim y^j$ for $j = 1, 2, \ldots, m-1$, it cannot be true that $x^m \succ y^m$. This condition must hold for all $m = 2, 3, \ldots$.

Definition 27 (Condition C^m)
We say that condition C^m holds if

$$[x^j \succsim y^j \text{ for } j = 1, 2, \ldots, m-1] \Rightarrow Not[\, x^m \succ y^m \,]$$

for all $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$ such that

$$(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m).$$

Remark 28
It is not difficult to check that:

- $C^{m+1} \Rightarrow C^m$,
- $C^2 \Rightarrow \succsim$ is independent,
- $C^3 \Rightarrow \succsim$ is transitive. •

We already observed that C^m was implied by the existence of an additive representation. The main result for the finite case states that requiring that $\succsim$ is complete and that C^m holds for $m = 2, 3, \ldots$ is also sufficient. Proofs can be found in [52, 115].

Theorem 29
Let $\succsim$ be a binary relation on a finite set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$. There are real-valued functions v_i on X_i such that (2) *holds if and only if $\succsim$ is complete and satisfies C^m for $m = 2, 3, \ldots$.*

Remark 30
Contrary to the "rich" case envisaged in the preceding section, we have here necessary and sufficient conditions for the additive value model (2). However, it is important to notice that the above result uses a denumerable scheme of conditions. It is shown in [163] that this denumerable scheme cannot be truncated: for all $m \geq 2$, there is a relation $\succsim$ on a finite set X such that C^m holds but violating C^{m+1}. This is studied in more detail in [125, 183, 199]. Therefore, no finite scheme of axioms is sufficient to characterize model (2) for all finite sets X.

Given a finite set X *of given cardinality*, it is well-known that the denumerable scheme of condition can be truncated. The precise relation between the cardinality of X and the number of conditions needed raises difficult combinatorial questions that are studied in [70, 71]. •

Remark 31
It is clear that, if a relation $\succsim$ has a representation in model (2) with functions v_i, it also has a representation using functions $v_i' = \alpha v_i + \beta_i$ with $\alpha > 0$. Contrary to the rich case, the uniqueness of the functions v_i is more complex as shown by the following example.

Example 32
Let $X = X_1 \times X_2$ with $X_1 = \{a, b, c\}$ and $X_1 = \{d, e\}$. Consider the weak order on X such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ cd \succ be \succ ce.$$

This relation has a representation in model (2) with

$$v_1(a) = 3, v_1(b) = 1, v_1(c) = 0, v_2(d) = 3, v_2(e) = 0.5.$$

An equally valid representation would be given taking $v_1(b) = 2$. Clearly this new representation cannot be deduced from the original one applying a positive affine transformation. ◇

Remark 33
Theorem 29 has been extended to the case of an arbitrary set X in [100]. The resulting conditions are however quite complex. This explains why we spent time on this "rich" case in the preceding section. •

Remark 34
The use of a denumerable scheme of conditions in theorem 29 does not facilitate the interpretation and the test of conditions. However it should be noticed that, on a given set X, the test of the C^m conditions amounts to finding if a system of finitely many linear inequalities has a solution. It is well-known that Linear Programming techniques are quite efficient for such a task. •

4.2 Implementation: LP-based assessment

We show how to use LP techniques in order to assess an additive value model (2), without supposing that the sets X_i are rich. For practical purposes, it is not restrictive to assume that we are only interested in assessing a model for a limited range on each X_i. We therefore assume that the sets X_i are bounded so that, using independence, there is a worst value x_{i*} and a most preferable value x_i^*. Using the uniqueness properties of model (2), we may always suppose, after an appropriate normalization, that:

$$v_1(x_{1*}) = v_2(x_{2*}) = \ldots = v_n(x_{n*}) = 0 \text{ and} \tag{9}$$

$$\sum_{i=1}^{n} v_i(x_i^*) = 1. \tag{10}$$

Two main cases arise (see Figures 10 and 11):

- attribute $i \in N$ is discrete so that the evaluation of any conceivable alternative on this attribute belongs to a finite set. We suppose that $X_i = \{x_{i*}, x_i^1, x_i^2, \ldots, x_i^{r_i}, x_i^*\}$. We therefore have to assess $r_i + 1$ values of v_i,

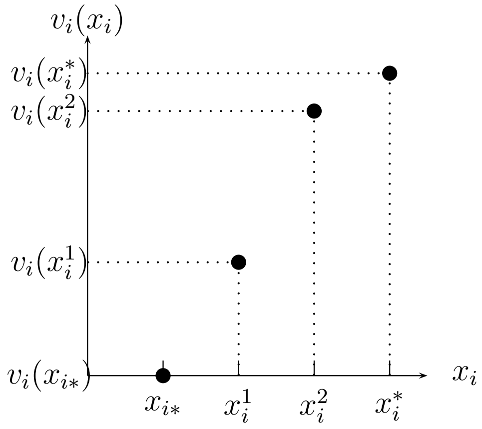


Figure 10: Value function when X_i is discrete.

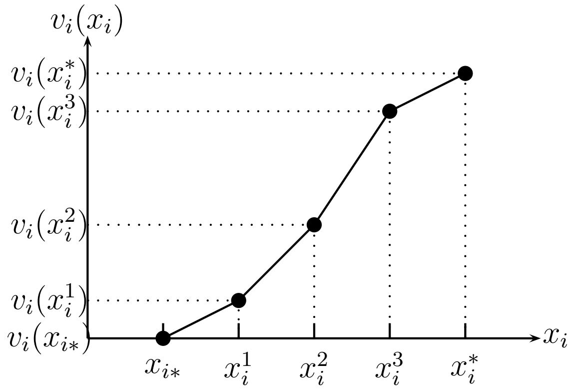


Figure 11: Value function when X_i is continuous.

- the attribute $i \in N$ has an underlying continuous structure. It is hardly restrictive in practice to suppose that $X_i \subset \mathbb{R}$, so that the evaluation of an alternative on this attribute may take any value between x_{i*} and x_i^*. In this case, we may opt for the assessment of a piecewise linear approximation of v_i partitioning the set X_i in $r_i + 1$ intervals and supposing that v_i is linear on each of these intervals. Note that the approximation of v_i can be made more precise simply by increasing the number of these intervals.

With these conventions, the assessment of the model (2) amounts to giving a value to $\sum_{i=1}^{n}(r_i + 1)$ unknowns. Clearly any judgment of preference linking x and y translate into a *linear inequality* between these unknowns. Similarly any judgment of indifference linking x and y translate into a *linear equality.* Linear Programming (LP) offers a powerful tool for testing whether such a system has solutions. Therefore, an assessment procedure can be conceived on the following basis:

- obtain judgments in terms of preference or indifference linking several alternatives in X,
- convert these judgments into linear (in)equalities,
- test, using LP, whether this system has a solution.

If the system has no solution then one may envisage either to propose a solution that will be "as close as possible" from the information obtained, e.g. violating the minimum number of (in)equalities or to suggest the reconsideration of certain judgements. If the system has a solution, one may explore the set of all solutions to this system since they are all candidates for the establishment of model (2). These various techniques depend on:

- the choice of the alternatives in X that are compared: they may be real or fictitious, they may differ on a different number of attributes,
- the way to deal with the inconsistency of the system and to eventually propose some judgments to be reconsidered,
- the way to explore the set of solutions of the system and to use this set as the basis for deriving a prescription.

Linear programming offers of simple and versatile technique to assess additive value functions. All restrictions generating linear constraints of the coefficient of the value function can easily be accommodated. This idea has been often exploited, see [15]. We present below two techniques using it. It should be noticed that rather different techniques have been proposed in the literature on Marketing [32, 92, 93, 101, 118].

4.2.1 UTA [99]

UTA ("UTilité Additive", i.e. additive utility in French) is one of the oldest technique belonging to this family. It is supposed in UTA that there is a subset $Ref \subset X$ of reference alternatives that the decision-maker knows well either because he/she has experienced them or because they have received particular attention. The technique amounts to asking the DM to provide a weak order on Ref. Each preference or indifference relation contained in this weak order is then translated into a linear constraint:

- $x \sim y$ gives an equality $v(x) - v(y) = 0$ and

- $x \succ y$ gives an inequality $v(x) - v(y) > 0$,

where $v(x)$ and $v(y)$ can be expressed as a linear combination of the unknowns as remarked earlier. Strict inequalities are then translated into large inequalities as is usual in Linear Programming, i.e. $v(x) - v(y) > 0$ becomes $v(x) - v(y) \geq \epsilon$ where $\epsilon > 0$ is a very small positive number that should be chosen according to the precision of the arithmetics used by the LP package.

The test of the existence of a solution to the system of linear constraints is done via standard Goal Programming techniques [33] adding appropriate deviation variables. In UTA, each equation $v(x) - v(y) = 0$ is translated into an equation $v(x) - v(y) + \sigma_x^+ - \sigma_x^- + \sigma_y^+ - \sigma_y^- = 0$, where $\sigma_x^+, \sigma_x^-, \sigma_y^+$ and σ_y^- are nonnegative deviation variables. Similarly each inequality $v(x) - v(y) \geq \epsilon$ is written as $v(x) - v(y) + \sigma_x^+ - \sigma_x^- + \sigma_y^+ - \sigma_y^- \geq \epsilon$. It is clear that there will exist a solution to the original system of linear constraints if there is a solution of the LP in which all deviation variables are zero. This can easily be tested using the objective function

$$\text{Minimize } Z = \sum_{x \in Ref} \sigma_x^+ + \sigma_x^- \tag{11}$$

Two cases arise. If the optimal value of Z is 0, there is an additive value function that represents the preference information. It should be observed that, except in exceptional cases (e.g. if the preference information collected is identical to the preference information collected with the standard sequence technique), there are infinitely many such additive value functions (that are not related via a simple change of origin and of unit, since we already fixed them through normalization (9-10)). The one given as the "optimal" one by the LP does not have a special status since it is highly dependent upon the arbitrary choice of the objective function; instead of minimizing the sum of the deviation variables, we could have as well, and still preserving linearity, minimized the largest of these variables. The whole polyhedron of feasible solutions of the original (in)equalities corresponds to adequate additive value functions: we have a whole set $\mathcal{V}$ of additive value functions representing the information collected on the set of reference alternatives *Ref*.

Using standard techniques in LP, several functions in $\mathcal{V}$ may be obtained, e.g. the ones maximizing or minimizing, within $\mathcal{V}$, $v_i(x_i^*)$ for each attribute [99]. The size of $\mathcal{V}$ is clearly dependent upon the choice of the alternatives in *Ref*. It is often interesting to present them to the decision-maker in the pictorial form of Figures 10 and 11.

If the optimal value of Z strictly greater than 0, there is no additive value function representing the preference information available. The solution given as optimal (note that it is not guaranteed that this solution leads to the minimum possible number of violations w.r.t. the information provided—this would require solving an integer linear programme) is, in general, highly dependent upon the choice of the objective function.

This absence of solution to the system might be due to several factors:

- the piecewise linear approximation of the v_i for the "continuous" attributes may be too rough. It is easy to test whether an increase in the number of linear pieces on some of these attributes may lead to a nonempty set of additive value functions.

- the information provided by the decision-maker may be of poor quality. It might then be interesting to present to the decision-maker one additive value function (e.g. one may present an average function after some post-optimality analysis) in the pictorial form of Figures 10 and 11 and to let him react to this information either by modifying his/her initial judgments or even by letting him/her react directly on the shape of the value functions. This is the solution implemented in the well-known PREFCALC system [97].

- the preference provided by the decision-maker might be inconsistent with the conditions implied by an additive value function. The system should then help locate these inconsistencies and allow the DM to reflect on them. Alternatively, since many alternative attribute description are possible, it may be worth investigating whether a different definition of the various attributes may lead to a preference model consistent with model (2). Several examples of such analysis may be found in [106, 108, 191]

When the above techniques fail, the optimal solution of the LP, even if not compatible with the information provided, may still be considered as an adequate model. Again, since the objective function introduced above is somewhat arbitrary and it is recommended in [99] to perform a post-optimality analysis, e.g. considering additive value functions that are "close" to the optimal solution through the introduction of a linear constraint:

$$Z \leq Z^* + \delta,$$

where Z^* is the optimal value of the objective function of the original LP and δ is a "small" positive number. As above, the result of the analysis is a set $\mathcal{V}$ of additive value functions defined by a set of linear constraints. A representative sample of additive value functions within $\mathcal{V}$ may be obtained as above.

It should be noted that many possible variants of UTA can be conceived building on the following comments. They include:

- the addition of monotonicity properties of the v_i with respect to the underlying continuous attributes,
- the addition of constraints on the shape of the marginal value functions v_i, e.g. requiring them to be concave, convex or S-shaped,
- the addition of constraints linked to a possible indication of preference intensity for the elements of *Ref* given by the DM, e.g. the difference between x and y is larger than the difference between z and w.

For applications of UTA-like techniques, we refer to [35, 42, 43, 94, 98, 133, 168, 169, 170, 171, 172, 173, 175, 178, 179, 200, 202, 201, 204, 203]. Variants of the method are considered in [18, 19, 174]. This method is also studied in detail in chapter XXX of this volume (Chapter 9, Siskos, UTA).

4.2.2 MACBETH [12]

It is easy to see that (9) and (10) may equivalently be written as:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} k_i u_i(x_i) \geq \sum_{i=1}^{n} k_i u_i(y_i), \tag{12}$$

where

$$u_1(x_{1*}) = u_2(x_{2*}) = \dots u_n(x_{n*}) = 0, \tag{13}$$
$$u_1(x_1^*) = u_2(x_2^*) = \dots u_n(x_n^*) = 1 \text{ and} \tag{14}$$
$$\sum_{i=1}^{n} k_i = 1. \tag{15}$$

With such an expression of an additive value function, it is tempting to break down the assessment into two distinct parts: an value function u_i is assessed on each attribute and, then, scaling constants k_i are assessed taking the shape of the value functions u_i as given. This is the path followed in MACBETH.

Remark 35
Again, note that we are speaking here of k_i as *scaling constants* and not as *weights*. As already mentioned weights that would reflect the "importance" of attributes are irrelevant to assess the additive value function model. Notice that, under (12-15) the ordering of the scaling constant k_i is dependent upon the choice of x_{i*} and x_i^*. Increasing the width of the interval $[x_{i*}, x_i^*]$ will lead to increasing the value of the scaling constant k_i. The value k_i has, therefore, nothing to do with the "importance" of attribute i. This point is

unfortunately too often forgotten when using a weighted average of some numerical attributes. Changing the units in which the attributes are measured should imply changing the "weights" accordingly. •

The assessment procedure of the u_i is conceived in such a way as to avoid comparing alternatives differing on more than one attribute. In view of what was said before concerning the standard sequence technique, this is clearly an advantage of the technique. But can it be done? The trick here is that MACBETH asks for judgments related to the difference between the desirability of alternatives and not only judgments in terms of preference or indifference. Partial value functions u_i are approximated in a similar way than in UTA: for discrete attributes, each point on the function is assessed, for continuous ones, a piecewise linear approximation is used.

MACBETH asks the DM to compare pairs of levels on each attribute. If no difference is felt between these levels, they receive an identical partial value level. If a difference is felt between x_i^k and x_i^r, MACBETH asks for a judgment qualifying the strength of this difference. The method and the associated software propose three different semantical categories:

Categories	Description
$C1$	weak
$C2$	strong
$C3$	extreme

with the possibility of using intermediate categories, e.g. between weak and strong (giving a total of six distinct categories). This information is then converted into linear inequations using the natural interpretation that if the "difference" between the levels x_i^k and x_i^r has been judged larger than the "difference" between $x_i^{k'}$ and $x_i^{r'}$ then it should follow that $u_i(x_i^k) - u_i(x_i^r) > u_i(x_i^{k'}) - u_i(x_i^{r'})$. Technically the six distinct categories are delimited by thresholds that are used in the establishment of the constraints of the LP. The software associated to MACBETH offers the possibility to compare all pairs of levels on each attribute for a total of $(r_i + 1)r_i/2$ comparisons. Using standard Goal Programming techniques, as in UTA, the test of the compatibility of a partial value function with this information is performed via the solution of a LP. If there is a partial value function compatible with the information, a "central" function is proposed to the DM who has the possibility to modify it. If not, the results of the LP are exploited in such a way to propose modifications of the information that would make it consistent.

The assessment of the scaling constant k_i is done using similar principles. The DM is asked to compare the following $(n+2)$ alternatives by pairs:

$$\begin{gathered}(x_{1*}, x_{2*}, \ldots, x_{n*}),\\ (x_1^*, x_{2*}, \ldots, x_{n*}),\\ (x_{1*}, x_2^*, \ldots, x_{n*}),\\ \ldots\\ (x_{1*}, x_{2*}, \ldots, x_n^*) \text{ and}\\ (x_1^*, x_2^*, \ldots, x_n^*),\end{gathered}$$

placing each pair in a category of difference. This information immediately translates into a set of linear constraints on the k_i. These constraints are processed as before. It should be noticed that, once the partial value functions u_i are assessed, it is not necessary to use the levels x_{i*} and x_i^* to assess the k_i since they may well lead to alternatives that are too unrealistic. The authors of MACBETH suggest to replace x_{i*} by a "neutral" level which appears neither desirable nor undesirable and x_i^* by a desirable level that is judged satisfactory. Although this clearly impacts the quality of the dialogue with the DM, this has no consequence on the underlying technique used to process information.

We refer to [6, 7, 8, 9, 10, 11] for applications of the MACBETH technique. This method is also studied in detail in chapter XXX of this volume (Chapter 10, Bana, MACBETH).

5 Extensions

The additive value model (2) is the central model for the application of conjoint measurement techniques to decision analysis. We envisage in this section various extensions to this model.

5.1 Transitive Decomposable models

The transitive decomposable model has been introduced in [115] as a natural generalization of model (2). It amounts to replacing the addition by a general function that is increasing in each of its arguments.

Definition 36
Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. The transitive decomposable model holds if, for all $i \in N$, there is a real-valued function v_i on X_i and a

real-valued function g on $\prod_{i=1}^{n} v_i(X_i)$ that is increasing in all its arguments such that:

$$x \succsim y \Leftrightarrow g(v_1(x_1), \ldots, v_n(x_n)) \geq g(v_1(y_1), \ldots, v_n(y_n)), \tag{16}$$

for all $x, y \in X$.

An interesting point with this model is that it admits an intuitively appealing simple characterization. The basic axiom for characterizing the above transitive decomposable model is weak independence, which is clearly implied by (16). The following theorem is proved in [115, ch. 7].

Theorem 37
A preference relation $\succsim$ on a finite or countably infinite set X has a representation in the transitive decomposable model iff $\succsim$ is a weakly independent weak order.

Remark 38
This result can be extended to sets of arbitrary cardinality adding a, necessary, condition implying that the weak order $\succsim$ has a numerical representation. •

The weak point of such a model is that the function g is left unspecified. Hence, the uniqueness results for v_i and g are clearly much less powerful than what we obtained with model (2), see [115, ch. 7]. Therefore, practical applications of this model generally imply specifying the type of function g, possibly by verifying further conditions on the preference that impose that g belongs to some parameterized family of functions, e.g. some polynomial function of the v_i. This is studied in detail in [115, ch. 7] and [125, 124, 140, 149, 184]. Since such models have, to the best of our knowledge, never been used in decision analysis, we do not analyze them further.

The structure of the decomposable model however suggests that assessment techniques for this model could well come from Artificial Intelligence with its "rule induction" machinery. Indeed the function g in model (16) may also be seen as a set of "rules". We refer to [86, 87, 89, 90] for a thorough study of the potentiality of such an approach.

Remark 39
A simple extension of the decomposable model consists in simply asking for a function g that would be nondecreasing in each of its arguments. The following result is proved in [29] (see also [89]) (it can easily be extended to cover the case of an arbitrary set X, adding a, necessary, condition implying that $\succsim$ has a numerical representation).

We say that $\succsim$ is weakly separable if, for all $i \in N$ and all $x_i, y_i \in X_i$, it is never true that $(x_i, z_{-i}) \succ (y_i, z_{-i})$ and $(y_i, w_{-i}) \succ (x_i, w_{-i})$, for some $z_{-i}, w_{-i} \in X_{-i}$. Clearly this is a weakening of weak independence that tolerates to have at the same time $(x_i, z_{-i}) \succ (y_i, z_{-i})$ and $(x_i, w_{-i}) \sim (y_i, w_{-i})$.

Theorem 40
A preference relation $\succsim$ on a finite or countably infinite set X has a representation in the weak decomposable model:

$$x \succsim y \Leftrightarrow g(u_1(x_1), \ldots, u_n(x_n)) \geq g(u_i(y_1), \ldots, u_i(y_n))$$

with g nondecreasing in all its arguments iff $\succsim$ is a weakly separable weak order.

A recent trend of research has tried to characterize special functional forms for g in the weakly decomposable model, such as max, min or some more complex forms. The main references include [24, 89, 91, 165, 177]. •

Remark 41
The use of "fuzzy integrals" as tools for aggregating criteria has recently attracted much attention [44, 79, 80, 82, 83, 84, 128, 130, 129, 131], the Choquet Integral and the Sugeno integral being among the most popular. It should be strongly emphasized that the very definition of these integrals requires to have at hand a weak order on $\cup_{i=1}^{n} X_i$, supposing w.l.o.g. that the sets X_i are disjoint. This is usually called a "commensurability hypothesis". Whereas this hypothesis is quite natural when dealing with an homogeneous Cartesian product, as in decision under uncertainty (see e.g. [193]), it is far less so in the area of multiple criteria decision making. A neat conjoint measurement analysis of such models and their associated assessment procedures is an open research question, see [81]. •

5.2 Intransitive indifference

Decomposable models form a large family of preferences though not large enough to encompass all cases that may be encountered when asking subjects to express preferences. A major restriction is that not all preferences may be assumed to be weak orders. The example of the sequence of cups of coffee, each differing from the previous one by an imperceptible quantity of sugar added [119], is famous; it leads to the notions of semiorder and interval order [4, 51, 60, 119, 144], in which indifference is not transitive, while strict preference is.

Ideally, taking intransitive indifference into account, we would want to arrive at a generalization of (2) in which:

$$x \sim y \Leftrightarrow |V(x) - V(y)| \leq \epsilon,$$
$$x \succ y \Leftrightarrow V(x) > V(y) + \epsilon,$$

where $\epsilon \geq 0$ and $V(x) = \sum_{i=1}^{n} v_i(x_i)$.

In the finite case, it is not difficult to extend the condition envisaged in section 4 to cover such a case. Indeed, we are still looking here for the solution to a system of linear constraints. Although this seems to have never been done, it would not be difficult to adapt the LP-based assessment techniques to this case.

On the contrary, extending the standard sequence technique of section 3 is a formidable challenge. Indeed, remember that these techniques crucially rest on indifference judgments which lead to the determination of "perfect copies" of a given preference interval. As soon as indifference is not supposed to be transitive, "perfect copies" are not so perfect and much trouble is expected. We refer to [75, 114, 120, 144, 180] for a study of these models.

Remark 42
Even if the analysis of such models proves difficult, it should be noted that the semi-ordered version of the additive utility model may be interpreted as having a "built-in" sensitivity analysis via the introduction of the threshold ϵ. Therefore, in practice, we may usefully view ϵ not as a parameter to be assessed but as a simple trick to avoid undue discrimination, because of the imprecision inevitably involved in our assessment procedures, between close alternatives •

Remark 43
Clearly the above model can be generalized to cope with a possibly non-constant threshold. The literature on the subject remains minimal however [144]. •

5.3 Nontransitive preferences

Many authors [132, 185] have argued that the reasonableness of supposing that strict preference is transitive is not so strong when it comes to comparing objects evaluated on several attributes. As soon as it is supposed that subjects may use an "ordinal" strategy for comparing objects, examples inspired from the well-known Condorcet paradox [159, 166] show that intransitivities will be difficult to avoid. Indeed it is possible to observe predictable intransitivities of strict preference in carefully controlled experiments [185]. There may therefore be a descriptive interest to studying such models. Now, when it comes to decision analysis, intransitive preferences are often dismissed on two grounds:

- on a practical level, it is not easy to build a recommendation on the basis of a binary relation in which $\succ$ would not be transitive. Indeed, social choice theorists, facing a similar problem, have devoted much effort to devising what could be called reasonable procedures to deal with such preferences [38, 56, 116, 117, 134, 142, 161]. This literature does not lead, as was expected, to the emergence of a single suitable procedure in all situations.

- on a more conceptual level, many others have questioned the very rationality of such preferences using some version of the famous "money pump" argument [123, 147].

P. C. Fishburn has forcefully argued [67] that these arguments might not be as decisive as they appear at first sight. Furthermore some MCDM techniques make use of such intransitive models, most notably the so-called outranking methods [23, 155, 186, 187]. Besides the intellectual challenge, there might therefore be a real interest in studying such models.

A. Tversky [185] was one of the first to propose such a model generalizing (2), known as the *additive difference model*, in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \tag{17}$$

where Φ_i are increasing and odd functions.

It is clear that (17) allows for intransitive $\succsim$ but implies its completeness. Clearly, (17) implies that $\succsim$ is independent. This allows to unambiguously define marginal preferences $\succsim_i$. Although model (17) can accommodate intransitive $\succsim$, a consequence of the increasingness of the Φ_i is that the marginal preference relations $\succsim_i$ are weak orders. This, in particular, excludes the possibility of any perception threshold on each attribute which would lead to an intransitive indifference relation on each attribute. Imposing that Φ_i are nondecreasing instead of being increasing allows for such a possibility. This gives rise to what is called the "weak additive difference model" in [20]

As suggested in [20, 64, 63, 66, 188], the subtractivity requirement in (17) can be relaxed. This leads to *nontransitive additive* conjoint measurement models in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0 \tag{18}$$

where the p_i are real-valued functions on X_i^2 and may have several additional properties (e.g. $p_i(x_i, x_i) = 0$, for all $i \in \{1, 2, \dots, n\}$ and all $x_i \in X_i$).

This model is an obvious generalization of the (weak) additive difference model. It allows for intransitive and incomplete preference relations $\succsim$ as well as for intransitive and incomplete marginal preferences $\succsim_i$. An interesting specialization of (18) obtains when p_i are required to be *skew symmetric* i.e. such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$. This skew symmetric nontransitive additive conjoint measurement model implies that $\succsim$ is complete and independent.

An excellent overview of these nontransitive models is [67]. Several axiom systems have been proposed to characterize them. P.C. Fishburn gave [64, 63, 66] axioms for the skew symmetric version of (18) both in the finite and the infinite case. Necessary and sufficient conditions for a nonstandard version of (18) are presented in [69]. [188] gives axioms for (18) with $p_i(x_i, x_i) = 0$ when $n \geq 4$. [20] gives necessary and sufficient conditions for (18) with and without skew symmetry in the denumerable case when $n = 2$.

The additive difference model (17) was axiomatized in [68] in the infinite case when $n \geq 3$ and [20] gives necessary and sufficient conditions for the weak additive difference model in the finite case when $n = 2$. Related studies of nontransitive models include [36, 58, 122, 137]. The implications of these models for decision-making under uncertainty were explored in [65] (for a different path to nontransitive models for decision making under risk and/or uncertainty, see [59, 61]).

It should be noticed that even the weakest form of these models, i.e. (18) without skew symmetry, involves an addition operation. Therefore it is unsurprising that the axiomatic analysis of these models share some common features with the additive value function model (2). Indeed, except in the special case in which $n = 2$, this case relating more to ordinal than to conjoint measurement (see [115]), the various axiom systems that have been proposed involve either:

- a denumerable set of cancellation conditions in the finite case or
- a finite number of cancellation conditions together with unnecessary structural assumptions in the general case (these structural assumptions generally allow us to obtain nice uniqueness results for (18): the functions p_i are unique up to the multiplication by a common positive constant).

A different path to the analysis of nontransitive conjoint measurement models has recently been proposed in [29, 27, 30]. In order to get a feeling for these various models, it is useful to envisage the various strategies that are likely to be implemented when comparing objects differing on several dimensions [37, 135, 136, 158, 182, 185].

Consider two alternatives x and y evaluated on a family of n attributes so that $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.

A first strategy that can be used in order to decide whether or not it can be said that "x is at least as good as y" consists in trying to measure the "worth" of each alternative on each attribute and then to combine these evaluations adequately. Abandoning all idea of transitivity and completeness, this suggests a model in which:

$$x \succsim y \Leftrightarrow F(u_1(x_1), \dots, u_n(x_n), u_1(y_1), \dots, u_n(y_n)) \geq 0 \tag{19}$$

where u_i are real-valued functions on the X_i and F is a real-valued function on $\prod_{i=1}^{n} u_i(X_i)^2$. Additional properties on F, e.g. its nondecreasingness (resp. nonincreasingness) in its first (resp. last) n arguments, will give rise to a variety of models implementing this first strategy.

A second strategy relies on the idea of measuring "preference differences" separately on each attribute and then combining these (positive or negative) differences in order to know whether the aggregation of these differences leads to an advantage for x over y. More formally, this suggests a model in which:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \dots, p_n(x_n, y_n)) \geq 0 \tag{20}$$

where p_i are real-valued functions on X_i^2 and G is a real-valued function on $\prod_{i=1}^{n} p_i(X_i^2)$. Additional properties on G (e.g. its oddness or its nondecreasingness in each of its arguments) or on p_i (e.g. $p_i(x_i, x_i) = 0$ or $p_i(x_i, y_i) = -p_i(y_i, x_i)$) will give rise to a variety of models in line with the above strategy.

Of course these two strategies are nor incompatible and one may well envisage to use the "worth" of each alternative on each attribute to measure "preference differences". This suggests a model in which:

$$x \succsim y \Leftrightarrow H(\phi_1(u_1(x_1), u_1(y_1)), \dots, \phi_n(u_n(x_n), u_n(y_n))) \geq 0 \tag{21}$$

where u_i are real-valued functions on X_i, ϕ_i are real-valued functions on $u_i(X_i)^2$ and H is a real-valued function on $\prod_{i=1}^{n} \phi_i(u_i(X_i)^2)$.

Clearly the use of very general functional forms, instead of additive ones, greatly facilitate the axiomatic analysis of these models. It mainly relies on the study of various kinds of *traces* induced by the preference relation on coordinates and does not require a detailed analysis of tradeoffs between attributes.

The price to pay for such an extension of the scope of conjoint measurement is that none of these models is likely to possess any remarkable uniqueness properties. Therefore, although proofs are constructive, these results will not give direct hints on how to devise assessment procedures. The general idea here is to use numerical representations as guidelines to understand the consequences of a limited number of cancelation conditions, without imposing any transitivity or completeness requirement on the preference relation and any structural assumptions on the set of objects. Such models have proved useful to:

- understand the ordinal character of some aggregation models proposed in the literature [153, 155], known as the "outranking methods" (see Chapters 3, 4, 5 of this Volume) as shown in [26],
- understand the links between aggregation models aiming at enriching a dominance relation and more traditional conjoint measurement approaches [29],
- to include in a classical conjoint measurement framework, noncompensatory preference in the sense of [20, 31, 50, 54, 55] as shown in [26, 28, 88].

References

[1] E. W. Adams. Elements of a theory of inexact measurement. *Philosophy of Science*, 32:205–228, 1965.

[2] E. W. Adams and R. F. Fagot. A model of riskless choice. *Behavioral Science*, 4:1–10, 1959.

[3] L. Adelman, P. J. Sticha, and M. L. Donnell. An experimental investigation of the relative effectiveness of two techniques for structuring multiattributed hierarchies. *Organizational Behavior and Human Decision Processes*, 37:188–196, 1986.

[4] F. Aleskerov and B. Monjardet. *Utility Maximization, Choice and Preference*. Springer-Verlag, Heidelberg, 2002.

[5] A. B. Atkinson. On the measurement of inequality. *Journal of Economic Theory*, 2:244–263, 1970.

[6] C. A. Bana e Costa. The use of multi-criteria decision analysis to support the search for less conflicting policy options in a multi-actor context: case study. *Journal of Multi-Criteria Decision Analysis*, 10(2):111–125, 2001.

[7] C. A. Bana e Costa, É. C. Corrêa, J.-M. De Corte, and J.-C. Vansnick. Facilitating bid evaluation in public call for tenders: a socio-technical approach. *Omega*, 30(3):227–242, April 2002.

[8] C. A. Bana e Costa, M. L. Costa-Lobo, I. A. J. Ramos, and J.-C. Vansnick. Multicriteria approach for strategic town planning: the case of Barcelos. In D. Bouyssou, É. Jacquet-Lagrèze, P. Perny, R. Słowiński, D. Vanderpooten, and Ph. Vincke, editors, *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, pages 429–456. Kluwer, Dordrecht, 2002.

[9] C. A. Bana e Costa, L. Ensslin, É. C. Corrêa, and J.-C. Vansnick. Decision support systems in action: integrated application in a multicriteria decission aid process. *European Journal of Operational Research*, 113(2):315–335, March 1999.

[10] C. A. Bana e Costa, F. Nunes da Silva, and J.-C. Vansnick. Conflict dissolution in the public sector: A case-study. *European Journal of Operational Research*, 130(2):388–401, April 2001.

[11] C. A. Bana e Costa and R. C. Oliveira. Assigning priorities for maintenance and repair and refurbishment in managing a municipal housing stock. *European Journal of Operational Research*, 138:380–91, 2002.

[12] C. A. Bana e Costa and J.-C. Vansnick. MACBETH – an interactive path towards the construction of cardinal value functions. *International Transactions in Operational Research*, 1:489–500, 1994.

[13] A. F. Beardon, J. C. Candeal, G. Herden, E. Induráin, and G. B. Mehta. The non-existence of a utility function and the structure of non-representatble preference relations. *Journal of Mathematical Economics*, 37:17–38, 2002.

[14] V. Belton, F. Ackermann, and I. Shepherd. Integrated support from problem structuring through alternative evaluation using COPE and V•I•S•A. *Journal of Multi-Criteria Decision Analysis*, 6:115–130, 1997.

[15] V. Belton and Th. Stewart. *Multiple Criteria Decision Analysis: An Integrated Approach*. Kluwer, Dordrecht, 2001.

[16] E. Ben-Porath and I. Gilboa. Linear measures, the Gini index and the income-equality tradeoff. *Journal of Economic Theory*, 64:443–467, 1994.

[17] E. Ben-Porath, I. Gilboa, and D. Schmeidler. On the measurement of inequality under uncertainty. *Journal of Economic Theory*, 75:194–204, 1997.

[18] M. Beuthe and G. Scannella. Applications comparées des méthodes d'analyse multicritère UTA. *RAIRO Recherche Opérationnelle / Operations Research*, 30(3):293–315, 1996.

[19] M. Beuthe and G. Scannella. Comparative analysis of UTA multicriteria methods. *European Journal of Operational Research*, 130(2):246–262, 2001.

[20] D. Bouyssou. Some remarks on the notion of compensation in MCDM. *European Journal of Operational Research*, 26:150–160, 1986.

[21] D. Bouyssou. Modelling inaccurate determination, uncertainty, imprecision using multiple criteria. In A. G. Lockett and G. Islei, editors, *Improving Decision Making in Organisations*, pages 78–87. Springer-Verlag, Heidelberg, 1989.

[22] D. Bouyssou. Builing criteria: a prerequisite for MCDA. In C. A. Bana e Costa, editor, *Readings in Multiple Criteria Decision Aid*, pages 58–81. Springer-Verlag, 1990.

[23] D. Bouyssou. Outranking methods. In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of optimization*. Kluwer, 2001.

[24] D. Bouyssou, S. Greco, B. Matarazzo, M. Pirlot, and R. Słowiński. Characterization of 'max', 'min' and 'order statistics' multicriteria aggregation functions. Communication to *IFORS'2002*, 8 – 12 July, 2002, Edinburgh, U.K., July 2002.

[25] D. Bouyssou, Th. Marchant, M. Pirlot, P. Perny, A. Tsoukiàs, and Ph. Vincke. *Evaluation and Decision models: a critical perspective*. Kluwer, Dordrecht, 2000.

[26] D. Bouyssou and M. Pirlot. A characterization of strict concordance relations. In D. Bouyssou, É. Jacquet-Lagrèze, P. Perny, R. Słowiński, D. Vanderpooten, and Ph. Vincke, editors, *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, pages 121–145. Kluwer, Dordrecht, 2002.

[27] D. Bouyssou and M. Pirlot. Nontransitive decomposable conjoint measurement. *Journal of Mathematical Psychology*, 46:677–703, 2002.

[28] D. Bouyssou and M. Pirlot. Notes on concordance relations. Working Paper, LAMSADE, Université Paris-Dauphine, 2002.

[29] D. Bouyssou and M. Pirlot. Preferences for multiattributed alternatives: Traces, dominance, and numerical representations. Working Paper, submitted, 2002.

[30] D. Bouyssou and M. Pirlot. 'Additive difference' models without additivity and subtractivity. Working Paper, submitted, 2003.

[31] D. Bouyssou and J.-C. Vansnick. Noncompensatory and generalized noncompensatory preference structures. *Theory and Decision*, 21:251–266, 1986.

[32] J. D. Carroll and P. E. Green. Psychometric methods in marketing research. Part 1. Conjoint analysis. *Journal of Marketing Research*, 32:385–391, 1995.

[33] A. Charnes and W. W. Cooper. *Management Models and Industrial Applications of Linear Programming*. Wiley, New York, 1961.

[34] A. Chateauneuf and P. P. Wakker. From local to global additive representation. *Journal of Mathematical Economics*, 22(6):523–545, 1993.

[35] J.-C. Cosset, Y. Siskos, and C. Zopounidis. The evaluation of country risk: a decision support approach. *Global Finance Journal*, 3:79–95, 1992.

[36] M. A. Croon. The axiomatization of additive difference models for preference judgements. In E. Degreef and G. van Buggenhaut, editors, *Trends in Mathematical Psychology*, pages 193–227, Amsterdam, 1984. North-Holland.

[37] V. Dahlstrand and H. Montgomery. Information search and evaluation processes in decision-making: A computer-based process tracking study. *Acta Psychologica*, 56:113–123, 1984.

[38] P. de Donder, M. Le Breton, and M. Truchon. Choosing from a weighted tournament. *Mathematical Social Sciences*, 40:85–109, 2000.

[39] G. Debreu. Representation of a preference ordering by a numerical function. In R. Thrall, C. H. Coombs, and R. Davies, editors, *Decision Processes*, pages 159–175, New York, 1954. Wiley.

[40] G. Debreu. *Theory of value: an axiomatic analysis of economic equilibrium.* John Wiley & Sons, New York, 1959.

[41] G. Debreu. Topological methods in cardinal utility theory. In S. Karlin K. J. Arrow and P. Suppes, editors, *Mathematical methods in the Social Sciences*, pages 16–26. Stanford University Press, Stanford, 1960.

[42] A. I. Dimitras, C. Zopounidis, and C. Hurson. Assessing financial risks using a multicriteria sorting procedure: the case of country risk assessment. *Foundations of Computing and Decision Sciences*, 29:97–109, 2001.

[43] M. Doumpos, S. H. Zanakis, and C. Zopounidis. Multicriteria preference disaggregation for classification problems with an application to global investing risk. *Decision Sciences*, 32(2):333–385, 2001.

[44] D. Dubois, J. L. Marichal, H. Prade, M. Roubens, and R. Sabbadin. The use of the discrete Sugeno integral in decision-making: a survey. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9(5):539–561, 2001.

[45] J. S. Dyer and R. K. Sarin. Measurable multiattribute value functions. *Operations Research*, 27:810–822, 1979.

[46] W. Edwards. Social utilities. *Engineering Economist*, 6:119–129, 1971.

[47] W. Edwards. How to use multiattribute utility measurement for social decision making. *IEEE Transactions on Systems, Man and Cybernetics*, 7(5):326–340, 1977.

[48] W. Edwards and F. Hutton Barron. SMART and SMARTER: Improved simple methods for multiattribute utility measurement. *Organizational Behavior and Human Decision Processes*, 60:306–325, 1994.

[49] J.-C. Falmagne. On a class of probabilistic conjoint measurement models: some diagnostic properties. *Journal of Mathematical Psychology*, 19(2):73–88, 1979.

[50] H. Fargier and P. Perny. Modélisation des préférences par une règle de concordance généralisée. In A. Colorni, M. Paruccini, and B. Roy, editors, A-MCD-A*, Aide Multicritère à la Décision/Multiple Criteria Decision Aid*, pages 99–115. European Commission, Joint Research Centre, 2001.

[51] P. C. Fishburn. Intransitive indifference in preference theory: A survey. *Operations Research*, 18(2):207–228, 1970.

[52] P. C. Fishburn. *Utility theory for decision-making*. Wiley, New-York, 1970.

[53] P. C. Fishburn. Lexicographic orders, utilities and decision rules: A survey. *Management Science*, 20(11):1442–1471, 1974.

[54] P. C. Fishburn. Axioms for lexicographic preferences. *Review of Economic Studies*, 42:415–419, 1975.

[55] P. C. Fishburn. Noncompensatory preferences. *Synthese*, 33:393–403, 1976.

[56] P. C. Fishburn. Condorcet social choice functions. *SIAM Journal on Applied Mathematics*, 33:469–489, 1977.

[57] P. C. Fishburn. A survey of multiattribute/multicriteria evaluation theories. In S. Zionts, editor, *Multicriteria problem solving*, pages 181–224. Springer Verlag, Berlin, 1978.

[58] P. C. Fishburn. Lexicographic additive differences. *Journal of Mathematical Psychology*, 21:191–218, 1980.

[59] P. C. Fishburn. Nontransitive measurable utility. *Journal of Mathematical Psychology*, 26:31–67, 1982.

[60] P. C. Fishburn. *Interval orders and intervals graphs*. Wiley, New-York, 1985.

[61] P. C. Fishburn. *Nonlinear preference and utility theory*. Johns Hopkins University Press, Baltimore, 1988.

[62] P. C. Fishburn. Normative theories of decision making under risk and under uncertainty. In *Nonconventional preference relations in decision making*, pages 1–21. Springer, Berlin, 1988.

[63] P. C. Fishburn. Additive non-transitive preferences. *Economic Letters*, 34:317–321, 1990.

[64] P. C. Fishburn. Continuous nontransitive additive conjoint measurement. *Mathematical Social Sciences*, 20:165–193, 1990.

[65] P. C. Fishburn. Skew symmetric additive utility with finite states. *Mathematical Social Sciences*, 19:103–115, 1990.

[66] P. C. Fishburn. Nontransitive additive conjoint measurement. *Journal of Mathematical Psychology*, 35:1–40, 1991.

[67] P. C. Fishburn. Nontransitive preferences in decision theory. *Journal of Risk and Uncertainty*, 4:113–134, 1991.

[68] P. C. Fishburn. Additive differences and simple preference comparisons. *Journal of Mathematical Psychology*, 36:21–31, 1992.

[69] P. C. Fishburn. On nonstandard nontransitive additive utility. *Journal of Economic Theory*, 56:426–433, 1992.

[70] P. C. Fishburn. Finite linear qualitative probability. *Journal of Mathematical Psychology*, 40:21–31, 1996.

[71] P. C. Fishburn. Cancellation conditions for multiattribute preferences on finite sets. In M. H. Karwan, J. Spronk, and J. Wallenius, editors, *Essays in Decision Making*, pages 157–167, Berlin, 1997. Springer Verlag.

[72] S. French. *Decision theory – An introduction to the mathematics of rationality.* Ellis Horwood, London, 1993.

[73] D. G. Fryback and R. L. Keeney. Constructing a complex judgmental model: an index of trauma severity. *Management Science*, 29:869–883., 1983.

[74] D. Gale. *The theory of linear economic models.* McGraw-Hill, New York, 1960.

[75] I. Gilboa and R. Lapson. Aggregation of semiorders: Intransitive indifference makes a difference. *Economic Theory*, 5:109–126, 1995.

[76] Ch. Gonzales. Additive utilities when some components are solvable and others not. *Journal of Mathematical Psychology*, 40:141–151, 1996.

[77] Ch. Gonzales. *Utilités additives : existence et construction.* Thèse de doctorat, Université Paris 6, Paris, France, 1996.

[78] Ch. Gonzales. Two factor additive conjoint measurement with one solvable component. *Journal of Mathematical Psychology*, 44:285–309, 2000.

[79] M. Grabisch. The application of fuzzy integrals to multicriteria decision making. *European Journal of Operational Research*, 89:445–456, 1996.

[80] M. Grabisch. Fuzzy integral as a flexible and interpretable tool of aggregation. In *Aggregation and fusion of imperfect information*, pages 51–72. Physica, Heidelberg, 1998.

[81] M. Grabisch, C. Labreuche, and J.-C. Vansnick. On the extension of pseudo-boolean functions for the aggregation of interacting criteria. *European Journal of Operational Research*, 148(1):28–47, 2003.

[82] M. Grabisch, S. A. Orlovski, and R. R. Yager. Fuzzy aggregation of numerical preferences. In R. Słowiński, editor, *Fuzzy sets in decision analysis, operations research and statistics*, pages 31–68. Kluwer, Boston, MA, 1998.

[83] M. Grabisch and P. Perny. Agrégation multicritère. In B. Bouchon and C. Marsala, editors, *Utilisations de la logique floue*. Hermès, 1999.

[84] M. Grabisch and M. Roubens. Application of the Choquet integral in multicriteria decision making. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy measures and integrals*, pages 348–374. Physica Verlag, Heidelberg, 2000.

[85] N. Grassin. Constructing criteria population for the comparison of different options of high voltage line routes. *European Journal of Operational Research*, 26:42–47, 1886.

[86] S. Greco, B. Matarazzo, and R. Słowiński. Rough approximation of a preference relation by dominance relations. *European Journal of Operational Research*, 117:63–83, 1999.

[87] S. Greco, B. Matarazzo, and R. Słowiński. The use of rough sets and fuzzy sets in MCDM. In T. Gal, T. Hanne, and T. Stewart, editors, *Multicriteria decision making, Advances in* MCDM *models, algorithms, theory and applications*, pages 14.1–14.59, Dordrecht, 1999. Kluwer.

[88] S. Greco, B. Matarazzo, and R. Słowiński. Axiomatic basis of noncompensatoty preferences. Communication to *FUR X*, 30 May–2 June, Torino, Italy, 2001.

[89] S. Greco, B. Matarazzo, and R. Słowiński. Conjoint measurement and rough set approach for multicriteria sorting problems in presence of ordinal criteria. In A. Colorni, M. Paruccini, and B. Roy, editors, A-MCD-A*, Aide Multicritère à la Décision/Multiple Criteria Decision Aid*, pages 117–144. European Commission, Joint Research Centre, 2001.

[90] S. Greco, B. Matarazzo, and R. Słowiński. Rough sets theory for multicriteria decision analysis. *European Journal of Operational Research*, 129:1–7, 2001.

[91] S. Greco, B. Matarazzo, and R. Słowiński. Preference representation by means of conjoint measurement and decision rule model. In D. Bouyssou, É. Jacquet-Lagrèze, P. Perny, R. Słowiński, D. Vanderpooten, and Ph. Vincke, editors, *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, pages 263–313. Kluwer, Dordrecht, 2002.

[92] P. E. Green and V. Srinivasan. Conjoint analysis in consumer research: issues and outlook. *Journal of Consumer Research*, 5:103–152, 1978.

[93] P. E. Green, D. S. Tull, and G. Albaum. *Research for marketing decisions*. Englewood Cliffs, 1988.

[94] E. Grigoroudis, Y. Siskos, and O. Saurais. TELOS: a customer satisfaction evaluation software. *Computers and Operations Research*, 27(7–8):799–817, 2000.

[95] F. Gul. Savage's theorem with a finite number of states. *Journal of Economic Theory*, 57:99–110, 1992.

[96] G. Iverson and J.-C. Falmagne. Statistical issues in measurement. *Mathematical Social Sciences*, 10:131–153, 1985.

[97] É. Jacquet-Lagrèze. Interactive assessment of preferences using holistic judgments. The PREFCALC system. In C. A. Bana e Costa, editor, *Readings in multiple criteria decision aid*, pages 335–350. Springer Verlag, Berlin, 1990.

[98] É. Jacquet-Lagrèze. An application of the UTA discriminant model for the evaluation of R&D projects. In P. Pardalos, Y. Siskos, and C. Zopounidis, editors, *Advances in Multicriteria Analysis*, pages 203–211. Kluwer Academic Publishers, Dordrecht, 1995.

[99] É. Jacquet-Lagrèze and J. Siskos. Assessing a set of additive utility functions for multicriteria decision making: The UTA method. *European Journal of Operational Research*, 10:151–164, 1982.

[100] J.-Y. Jaffray. On the extension of additive utilities to infinite sets. *Journal of Mathematical Psychology*, 11:431–452, 1974.

[101] R. M. Johnson. Trade-off analysis of consumer values. *Journal of Marketing Research*, 11:121–127, 1974.

[102] E. Karni and Z. Safra. The hexagon condition and additive representation for two dimensions: An algebraic approach. *Journal of Mathematical Psychology*, 42:393–399, 1998.

[103] R. L. Keeney. Measurement scales for quantifying attributes. *Behavioral Science*, 26:29–36, 1981.

[104] R. L. Keeney. Building models of values. *European Journal of Operational Resesearch*, 37(2):149–157, 1988.

[105] R. L. Keeney. Structuring objectives for problems of public interest. *Operations Research*, 36:396–405, 1988.

[106] R. L. Keeney. *Value-focused thinking*. Harvard University Press, Cambridge, MA., 1992.

[107] R. L. Keeney, J. S. Hammond, and H. Raiffa. *Smart choices: A guide to making better decisions*. Harvard University Press, Boston, 1999.

[108] R. L. Keeney and H. Raiffa. *Decisions with multiple objectives: Preferences and value tradeoffs*. Wiley, New-York, 1976.

[109] R. L. Keeney and G. A. Robillard. Assessing and evaluating environmental impacts at proposed nuclear power plant sites. *Journal of Environmental Economics and Management*, 4:153–166, 1977.

[110] V. Köbberling. Comment on: Edi Karni & Zvi Safra (1998) The hexagon condition and additive representation for two dimensions: An algebraic approach. *Journal of Mathematical Psyvhology*, 47(3):370, 2003.

[111] T. C. Koopmans. Stationary ordinal utility and impatience. *Econometrica*, 28:287–309, 1960.

[112] T. C. Koopmans. Representation of prefernce orderings over time. In C. B. McGuire and R. Radner, editors, *Decision and Organization*, pages 57–100. Noth-Holland, Amsterdam, 1972.

[113] D. H. Krantz. Conjoint measurement: the Luce-Tukey axiomatization and some extensions. *Journal of Mathematical Psychology*, 1:248–277, 1964.

[114] D. H. Krantz. Extensive measurement in semiorders. *Philos. Sci.*, 34:348–362, 1967.

[115] D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky. *Foundations of measurement,* vol. 1: *Additive and polynomial representations.* Academic Press, New-York, 1971.

[116] G. Laffond, J.-F. Laslier, and M. Le Breton. Condorcet choice correspondences: A set-theoretical comparison. *Mathematical Social Sciences*, 30:23–36, 1995.

[117] J.-F. Laslier. *Tournament solutions and majority voting.* Springer-Verlag, Berlin, 1997.

[118] J. Louvière. *Analyzing Decision Making: Metric Conjoint Analysis.* Sage, Park, CA, 1988.

[119] R. D. Luce. Semiorders and a theory of utility discrimination. *Econometrica*, 24:178–191, 1956.

[120] R. D. Luce. Three axiom systems for additive semiordered structures. *SIAM Journal of Applied Mathematics*, 25:41–53, 1973.

[121] R. D. Luce. Conjoint measurement: A brief survey. In David E. Bell, Ralph L. Keeney, and Howard Raiffa, editors, *Conflicting objectives in decisions*, pages 148–171. Wiley, New York, 1977.

[122] R. D. Luce. Lexicographic tradeoff structures. *Theory and Decision*, 9:187–193, 1978.

[123] R. D. Luce. *Utility of gains and losses: Measurement-theoretical and experimental approaches.* Lawrence Erlbaum Publishers, Mahwah, New Jersey, 2000.

[124] R. D. Luce and M. Cohen. Factorizable automorphisms in solvable conjoint structures. I. *Journal of Pure and Applied Algebra*, 27(3):225–261, 1983.

[125] R. D. Luce, D. H. Krantz, P. Suppes, and A. Tversky. *Foundations of measurement,* vol. 3: *Representation, axiomatisation and invariance.* Academic Press, New-York, 1990.

[126] R. D. Luce and A. A. J. Marley. Extensive measurement when concatenation is restricted. In S. Morgenbesser, P. Suppes, and M. G. White, editors, *Philosophy, science and method: Essays in honor of Ernest Nagel*, pages 235–249. St. Martin's Press, New York, 1969.

[127] R. D. Luce and J. W. Tukey. Simultaneous conjoint measurement: a new type of fundamental measurement. *Journal of Mathematical Psychology*, 1:1–27, 1964.

[128] J.-L. Marichal. An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. *IEEE Transactions on Fuzzy Systems*, 8, 2000.

[129] J.-L. Marichal. On Choquet and Sugeno integrals as aggregation functions. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy measures and integrals*, pages 247–272. Physica Verlag, Heidelberg, 2000.

[130] J.-L. Marichal. On Sugeno integralsas an aggregation function. *Fuzzy Sets and Systems*, 114, 2000.

[131] J.-L. Marichal and M. Roubens. Determination of weights of interacting criteria from a reference set. *European Journal of Operational Research*, 124:641–50, 2000.

[132] K. O. May. Intransitivity, utility and the aggregation of preference patterns. *Econometrica*, 22:1–13, 1954.

[133] G. Mihelis, E. Grigoroudis, Y. Siskos, Y. Politis, and Y. Malandrakis. Customer satisfaction measurement in the private bank sector. *European Journal of Operational Research*, 130(2):347–360, April 2001.

[134] N. R. Miller. Graph theoretical approaches to the theory of voting. *American Journal of Political Science*, 21:769–803, 1977.

[135] H. Montgomery. A study of intransitive preferences using a think aloud procedure. In H. Jungerman and G. de Zeeuw, editors, *Decision-making and Change in Human Affairs*, pages 347–362, Dordrecht, 1977. D. Reidel.

[136] H. Montgomery and O. Svenson. On decision rules and information processing strategies for choice among multiattribute alternatives. *Scandinavian Journal of Psychology*, 17:283–291, 1976.

[137] Y. Nakamura. Lexicographic additivity for multi-attribute preferences on finite sets. *Theory and Decision*, 42:1–19, 1997.

[138] Y. Nakamura. Additive utility on densely ordered sets. *Journal of Mathematical Psychology*, 46:515–530, 2002.

[139] L. Narens. *Abstract measurement theory.* MIT press, Cambridge, Mass., 1985.

[140] L. Narens and R. D. Luce. The algebra of measurement. *Journal of Pure and Applied Algebra*, 8(2):197–233, 1976.

[141] A. Ostanello. Action evaluation and action structuring – Different decision aid situations reviewed through two actual cases. In C. A. Bana e Costa, editor, *Readings in multiple criteria decision aid*, pages 36–57. Springer Verlag, Berlin, 1990.

[142] J. E. Peris and B. Subiza. Condorcet choice correspondences for weak tournaments. *Social Choice and Welfare*, 16:217–231, 1999.

[143] M. Pirlot and Ph. Vincke. Lexicographic aggregation of semiorders. *Journal of Multicriteria Decision Analysis*, 1:47–58, 1992.

[144] M. Pirlot and Ph. Vincke. *Semiorders. Properties, representations, applications.* Kluwer, Dordrecht, 1997.

[145] J.-Ch. Pomerol and S. Barba-Romero. *Multicriterion Decision in Management, Principles and Practice.* Kluwer, Dordrecht, 2000.

[146] M. Pöyhönen, H. Vrolijk, and R. P. Hämäläinen. Behavioral and procedural consequences of structural variations in value trees. *European Journal of Operational Research*, 134(1):216–227, October 2001.

[147] H. Raiffa. *Decision analysis – Introductory lectures on choices under uncertainty.* Addison-Wesley, Reading, Mass., 1968.

[148] H. Raiffa. Preference for multi-attributed alternatives. RAND Memorandum, RM-5868-DOT/RC, December 1968.

[149] M. K. Richter. Rational choice and polynomial measurement theory. *Journal of Mathematical Psychology*, 12:99–113, 1975.

[150] F. S. Roberts. *Measurement theory with applications to decision making, utility and the social sciences.* Addison-Wesley, Reading, 1979.

[151] F. S. Roberts and R. D. Luce. Axiomatic thermodynamics and extensive measurement. *Synthese*, 18:311–326, 1968.

[152] M. Roubens and Ph. Vincke. *Preference modelling.* Springer Verlag, Berlin, 1985.

[153] B. Roy. Classement et choix en présence de points de vue multiples (la méthode ELECTRE). *RIRO*, 2:57–75, 1968.

[154] B. Roy. Main sources of inaccurate determination, uncertainty and imprecision in decision models. *Mathematical and Computer Modelling*, 12(10–11):1245–1254, 1989.

[155] B. Roy. The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31:49–73, 1991.

[156] B. Roy. *Multicriteria methodology for decision aiding*. Kluwer, Dordrecht, 1996. Original version in French "*Méthodologie multicritère d'aide à la décision*", Economica, Paris, 1985.

[157] B. Roy and D. Bouyssou. *Aide multicritère à la décision : Méthodes et cas*. Economica, Paris, 1993.

[158] J. E. Russo and B. A. Dosher. Strategies for multiattribute binary choice. *Journal of Experimental Psychology: Learning, Memory and Cognition*, 9:676–696, 1983.

[159] D. G. Saari. Connecting and resolving Sen's and Arrow's theorems. *Social Choice and Welfare*, 15:239–261, 1998.

[160] L. J. Savage. *The Foundations of Statistics*. John Wiley and Sons, New York, 1954.

[161] T. Schwartz. *The logic of collectice choice*. Columbia University Press, 1986.

[162] D. Scott. Measurement structures and linear inequalities. *Journal of Mathematical Psychology*, 1:233–247, 1964.

[163] D. Scott and P. Suppes. Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, 23:113–128, 1958.

[164] U. Segal. A sufficient condition for additively separable functions. *Journal of Mathematical Economics*, 23:295–303, 1994.

[165] U. Segal and J. Sobel. Min, Max and Sum. *Journal of Economic Theory*, 106:126–150, 2002.

[166] A. K. Sen. Social choice theory. In K. J. Arrow and M. D. Intriligator, editors, *Handbook of mathematical economics*, volume 3, pages 1073–1181. North-Holland, Amsterdam, 1986.

[167] L. Shapiro. Conditions for expected uility maximization. *Annals of Statistics*, 7:1288–1302, 1979.

[168] Y. Siskos. Evaluating a system of furniture retail outlets using an interactive ordinal regression model. *European Journal of Operational Research*, 23:179–193, 1986.

[169] Y. Siskos and D. K. Despotis. A DSS oriented method for multiobjective linear programming problems. *Decision Support Systems*, 5(1):47–56, 1989.

[170] Y. Siskos, D. K. Despotis, and M. Ghediri. Multiobjective modelling for regional agricultural planning: case study in Tunisia. *European Journal of Operational Research*, 77(3):375–391, 1993.

[171] Y. Siskos, E. Grigoroudis, C. Zopounidis, and O. Saurais. Measuring customer satisfaction using a collective preference disaggregation model. *Journal of Global Optimization*, 12(2):175–195, 1998.

[172] Y. Siskos, N. F. Matsatsinis, and G. Baourakis. Multicriteria analysis in agricultural marketing: The case of french olive oil market. *European Journal of Operational Research*, 130(2):315–331, April 2001.

[173] Y. Siskos, A. Spyridakos, and D. Yannacopoulos. Using artificial intelligence and visual techniques into preference disaggregation analysis: the MIIDAS system. *European Journal of Operational Research*, 113(2):281–299, March 1999.

[174] Y. Siskos and D. Yannacopoulos. UTASTAR, An ordinal regression method for building additive value functions. *Investigaçao Operacional*, 5(1):39–53, 1985.

[175] Y. Siskos and C. Zopounidis. The evaluation of venture capital investment activity: an interactive assessment. *European Journal of Operational Research*, 31(3):304–313, 1987.

[176] H. J. Skala. *Non-Archimedean Utility Theory*. Kluwer, Dordrecht, 1975.

[177] J. Sounderpandian. Value functions when decision criteria are not totally substitutable. *Operations Research*, 39:592–600, 1991.

[178] A. Spyridakos, Y. Siskos, D. Yannacopoulos, and A. Skouris. Multicriteria job evaluation for large organizations. *European Journal of Operational Research*, 130(2):375–387, April 2001.

[179] Th. Stewart. An interactive multiple objective linear programming method based on piecewise linear additive value functions. *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-17(5):799–805, 1987.

[180] P. Suppes, D. H. Krantz, R. D. Luce, and A. Tversky. *Foundations of measurement,* vol. 2: *Geometrical, threshold, and probabilistic representations.* Academic Press, New York, 1989.

[181] P. Suppes and M. Winet. An axiomatization of utility based on the notion of utility difference. *Management Science*, 1:259–270, 1955.

[182] O. Svenson. Process description of decision making. *Organizational Behavior and Human Performance*, 23:86–112, 1979.

[183] R. J. Titiev. Measurement structures in classes that are not universally axiomatizable. *Journal of Mathematical Psychology*, 9:200–205, 1972.

[184] A. Tversky. A general theory of polynomial conjoint measurement. *Journal of Mathematical Psychology*, 4:1–20, 1967.

[185] A. Tversky. Intransitivity of preferences. *Psychological Review*, 76:31–48, 1969.

[186] Ph. Vincke. *Multi-criteria decision aid.* Wiley, New York, 1992. Original version in French "*L'Aide Multicritère à la Décision*", Éditions de l'Université de Bruxelles-Éditions Ellipses, Brussels, 1989.

[187] Ph. Vincke. Outranking approach. In T. Gal, T. Stewart, and T. Hanne, editors, *Multicriteria decision making, Advances in* MCDM *models, algorithms, theory and applications*, pages 11.1–11.29. Kluwer, 1999.

[188] K. Vind. Independent preferences. *Journal of Mathematical Economics*, 20:119–135, 1991.

[189] J. von Neumann and O. Morgenstern. *Theory of games and economic behavior.* Wiley, 2nd edition, 1947.

[190] R. von Nitzsch and M. Weber. The effect of attribute ranges on weights in multiattribute utility measurements. *Management Science*, 39(8):937–943, August 1993.

[191] D. von Winterfeldt and W. Edwards. *Decision analysis and behavioral research.* Cambridge University Press, Cambridge, 1986.

[192] P. P. Wakker. Cardinal coordinate independence for expected utility. *Journal of Mathematical Psychology*, 28(1):110–117, 1984.

[193] P. P. Wakker. *Additive representations of preferences: A new foundation of decision analysis*. Kluwer, Dordrecht, 1989.

[194] P. P. Wakker. Additive representations of preferences, a new foundation of decision analysis; the algebraic approach. In J.-P. Doignon and J.-C. Falmagne, editors, *Mathematical Psychology: Current Developments*, pages 71–87, Berlin, 1991. Springer Verlag.

[195] P. P. Wakker. Additive representations on rank-ordered sets. I. The algebraic approach. *Journal of Mathematical Psychology*, 35(4):501–531, 1991.

[196] P. P. Wakker. Additive representations on rank-ordered sets. II. The topological approach. *Journal of Mathematical Economics*, 22(1):1–26, 1993.

[197] M. Weber, F. Eisenfuhr, and D. von Winterfeld. The effects of splitting attributes on weights in multiattribute utility measurement. *Management Science*, 34(4):431–445, 1988.

[198] J. A. Weymark. Generalized Gini inequality indices. *Mathematical Social Sciences*, 1:409–430, 1981.

[199] U. Wille. Linear measurement models—Axiomatizations and axiomatizability. *Journal of Mathematical Psychology*, 44:617–650, 2000.

[200] C. Zopounidis and M. Doumpos. Stock evaluation using a preference disaggregation methodology. *Decision Science*, 30:313–336, 1999.

[201] C. Zopounidis and M. Doumpos. Building additive utilities for multi-group hierarchical discrimination: The MHDIS method. *Optimization Methods & Software*, 14(3):219–240, 2000.

[202] C. Zopounidis and M. Doumpos. PREFDIS: a multicriteria decision support system for sorting decision problems. *Computers & Operations Research*, 27(7-8):779–797, June 2000.

[203] C. Zopounidis and M. Doumpos. Multicriteria preference disaggregation for classification problems with an application to global investing risk. *Decision Sciences*, 32(2):333–385, 2001.

[204] C. Zopounidis and M. Doumpos. A preference disaggregation decision support system for financial classification problems. *European Journal of Operational Research*, 130(2):402–413, April 2001.