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Ranking objects from a preference relation over their subsets

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Abstract: In many every day situations it can happen that one wants to rank the elements of a group of people having the possibility to observe them collaborating in subgroups. In this paper we propose a general way to get this ranking over the elements of a group N, starting from an arbitrary preference relation over the subsets of N and taking into account the information provided by this ranking over the subsets. To discuss this issue, we use the standard approach to this type of problems, very common in the social choice framework: we single out some properties that a general solution should have, and we prove that these properties characterize a unique function. Given the very general type of problems that this model can represent, we believe that this is only a starting point for a more extended analysis. In particular, it is clear that different contexts can suggest different properties, thus identifying other ranking methods.

1 Introduction

Suppose the director of a department has to rank its professors on the basis of their scientific results, obtained by collaborating together in different research groups. The collaboration among the professors can be observed looking at the papers published, the workshops organized and so on; thus, we suppose there is a ranking over these subgroups, maybe provided by some external institution or defined according to some previously defined criteria. For instance, the director could be interested in combining, for each group, different bibliometric indicators about the quality and the quantity of its scientific contribution, like the number of papers co-authored by the members of a group, the number of citations of their joint publications, the aggregate impact factors of journals and conferences on which the group's members publish their results, etc. We can also suppose that, alternatively, some national institution provides this type of information, as it happens in several countries. However the director must

provide its institution with a ranking of the professors by themselves, reflecting their overall contribution over all possible research groups. How can she do this? It is clear that, if not only a preference relation on the subsets of the agents is available, but also a utility function is associated to it, it is possible and natural to use cooperative game theory to solve the problem (even getting supplementary information). This is due to the fact that such a utility function actually represents a Transferable Utility (TU) game, and thus some power index, f.i. the Shapley value [11] or the Banzhaf index [1, 10], can be used to provide not only the ranking of the players, but also a numerical estimation of their average marginal contribution over all possible coalitions. However, in general, changing power index can change the ranking (see for instance [3, 4]) and more importantly, quite often it is conceivable that a natural relation over the groups is easy to get, while a specific utility function is not easily made available. For instance, even if one may (questionably) expect that some bibliometric index could be somehow used to establish that a group is more productive than another one, on the other hand it seems absolutely unrealistic to imagine a procedure better than random guessing to assess a numeric evaluation to the absolute scientific value of a group. On the other hand, it would be a bad idea to select randomly such a utility function, since it is well known that the same ordering on the subsets, when described by different utility functions, can provide different ranking among players [7]. For all these reasons, we want to construct a ranking procedure which is purely ordinal. Thus the problem becomes to define a suitable function, defined on the set of the complete pre-orders on the non-empty subsets of a given finite set N, and valued on the complete pre-orders on N. To do this, we use a classical method in Social Choice and Game Theory: we propose a small set of properties that such a function should satisfy, and we prove that these properties are enough to identify a unique function.

Two of the properties we consider are, in some sense, classical properties in the contexts of Social Choice and Game Theory: they recall some form of anonymity and monotonicity. To these properties we add a new one that reflects the *philosophy* underlying our procedure and our idea of solution. It is clear that different criteria can affect the ranking, since the focus can be put on different aspects: for instance, one can decide that outstanding performance of a group should be very important, thus people in the first groups should also be ranked in the first positions; dually, it could be preferable to punish groups with bad performances and rank their members in the last positions. Also, according to the given context, it is conceivable that the size of the groups plays a role. For instance groups neither too small nor too large could be encouraged, and thus people participating in groups of a prescribed size and well ranked should get a good individual ranking. We want to underline this, since we are convinced that the function we propose is interesting, but for sure it is not the only one potentially available; rather, it is likely that our function is not the best one in every possible contexts. As for TU games we have many solution concepts, and each one of them can be of interest and preferable to others in some situations, in the same way in this context we believe that several useful functions can be defined and studied.

Related literature. A similar problem has been investigated in the paper [8], where the author studies an alternative notion of power index for TU games that is invariant to the choice of the characteristic function representing the ranking over the coalitions. However, such invariant power index is properly

defined for a limited class of total pre-orders (over the set of all the coalitions). Alternatively, in [9] the authors analyse the individual ranking problem using a property-driven approach, and they prove several impossibility theorems showing that no ranking function satisfies a given set of attractive properties. To the best of our knowledge, this paper is then the first one providing an axiomatic characterization of a ranking function.

It is also worthy to mention that the problem investigated in this paper can be seen as the "inverse" of how to derive a ranking over the set of all subsets of N in a way that is "compatible" with a primitive ranking over the single elements of N, which is generally known as the *ranking sets of objects problem* in the literature related to problem (see, for instance, [6] and the survey [2]).

Following an alternative interpretation, our problem is also connected to the *choice-based welfare* problem of deriving a linear welfare ordering from a *choice function*, where a choice function over a set N is defined as a map selecting a single alternative from every subset $B \subseteq N$, as recently introduced in the paper [5]. As an interesting issue for future research, the authors of [5] suggest to investigate a non-canonical setting dealing with the problem of finding a linear welfare ordering when the agent chooses multiple alternatives, i.e. the agent's behaviour is described by a *choice correspondence* that selects a (nonempty) subset of alternatives in S, for every subset $S \subseteq N$. If an agent's behaviour is fully rational over her multiple selections, i.e. her multiple selections are compatible with the information provided by a preference relation over the subsets of N, the choice-based welfare problem with choice correspondence can be represented as a problem of ranking objects, as defined in this work.

The roadmap of the paper is as follows. Section 2 is dedicated to some preliminaries and to the main definitions. In Section 3 we introduce some properties a ranking function should satisfy. Section 4 is the core of the paper: the function we are proposing, called the lexicographic excellence solution (shortly, the lex-cel solution) is defined, then it is proved that the function is characterized by the axioms introduced in Section 3. Section 5 focuses on a dual definition of the lex-cel solution, and on its characterization. The independence of the axioms characterizing both ranking functions is studied in Section 6. Finally, Section 7 draws some conclusion and suggests directions for future research.

2   Notation and preliminaries

Let us introduce the main definitions for the development of our work. A *binary relation* $\succcurlyeq$ on a finite set X is a subset of the cartesian product $X \times X$. For each $x, y \in X$, the notation $x \succcurlyeq y$ will be preferably used instead of the more formal $(x, y) \in \succcurlyeq$. The following are some standard properties for a binary relation $\succcurlyeq$:

- *reflexivity*: for each $x \in X$, $x \succcurlyeq x$;

- *transitivity*: for each $x, y, z \in X$, $x \succcurlyeq y$ and $y \succcurlyeq z \Rightarrow x \succcurlyeq z$;

- *completeness*: for each $x, y \in X$, $x \neq y \Rightarrow$ either $x \succcurlyeq y$ or $x \succcurlyeq y$;

- *antisymmetry*: for each $x, y \in X$, $x \succcurlyeq y$ and $y \succcurlyeq x \Rightarrow x = y$.

A reflexive, transitive and complete binary relation on X is called a *complete pre-order* or also, indifferently, a *preference relation* or a *ranking* over X.

A reflexive, transitive, complete and antisymmetric binary relation on X is called a *complete order* on X or also a *strict ranking* over X. $\mathcal{R}(X)$ denotes the set of rankings (or complete pre-orders) on a given set X. Given a ranking $\succcurlyeq \in \mathcal{R}(X)$, in general, there are elements $x \neq y$ such that both $x \succcurlyeq y$ and $y \succcurlyeq x$ hold: in this case we say that x and y are indifferent, and we write $x \sim y$.

In the following, we consider a *finite* set N of n elements that should be ranked and the set of its non-empty subsets is denoted by $\mathcal{P}(N)$

Definition 1. A *social ranking solution under coalition information*, or briefly a *solution*, is a function

$$r : \mathcal{R}(\mathcal{P}(N)) \to \mathcal{R}(N).$$

In other words, r is a function providing a ranking of the objects of N, starting from any possible ranking on subsets of N.

In the sequel, we will reserve the notation $\succcurlyeq$ to denote a ranking over the non-empty subsets of N, i.e. a complete pre-order $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$. However, in order to facilitate the reading, with a little abuse of notation, given a ranking $\succcurlyeq$ and a ranking function r we shall write

$$r(x) \succcurlyeq r(y)$$

instead of $x\ r(\succcurlyeq)\ y$ to indicate that, starting from the given ranking $\succcurlyeq$ on $\mathcal{P}(N)$, x is in relation with y according to the solution r. Analogously we write

$$r(x) \succ r(y)$$

to denote that r ranks x strictly better than y over the ranking $\succcurlyeq$.

A trivial example of a solution is the function p that ranks the elements in N according to the ranking of the singletons in $\mathcal{P}(N)$. Given a ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ we have that

$$p(x) \succcurlyeq p(y) \iff \{x\} \succcurlyeq \{y\}.$$

However, this solution is extremely unsophisticated since all the information provided by the ranking over coalitions is not taken into account at all.

We introduce another notation we will use in the following. Suppose we have a ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ of the form

$$S_1 \succcurlyeq S_2 \succcurlyeq S_3 \succcurlyeq \cdots \succcurlyeq S_{2^n-1}.$$

Unless the ranking is a complete order, some indifferences are present. We associate to this ranking the *quotient order* denoted as follows

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

in which the subsets S_j have been grouped in the *equivalence classes* Σ_k generated by the indifference relation $\sim$ associated to $\succcurlyeq$. We use this notation to stress the strict ranking among some groups and the indifferences among the others. This means that all the sets in Σ_1 are indifferent to S_1 and are strictly better then the sets in Σ_2 and so on. Thus, for every j all coalitions in Σ_j are

ranked at the same level, and are strictly better than any coalition in Σ_{j+1}. It is clear that $\succcurlyeq$ offers, for some $l = 1, \ldots, 2^n - 1$, l different levels of satisfaction with respect to the coalitions, with the first level, i.e. coalitions in Σ_1 are the best, and so on. For example, such equivalence classes could represent the levels of scientific productivity reached by different groups of researchers (f.i., given by the best ranked journal on which a group has published articles). Note that if $\succcurlyeq$ is a complete order, then the two relations are the same and $\Sigma_i = S_i$ for any $i = 1, \ldots, 2^n - 1$.

3   Properties for a ranking function

In this section we introduce some properties that in our opinion a ranking function should reasonably satisfy, and we discuss their importance and interpretation.

Let us consider a ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\succ$:

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

and an element $x \in N$. We denote by x_k the number of sets in Σ_k containing x:

$$x_k = |\{S \in \Sigma_k : x \in S\}|.$$

The first property we introduce is the following

Axiom 1 (Per-class Equal Power (PEP)). We say that a solution r satisfies the *Per-class Equal Power* if, for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\succ$ and any $x, y \in N$ such that $x_k = y_k$ for any $k = 1, \ldots, l$, then

$$r(x) \sim r(y).$$

The Per-class Equal Power property requires that all subsets belonging to the same equivalence class count the same. To be more specific, what the above property want to stress is the following. The ranking $\succcurlyeq$ provides a partition of the groups on l different levels of satisfaction and at each of these levels the objects x and y are present in the same number of groups. Thus, since we do not assume that some groups are more desirable than others (f.i. according to some rule related to their cardinality), the property requires that there cannot be a strict preference between x and y. For the sake of the example about professors' evaluation, let's say that if two professors belong to an equal number of equivalent groups for each level of scientific productivity, then the PEP property imposes to them the same ranking.

The next axiom is a kind of monotonicity property, and it serves, as it is clear from its definition, to break ties in a consistent way. In order to introduce it, we need some extra notion. For any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\succ$:

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_u \succ \Sigma_{u+1} \succ \cdots \succ \Sigma_l$$

and for any $x, y \in N$, let $\Sigma \subseteq \Sigma_{u+1}$, be a class of subsets of N such that $\{x, y\} \cap S = \{x\}$ for every $S \in \Sigma$. Define another ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \quad \Sigma_u \cup \Sigma \quad \sqsupset \quad \Sigma_{u+1} \setminus \Sigma \quad \sqsupset \cdots \sqsupset \Sigma_l.$$

We say that $\sqsupseteq$ is *x-improving* and *y-invariant with respect to* $\succsim$. In words, in $\sqsupseteq$ some subsets containing x but not y are strictly better ranked than in $\succsim$, and no subset containing y has changed its ranking position with respect to $\succsim$.

Axiom 2 (Monotonicity)**.** We say that a solution r is *monotone* if for any ranking $\succsim \in \mathcal{R}(\mathcal{P}(N))$, every $x, y \in N$ such that $r(x) \sim r(y)$ and any ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ which is x-improving and y-variant with respect to $\succsim$, then it holds $r(x) \sqsupset r(y)$.

In words, if x and y are indifferent in the ranking given by r on $\succsim$, then in the ranking given by r on $\sqsupseteq$, x is ranked strictly better than y. The requirement is clear. Suppose that, for a given preference order, the ranking function ranks the two objects x and y at the same level of satisfaction (e.g., the two professors based on their scientific performance over different teams). Then for every pre-order obtained by the given one just strictly improving the ranking of some coalition containing x and not y (e.g., due to the fact that the educational quality is considered as a secondary criterion for the groups' evaluation, and the educational offer provided by the group containing professor x and not y is much more differentiated than the one of other groups with an equivalent scientific productivity), the ranking function now ranks x strictly better than y. In other words, since x in the new preference system has an improvement (because it belongs to a coalition that now is strictly better ranked than before), while y is exactly in the same situation as before, the tie present before between x and y can now be broken. We observe that this looks like a very reasonable criterion to break ties.

The two previous properties are general request to a ranking function and do not refer to any specific idea behind the way how to rank people looking at their performances when working in different groups. As far as the next property is concerned, on the contrary, a precise underlying idea on how to drive the ranking pops up: to consider the good performances more important than the bad ones.

Axiom 3 (Independence from the Worst Set (IWS))**.** We say that a ranking function r is *independent from the worst set* if for any ranking $\succsim \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

with $l \geq 2$, $x, y \in N$ such that

$$r(x) \succ r(y)$$

then, it holds

$$r(x) \sqsupset r(y)$$

for any partition $T_1, \ldots, T_m$ of Σ_l and for any ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset T_1 \sqsupset \cdots \sqsupset T_m.$$

We can reformulate the IWS property as follows. Suppose according to a certain solution, a strict ranking is already reached on a pair of elements of N,

starting from a given ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$, with the associated quotient order $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l.$$

Suppose now to consider a new ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ with the property that it is built from the previous one just refining *only* the information in the set Σ_l. Then the property simply requires that the ranking of the two objects provided by the solution over $\sqsupseteq$ does not change. Thus the property proposes to consider as irrelevant a further refinement of the preferences on the worst group, and thus to ignore this further information, as a principle of coherence with the highest standard of quality used for groups evaluation, and when a strict ranking among two elements is already defined. In our example, once that a strict ranking between two professors is established on the basis of their scientific productivity over all groups, the possible use of a secondary criterion for groups' evaluation (e.g., the educational offer of a team) affecting only coalitions with the lowest scientific productivity, may not impact a strict ranking defined according to the most important evaluation's criteria.

We again stress the fact that, differently from the two previous properties, that should be fulfilled by any reasonable ranking function, this last property privileges an idea that, though very reasonable, is clearly not the unique reasonable criterion. In a subsequent section we shall see how it is possible to *dualize* the above property to get another, different but equally reasonable criterion, characterizing a different ranking function.

4   The lexicographic excellence solution

We can now introduce and characterize the solution we propose. First of all, let us recall that given a ranking $\succcurlyeq$ and any element $x \in N$, x_k is the number of sets containing x in Σ_k, that is

$$x_k = |\{S \in \Sigma_k : x \in S\}|$$

for $k = 1, \ldots, l$. Now, let $\theta_{\succcurlyeq}(x)$ be the l-dimensional vector $\theta_{\succcurlyeq}(x) = (x_1, \ldots, x_l)$ associated to $\succcurlyeq$. Consider the lexicographic order among vectors:

$$\mathbf{x} \geq_L \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \quad \text{or } \exists j : x_i = y_i, i = 1, \ldots, j-1 \ \wedge x_j > y_j.$$

As it is well known, and easy to see, $\geq_L$ defines a complete order on $\mathbb{R}^l$. Now, we are ready for the main definition.

Definition 2. The *lexicographic excellence (lex-cel) solution* is the function $\boldsymbol{e} : \mathcal{R}(\mathcal{P}(N)) \to \mathcal{R}(N)$ defined for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ as

$$\boldsymbol{e}(x) \succcurlyeq \boldsymbol{e}(y) \qquad \text{if} \qquad \theta_{\succcurlyeq}(x) \ \geq_L \ \theta_{\succcurlyeq}(y).$$

Note that, in general, $\boldsymbol{e}$ provides a pre-order and not an order, even if $\geq_L$ is an order; this is due to the fact that $\theta_{\succcurlyeq}$ in general is not one-to-one; however, if $\succcurlyeq$ is a complete order, actually $\boldsymbol{e}(\succcurlyeq)$ provides an order and the $(2^n - 1)$-dimensional vector $\theta_{\succcurlyeq}(x)$ is boolean, i.e. made by only zeros and ones.

In the remaining of this section we characterize the lex-cel solution as the unique one fulfilling the three properties: ETG, monotonicity and independence from the worst set.

Theorem 1. *The lex-cel solution e satisfies axioms 1, 2 and 3.*

Proof. The PEP property requires that for any ranking $\succcurlyeq$, if $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$ then $x \sim y$. This comes immediately from the definition of lex-cel solution.

Let us consider the monotonicity property. Given a ranking $\succcurlyeq$ if $e(x) \sim e(y)$, this means that $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$. Now consider a new ranking $\sqsupseteq$ that is x-improving and y-invariant with respect to y. We have that $\theta_{\sqsupseteq}(x) >_L \theta_{\succcurlyeq}(x)$, while $\theta_{\sqsupseteq}(y) =_L \theta_{\succcurlyeq}(y)$. So $\theta_{\sqsupseteq}(x) >_L \theta_{\sqsupseteq}(y)$ and $e(x) \sqsupseteq e(y)$.

Finally the IWS property obviously follows from the definition of e.

$\square$

Theorem 2. *Let φ be a ranking function that satisfies axioms 1, 2 and 3. Then for any $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ and $x, y \in N$*

$$e(x) \succ e(y) \iff \varphi(x) \succ \varphi(y).$$

Proof. ($\Rightarrow$) Let $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

and such that $e(x) \succ e(y)$. This means that $\theta_{\succcurlyeq}(x) >_L \theta_{\succcurlyeq}(y)$. Let k be the first index such that $x_k > y_k$, i.e., such that $x_j = y_j$ for all $j = 1, \ldots, k-1$, if $k > 1$.

Define $\Sigma^* = \{A \in \Sigma_k : x \in A, y \notin A\}$ and let Σ be any subset of Σ^* such that $|\Sigma| = x_k - y_k$. This means that in Σ there are only sets containing x and not containing y and if we remove Σ from Σ_k then the number of elements in Σ_k containing x is the same of the number of elements containing y. Note that Σ is not empty and well-defined due to the hypothesis $e(x) \succ e(y)$.

Now, define another ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \quad \Sigma_k \smallsetminus \Sigma \quad \sqsupset \Sigma \cup \Sigma_{k+1} \cup \cdots \cup \Sigma_l,$$

then since φ satisfies axiom 1 we have that x and y are indifferent according to φ on $\sqsupseteq$.

Consider now a still different ranking $\geq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $>$ in which Σ is moved up of one level:

$$\Sigma_1 > \Sigma_2 > \cdots > \quad \Sigma_k \quad > \Sigma_{k+1} \cup \cdots \cup \Sigma_l,$$

then since φ satisfies axiom 2, if we compare $\sqsupset$ and $>$ we have $\varphi(x) > \varphi(y)$. Finally, thanks to axiom 3, when we consider the original ranking $\succcurlyeq$ with the associated quotient ranking $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \quad \Sigma_k \quad \succ \cdots \succ \Sigma_l,$$

we have $\varphi(x) \succ \varphi(y)$.

($\Leftarrow$) Let $\varphi(x) \succ \varphi(y)$. Suppose that $e(x) \sim e(y)$, then $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$ but this is impossible since φ satisfies axiom 1.

Suppose that $e(x) \prec e(y)$, then we already proved that this would imply $\varphi(x) \prec \varphi(y)$. Thus the only possibility is that $e(x) \succ e(y)$. $\square$

Corollary 1. *There is one and only one ranking function satisfying the properties of ETG, monotonicity and independence from the worst set. This is the lex-cel solution e.*

Proof. The lex-cel solution fulfils the properties, according to Theorem 1. Theorem 2 takes care of the uniqueness argument, since it shows that a preference relation fulfilling the three properties denotes the same subset of $N \times N$ as the lex-cel solution. □

We conclude this section by introducing a new property, and proving that the lex-cel solution satisfies this property.

Property 1. The function $r : \mathcal{R}(\mathcal{P}(N)) \to \mathcal{R}(N)$ is said to satisfy the property of *independence from the irrelevant coalitions (IIC)* if for every $\succcurlyeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ such that there are $x, y \in N$ and for all S with $x, y \notin S$ it holds

$$S \cup \{x\} \succcurlyeq S \cup \{j\} \iff S \cup \{x\} \sqsupseteq S \cup \{j\}$$

then

$$r(x) \succcurlyeq r(y) \iff r(x) \sqsupseteq r(y).$$

Suppose we are considering two different complete pre-orders between the subsets of N, and that for two of these objects, denoted by x and y respectively, it happens that adding x to a set not containing both and y to the same subset not containing both *does not change* the mutual positions of the new subsets in the two rankings. The IIC property then requires that the mutual ranking of x and y must be the same in the two cases. In other words, the ranking of x, y depends only on the ranking of subsets in which there is only one of them, and thus does *not* depend from the subsets either not containing them or containing both of them.

The connection between this property and the celebrated condition of independence from the irrelevant alternatives in Arrow's impossibility theorem is quite transparent.

Lemma 1. *The lex-cel solution satisfies property 1.*

Proof. It is clear that the ranking provided by $\boldsymbol{e}$ over x and y does not depend on the ranking of the subsets of N in which there are not the two elements x and y. Moreover, if x, y are in the same subset then this information is not used to compare $\theta_{\succcurlyeq}(x)$ and $\theta_{\succcurlyeq}(y)$. This means that $\boldsymbol{e}$ satisfies property 1. □

5 The dual-lexicographic solution

It is possible to define a solution dual to the lex-cel solution, in the following way. Given two vectors $\mathbf{x}, \mathbf{y}$ we define the lexicographic* order $\geq_{L^*}$ as

$$\mathbf{x} \geq_{L^*} \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \text{ or } \exists j : x_i = y_i, \forall i > j \ \wedge x_j < y_j.$$

Definition 3. The *dual-lexicographic (dua-lex) solution* is the function $\boldsymbol{w} : \mathcal{R}(\mathcal{P}(N)) \to \mathcal{R}(N)$ defined for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(N))$ as

$$\boldsymbol{w}(x) \succcurlyeq \boldsymbol{w}(y) \quad \text{if} \quad \theta_{\succcurlyeq}(x) \ \geq_{L^*} \ \theta_{\succcurlyeq}(y).$$

It immediately appears that this ranking function acts, in a sense, dually to the lex-cel solution, since it ranks at the last places objects present in the worst ranked groups. Here mediocrity is punished, before excellence was rewarded.

Let us refer to our usual example: how the chairman of the department should rank her professors? Which property should choose? The answer can heavily depend on the way financial support of the research is distributed in the country. If the department is rewarded provided there are outstanding research teams, the independence from the worst set is a very reasonable assumption to make, thus the lex-cel solution would be a good choice. On the contrary, if a department is punished if there are groups with a very low scientific production, a competition at the lowest level is natural, as the dua-lex solution proposes, in order to enhance the level of quality of the worst ranked groups.

It is clear that in general the two ranking functions give different ranking among the elements, as the following example shows.

Example 1. *Consider the ranking $\succcurlyeq$ defined on the power set of $N = \{1, 2, 3\}$ as*

$$\{1, 2\} \sim \{1, 3\} \succ \{2\} \succ \{2, 3\} \succ \{3\} \sim \{1, 2, 3\} \succ \{1\}.$$

We have that $\theta_{\succcurlyeq}(1) = (2, 0, 0, 1, 1)$, $\theta_{\succcurlyeq}(2) = (1, 1, 1, 1, 0)$ and $\theta_{\succcurlyeq}(3) = (1, 0, 1, 2, 0)$. Then the excellence function ranks $\boldsymbol{e}(1) \succ \boldsymbol{e}(2) \succ \boldsymbol{e}(3)$, while the dua-lex solution ranks $\boldsymbol{w}(2) \succ \boldsymbol{w}(3) \succ \boldsymbol{w}(1)$.

The function $\boldsymbol{w}$ satisfies ETG and monotonicity. In order to characterize it we need to define an axiom dual to axiom 3.

Axiom 4 (Independence from the best set (IBS))**.** We say that a ranking function is *independent from the best set* if for any strict order $\succ$

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

and $x, y \in N$ such that

$$r(x) \succ r(y),$$

then it holds

$$r(x) \sqsupset r(y)$$

for any partition $T_1, \ldots T_m$ of Σ_1 and for any order $\sqsupset$

$$T_1 \sqsupset \cdots \sqsupset T_m \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma_l.$$

The requirement of the above property is that if a strict ranking is already reached on a pair of objects, then it cannot change by refining the best preferred equivalence class: the ranking remains the same even if Σ_1 is partitioned according to any possible partition. Clearly the requirement proposed by this property is to ignore the information provided from the best ranked coalitions.

Theorem 3. *The dua-lex solution $\boldsymbol{w}$ satisfies axioms 1, 2 and 4.*

Proof. Similar to the proof of Theorem 2, from the definition of the $\boldsymbol{w}$ function it is trivial to check that it satisfies the three axioms. $\qquad\square$

Theorem 4. *Let φ be a ranking function that satisfies axioms 1, 2 and 4. Then for any $\succcurlyeq$ and $x, y \in N$*

$$\boldsymbol{w}(x) \succ \boldsymbol{w}(y) \iff \varphi(x) \succ \varphi(y).$$

Corollary 2. *There is one and only one ranking function satisfying the properties of ETG, monotonicity and independence from the best set. This is the dua-lex solution $\boldsymbol{w}$.*

Proof. The dual ranking function fulfils the properties, according to Theorem 3. Theorem 4 takes care of the uniqueness argument, since it shows that a preference relation fulfilling the three properties denotes the same subset of $N \times N$ as the dual ranking function. □

6 Independence of the axioms

Let us prove that axioms 1, 2, 3 (and 4) are independent, thus they all are necessary in order to uniquely characterize the lexicographic excellence solution and its dual lexicographic solution.

PEP is not satisfied Given a finite set N, take any arbitrary order $<$ on it. Let $\boldsymbol{e}^*$ be the solution defined as $\boldsymbol{e}$ but if there is a ranking $\succcurlyeq$ and there are two elements x, y such that $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$ then $\boldsymbol{e}^*$ breaks the tie with $\boldsymbol{e}^*(x) \succ \boldsymbol{e}^*(y)$ if $x < y$.
For instance if we take $N = \{1, 2, \ldots, n\}$ and $>$ as the majority relation among integer numbers, then in case of indifference among all the subsets, $\boldsymbol{e}^*$ ranks the elements in increasing order.

This function satisfies Axioms 2 and 3, but does not satisfy PEP.

Monotonicity is not satisfied Let f be a solution defined as

$$f(x) \succcurlyeq f(y) \text{ if } x_j \geq y_j$$

where $j = \min\{k : \Sigma_k \cap \{x, y\} \neq \emptyset\}$. The function f ranks players only counting the number of sets in the first level containing those players.
It is clear that f satisfies Axioms 1 and 3, since the ranking only depends on the number of sets in Σ_j. However, this function does not satisfy monotonicity: if we improve the situation of one player but do not change Σ_j, then the ranking does not change. For instance consider the following ranking $\succcurlyeq$ on $N = \{1, 2\}$

$$\{1, 2\} \succ \{2\} \sim \{1\}$$

then $f(1) \sim f(2)$. Consider now the ranking

$$\{1, 2\} \sqsupset \{2\} \sqsupset \{1\},$$

then it still holds $f(1) - f(2)$, while monotonicity would require $f(1) \sqsupset f(2)$.

Independence from the worst set is not satisfied Consider the dua-lex solution $\boldsymbol{w}$ as defined in the previous section. By Theorem 2 it satisfies ETG and monotonicity. Of course, $\boldsymbol{w}$ does not satisfy independence from the worst set. For instance given the ranking

$$\cdots \succ \{1, 2\} \sim \{1\} \succ \{2\} \sim \{2, 3\} \sim \{1, 3\}$$

we have $\boldsymbol{w}(1) \succ \boldsymbol{w}(3) \succ \boldsymbol{w}(2)$. But if we change the last set in the following way

$$\cdots \sqsupset \{1, 2\} - \{1\} \sqsupset \{2\} - \{2, 3\} \sqsupset \{1, 3\}$$

we get a new ranking: $\boldsymbol{w}(2) \sqsubset \boldsymbol{w}(1) \sqsubset \boldsymbol{w}(3)$.

Independence from the best set is not satisfied   The lex-cel solution e satisfies ETG and monotonicity but of course, it is not independent from the best set. Thus, it is also clear that the three axioms characterizing the dua-lex solution are independent.

7   Conclusions

We have proposed in this paper a general way to rank objects whenever it is available a ranking between the subsets of these objects. Our approach was classical, in the sense that we identified a rule by requiring the fulfillment of some general properties. Having this characterization in mind, there are other interesting issues to consider: first of all, it is clear that in practical situations it can happen that having a ranking on *all* subsets of N is an unrealistic requirement: in our paradigmatic example of the ranking of the professors in a department, it is unreasonable to expect that all members collaborate forming every possible research team.

Moreover, there may be some specific reason to reward coalitions of a fixed size for instance or the collaborations with a specific element or group of elements. In all these situations, there may be a ranking over only a subset of $\mathcal{P}(N)$.

Of course, the lexicographic excellence solution can be defined, without any changes, on subsets of $\mathcal{P}(N)$. And clearly the properties that characterize the function hold if we restrict the domain. However uniqueness is not clear in this case. For instance, the equal treatment of groups property in a subset could be totally uninformative, since it is possible, f.i., that no pair of objects is present in the same number of coalitions. Thus a characterization in this case must depend from the type of coalitions that are actually ranked. This issue will be analyzed in a subsequent paper.

Secondly, as usual the main theorem we provide requires the solution to be defined on a very large set, namely the set of all pre-orders over the subsets of N. But it would be interesting to analyze the same problem of characterizing a solution defined only on (meaningful) subsets of $\mathcal{R}(\mathcal{P}(N))$. For instance, there could be only k different levels to judge the collaboration between the elements, thus we can consider the elements in $\mathcal{R}(\mathcal{P}(N))$ for which the number of indifference sets is fixed.

Moreover, about this second issue, a particularly interesting case is when we restrict our attention to orders over $\mathcal{P}(N)$, i.e. we do not consider the possibility of ties. In this case one would expect as an output a strict ranking, and actually the lex-cel solution and also its dual do make the job, but it is clear that the pool of properties characterizing them on the whole set of the rankings, where restricted to this subset, does not play a role. Actually the equal treatment of groups property does not apply, since in this case $\theta(x) = \theta(y)$ if and only if $x = y$. Moreover, monotonicity alone is unable to start any process of selection of a solution, since it appeals to indifference, which is impossible to establish in this context (and actually we never want to have indifference starting form a strict ranking, since we are looking for strict rankings). Finally, clearly independence from below (or above), is simply meaningless in the context of complete orders.

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