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Contents

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Abstract

We present a method called GRIP (*Generalized Regression with Intensities of Preference*) for ranking a set of actions evaluated on multiple criteria. GRIP builds a set of additive value functions compatible with preference information composed of a partial preorder and required intensities of preference on a subset of actions, called *reference actions*. It constructs not only the preference relation in the considered set of actions, but it also gives information about intensities of preference for pairs of actions from this set for a given Decision Maker (DM). Distinguishing necessary and possible consequences of prefernce information on the all set of actions, GRIP answers questions of robustness analysis. The proposed methodology can be seen as an extension of UTA method based on ordinal regression. GRIP can also be compared to AHP method, which requires pairwise comparison of all actions and criteria, and yields a priority ranking of actions. As for the preference information being used, GRIP can be compared, moreover, to MACBETH method which also takes into account a preference order of actions and intensity of preference for pairs of actions. The preference information used in GRIP does not need, however, to be complete: the DM is asked to provide comparisons of only those pairs of reference actions on particular criteria for which his/her judgment is sufficiently certain. This is an important advantage comparing to methods which, instead, require comparison of all possible pairs of evaluations on all the considered criteria. Moreover, GRIP works with a set of general additive value functions compatible with the preference information, while other methods use a single and less general value function, such as the weighted-sum.

Keywords: *Multiple criteria decision aiding, Preference model, Value function, Ordinal regression, Intensity of preference*

1 Introduction

Ranking a finite set of actions evaluated on a finite set of criteria is a problem of uttermost importance in many areas of real-world decision-making situations (see Figueira et al. 2005). Among many approaches that have been designed to deal with the ranking problem, two of them seem to prevail. The first one exploits the idea of assigning a score to each action, as it is the case of MAUT - Multi-Attribute Utility Theory (see Keneey and Raiffa 1976). The second relies on the principle of pairwise comparison of actions, as it is the case of outranking methods (see Roy 1996). The value function and the outranking relation are two preference models underlying these two main approaches. In order to build such models, preference information from the Decision Maker (DM) is required.

The preference information may be either direct or indirect, depending whether it specifies directly values of some parameters used in the preference model (e.g. trade-off weights, aspiration levels, discrimination thresholds, etc.) or whether it specifies some examples of holistic judgments from which compatible values of the preference model parameters are induced. Eliciting direct preference information from the DM can be counterproductive in real-world decision making situations because of a high cognitive effort required. Consequently, asking directly the DM to provide values for the parameters seems to make the DM uncomfortable. Eliciting indirect preference is less demanding of the cognitive effort. Indirect preference information is mainly used in the *ordinal regression paradigm*. According to this paradigm, a holistic preference information on a subset of some *reference* or *training actions* is known first and then a preference model compatible with the information is built and applied to the whole set of actions in order to rank them.

The ordinal regression paradigm is concordant with the *posterior rationality* postulated by March (1978). It has been known for at least fifty years in the field of multidimensional analysis. It is also concordant with the *induction principle* used in machine learning. This paradigm has been applied within the two main Multiple Criteria Decision Analysis (MCDA) approaches mentioned above: those using value function as preference model (Srinivasan and Shocker 1973, Pekelman and Sen 1974, Jacquet-Lagrèze and Siskos 1982, Siskos et al. 2005), and those using outranking relation as preference model (Kiss et al. 1994, Mousseau and Słowiński 1998). This paradigm has also been used since mid ninetieth's in MCDA methods involving a new, third family of preference models – a set of dominance decision rules induced from rough approximations of holistic preference relations (Greco et al. 1999, 2001, 2003, 2005, and Słowiński et al. 2005).

Recently, the ordinal regression paradigm has been revisited with the aim of considering the whole set of value functions compatible with the preference information provided by the DM, instead of a single compatible value function used, for example, in UTA-like methods (Jacquet-Lagrèze and Siskos 1982, and Siskos et al. 2005). This extension has been implemented in a method called UTA^{GMS} (Greco et al. 2003, 2005). This method is not revealing to the DM one compatible value function, but it is using the whole set of compatible (general, not piecewise linear only) additive value functions to set up a necessary weak preference relation and a possible weak preference relation in the whole set of considered actions. Moreover, in UTA^{GMS}, the preference information has the form of a partial preorder (instead of a complete preorder) in a subset of reference actions. The necessary and possible weak preference relations are exploited such that one finally obtains two rankings in the set of actions: the necessary ranking (partial preorder) identifying preference statements being true for all compatible value functions, and the possible ranking (complete and negatively transitive binary relation) identifying preference statements being true for at least one compatible value function. Distinguishing necessary and possible consequences of preference information on the all set of actions, UTA^{GMS} answers questions of robustness analysis (Roy 1998).

In this paper, we present a new method called GRIP (*Generalized Regression with Intensities of Preference*) which also belongs to the class of methods based on indirect preference information and the ordinal regression paradigm. GRIP generalizes both UTA and UTA^{GMS} methods by adopting all features of UTA^{GMS} and taking into account additional preference information in the form of comparisons of intensities of preference between some pairs of reference actions. These comparisons are expressed in two possible ways (not exclusive): comprehensively, i.e. on all criteria, and partially, i.e. on particular criteria.

GRIP can be compared to the AHP method (Saaty 2005), which requires, from the DM, preference information composed of pairwise comparisons of all actions and criteria on a fixed ratio scale, and constructs a weighted-sum value function producing a priority ranking of actions.

From the viewpoint of the type of preference information being used, the GRIP method can be also compared, moreover, to the MACBETH method (Bana e Costa and Vansnick 1994; Bana e Costa et al. 2005), which also involves a preference order in the set of actions and the intensity of preference for pairs of actions. Unlike in MACBETH, however, the preference information in GRIP does not need to be complete, i.e. the preference order may be partial and may concern just reference actions, not all actions, and, moreover, information about intensities of preference may also concern some pairs of reference actions, not all possible pairs of evaluations. This is an important feature of GRIP, answering to the current demand commonly addressed to decision aiding methods: "try to help DMs using incomplete although reliable information".

The paper is organized in the following way. In section 2, we present motivations that led us to built up a new method. In section 3, we recall some useful concepts concerning mathematical modelling of preferences, with an adequate notation. Sections 4 and 5 are devoted to the UTA and UTA$^{\mathrm{GMS}}$ methods, respectively. In section 6, we present the new GRIP method and we provide a theoretical comparison of GRIP with AHP and MACBETH. Finally, section 7 provides conclusions and avenues for future research.

2 Motivation

Apart from the reasons that motivated the proposal of UTA$^{\mathrm{GMS}}$ method (Greco et al. 2005), there are two major issues that led us to conduct research on this very topic:

- *Preference information.* There is a need, observed in practice, of handling information related to intensity of preferences that was not considered in the UTA family of methods. In many real-life decision-making situations, DMs are willing to provide more information than a preorder on a set of reference or training actions. Thus, it is frequent to observe assertions of the type "x is preferred to y at least as much as w is preferred to z", expressed on particular criteria (partially) and/or on all criteria together (comprehensively).

- *Technical aspects.* The additional constraints related to requirements about intensity of preference can reduce the feasible polyhedron of all value functions compatible with preference information - the polyhedron which, in general, can be quite large. This can be useful for both classical UTA method and UTA$^{\mathrm{GMS}}$ method.

Similarly to UTA$^{\mathrm{GMS}}$, it would be desirable to design an interactive procedure based on the GRIP methodology enabling progressive articulation of DM's preferences along with narrowing the range of compatible value functions, in the spirit of a constructive (or learning) process.

3 Mathematical background on preference modelling

This section aims to recall some basic concepts of MCDA and mathematical preference modelling, along with an adequate notation.

3.1 Elementary notation and problem statement

We are considering a multiple criteria decision problem where a finite set of actions $A = \{x, \ldots, y, \ldots, w, \ldots, z\}$ is evaluated on a family $F = \{g_1, g_2, \ldots, g_m\}$ of m criteria. Let $I = \{1, 2, \ldots, m\}$ denote the set of criteria indices. We assume, without loss of generality, that the greater $g_i(x)$, the better action x on criterion g_i, for all $i \in I$, $x \in A$. A DM is willing to rank the actions of A from the best to the worst, according to his/her preferences. The ranking can be complete or partial, depending on the preference information provided by the DM and on the way of exploiting this information. The family of criteria F is supposed to satisfy consistency conditions, i.e. completeness (all relevant criteria are considered), monotonicity (the better the evaluation of an action on considered criteria, the more it is preferable to another), and non-redundancy (no superfluous criteria are considered) (see Roy and Bouyssou 1993).

Such a decision-making problem statement is called *multiple criteria ranking problem*. It is known that the only information coming out from the formulation of this problem is the dominance ranking. Let us recall that in the dominance ranking, action $x \in A$ is preferred to action $y \in A$, $x \succ y$, if and only if $g_i(x) \geq g_i(y)$ for all $i \in I$, with at least one strict inequality. Moreover, x is indifferent to y, $x \sim y$, if and only if $g_i(x) = g_i(y)$ for all $i \in I$. Hence, for any two actions $x, y \in A$, one of the four situations may arise in the dominance ranking: $x \succ y$, $y \succ x$, $x \sim y$ and $x?y$, where the last one means that x and y are incomparable. Usually, the dominance ranking is very poor, i.e. the most frequent situation is $x?y$.

In order to enrich the dominance ranking, the DM has to provide preference information which is used to construct an aggregation model making the actions more comparable. Such an aggregation model is called preference model. It induces a preference structure on set A, whose proper exploitation permits to work out a ranking proposed to the DM.

In what follows, the evaluation of each action $x \in A$ on each criterion $g_i \in F$ will be denoted either by $g_i(x)$ or x_i.

3.2 Preference relation

Let G_i denote the value set (scale) of criterion g_i, $i \in I$. Consequently,

$$G = \prod_{i \in I} G_i$$

represents the evaluation space, and $x \in G$ denotes a profile of an action in such a space. We consider a weak preference relation $\succsim$ on A which means, for each pair of vectors, $x, y \in G$,

$$x \succsim y \Leftrightarrow \text{“}x \text{ is at least as good as } y\text{”}.$$

This weak preference relation can be decomposed into its asymmetric and symmetric parts, as follows,

1) $x \succ y \equiv [x \succsim y \text{ and not } y \succsim x] \Leftrightarrow$ “x is preferred to y”, and

2) $x \sim y \equiv [x \succsim y \text{ and } y \succsim x] \Leftrightarrow$ “x is indifferent to y”.

From a pragmatic point of view, it is reasonable to assume that $G_i \in \mathbb{R}$, for $i = 1, \ldots, m$. More specifically, we will assume that the evaluation scale on each criterion g_i is bounded, such that $G_i = [\alpha_i, \beta_i]$, where α_i, β_i, $\alpha_i < \beta_i$ are the worst and the best (finite) evaluations, respectively. Thus, $g_i : A \to G_i$, $i \in I$, therefore, each action $x \in A$ is associated with an evaluation vector denoted by $\underline{g}(x) = (x_1, x_2, \ldots, x_m) \in G$.

4 Ordinal regression: The foundations of the UTA method

This section presents an outline of the principle of the ordinal regression *via* linear programming, as proposed in the original UTA method (see Jacquet-Lagrèze and Siskos 1982).

4.1 Preference information

The preference information is given in the form of a complete preorder on a subset of reference actions $A^R \subseteq A$ (where $|A^R| = p$), called *reference preorder*. The reference actions are usually those contained in set A for which the DM is able to express holistic preferences. Let $A^R = \{a, b, c, \ldots\}$ be the set of reference actions.

4.2 An additive model

The additive value function is defined on A such that for each $\underline{g}(x) \in G$,

$$U(\underline{g}(x)) = \sum_{i \in I} u_i(g_i(x)), \tag{1}$$

where, u_i are non-decreasing marginal value functions, $u_i : G_i \rightarrow \mathbb{R}$, $i \in I$. For the sake of simplicity, we shall write (1) as follows,

$$U(x) = \sum_{i \in I} u_i(x_i) \quad (1$$

In the UTA method, the marginal value functions u_i are assumed to be piecewise linear functions. The ranges $[\alpha_i, \beta_i]$ are divided into $\gamma_i \geq 1$ equal sub-intervals,

$$[x_i^0, x_i^1], [x_i^1, x_i^2], \ldots [x_i^{\gamma_i - 1}, x_i^{\gamma_i}]$$

where,

$$x_i^j = \alpha_i + \frac{j}{\gamma_i}(\beta_i - \alpha_i), \;\; j = 0, \ldots, \alpha_i, \text{ and } i \in I.$$

The marginal value of an action $x \in A$ is obtained by linear interpolation,

$$u_i(x) = u_i(x_i^j) + \frac{x_i - x_i^j}{x_i^{j+1} - x_i^j}(u_i(x_i^{j+1}) - u_i(x_i^j)), \;\; \text{for } \; x_i \in [x_i^j, x_i^{j+1}]. \tag{2}$$

The piecewise linear additive model is completely defined by the marginal values at the breakpoints, i.e. $u_i(x_i^0) = u_i(\alpha_i), u_i(x_i^1), u_i(x_i^2), ..., u_i(x_i^{\gamma_i}) = u_i(\beta_i)$.

In what follows, the principle of the UTA method is described as it was recently presented by Siskos et al. (2005).

Therefore, a value function $U(x) = \sum_{i=1}^n u_i(x_i)$ is compatible if it satisfies the following set of constraints

$$\begin{array}{l} \left. \begin{array}{l} U(a) > U(b) \Leftrightarrow \; a \succ b \\ U(a) = U(b) \Leftrightarrow \; a \sim b \end{array} \right\} \quad \forall \;\; a, b \in A^R \\ u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0, \; i = 1, ..., n, \; j = 1, ..., \gamma_i - 1 \\ u_i(\alpha_i) = 0, \; i = 1, ..., n \\ \sum_{i=1}^n u_i(\beta_i) = 1 \end{array} \tag{3}$$

4.3 Checking for compatible value functions through linear programming

To verify if a compatible value function $U(x) = \sum_{i=1}^n u_i(x_i)$ restoring the reference preorder $\succsim$ on A^R exists, one can solve the following linear programming problem, where $u_i(x_i^j), i = 1, ..., n, \; j = 1, ..., \gamma_i$, are unknown, and $\sigma^+(a), \sigma^-(a), \; a \in A^R$, are auxiliary variables:

$$
\begin{array}{ll}
Min \rightarrow & F = \sum_{a \in A^R}^{m} \left(\sigma^+(a) + \sigma^-(a)\right) \\
\text{s.t.} & \\
& \left.\begin{array}{l}
U(a) + \sigma^+(a) - \sigma^-(a) \geq \\
\quad U(b) + \sigma^+(b) - \sigma^-(b) + \varepsilon \;\Leftrightarrow\; a \succ b \\
U(a) + \sigma^+(a) - \sigma^-(a) = \\
\quad U(b) + \sigma^+(b) - \sigma^-(b) \;\Leftrightarrow\; a \sim b
\end{array}\right\} \forall a, b \in A^R \\
& u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0,\; i = 1, ..., n,\; j = 1, ..., \gamma_i - 1 \\
& u_i(\alpha_i) = 0,\; i = 1, ..., n \\
& \sum_{i=1}^{n} u_i(\beta_i) = 1 \\
& \sigma^+(a),\; \sigma^-(a) \geq 0,\; \forall a \in A^R
\end{array}
\tag{4}
$$

where ε is an arbitrarily small positive value so that $U(a) + \sigma^+(a) - \sigma^-(a) > U(b) + \sigma^+(b) - \sigma^-(b)$ in case of $a \succ b$.

If the optimal value of the objective function of program (4) is equal to zero ($F^* = 0$), then there exists at least one value function $U(x) = \sum_{i=1}^{n} u_i(x_i)$ satisfying (3), i.e. compatible with the reference preorder on A^R. In other words, this means that the corresponding polyhedron (3) of feasible solutions for $u_i(x_i^j),\ i = 1, ..., n,\ j = 1, ..., \gamma_i$, is not empty.

Let us remark that the transition from the preorder $\succsim$ to the marginal value function exploits the ordinal character of the criterion scale G_i. Note, however, that the scale of the marginal value function is a conjoint interval scale. More precisely, for the considered additive value function, the admissible transformations on the marginal value functions $u_i(x_i)$ have the form $u_i^*(x_i) = k \times u_i(x_i) + h_i$, $h_i \in \mathbb{R},\ i = 1, \ldots, n,\ k > 0$, such that for all $[x_1, ..., x_n], [y_1, ..., y_n] \in \prod_{i=1}^{n} G_i$

$$
\sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) \Leftrightarrow \sum_{i=1}^{n} u_i^*(x_i) \geq \sum_{i=1}^{n} u_i^*(y_i).
$$

An alternative way of representing the same preference model is:

$$
U(x) = \sum_{i \in I} w_i \hat{u}_i(x) \;\text{ where }\; \hat{u}(\alpha_i) = 0,\; \hat{u}(\beta_i) = 1,\; w_i \geq 0\;\; \forall i \in I, \;\text{ and }\; \sum_{i \in I} w_i = 1 \tag{5}
$$

Note that the correspondence between (5) and (1′) is such that $w_i = u_i(\beta_i),\ \forall i \in G$. Due to the cardinal character of the marginal value function scale, the parameters w_i can be interpreted as tradeoff weights among marginal value functions $\hat{u}_i(x)$. We will use, however, the preference model (1′) with normalization constraints bounding $U(x)$ to the interval $[0, 1]$.

When the optimal value of the objective function of the program (4) is greater than zero ($F^* > 0$), then there is no value function $U(x) = \sum_{i \in I} u_i(x_i)$ compatible with the reference preorder on A^R. In such a case, three possible moves can be considered:

- increasing the number of linear pieces γ_i for one or several marginal value functions u_i could make it possible to find an additive value function compatible with the reference preorder on A^R,
- revising the reference preorder on A^R could lead to find an additive value function compatible with the new preorder,
- searching over the relaxed domain $F \leq F^* + \eta$ could lead to an additive value function giving a preorder on A^R sufficiently close to the reference preorder (in the sense of Kendall's τ).

5 On the UTA$^{\text{GMS}}$ method

The preference information provided by the DM is similar to that of UTA, but the output is quite different (see Greco et al. 2005).

In UTA$^{\text{GMS}}$, the preference information has the form of a partial preorder in a set of reference actions $A^R \subseteq A$ (i.e., a set of pairwise comparisons of reference actions). As a result, one obtains two rankings on set A, such that for any pair of actions $x, y \in A$,

i) in a *necessary ranking* (partial preorder): x is ranked at least as good as y if and only if, $U(x) \geq U(y)$ for all the value functions compatible with the preference information provided by the DM;

ii) in a *possible ranking* (strongly complete and negatively transitive relation): x is ranked at least as good as y if and only if, $U(x) \geq U(y)$ for at least one value function compatible with the preference information.

5.1 Main features

The main features of UTA$^{\text{GMS}}$ are the following.

- Possibility to deal with a general additive value function: a feasible space of value functions is identified and any additive function belonging to that set is called a compatible value function.
- The preference information can be given as a partial preorder on the set of reference actions.
- Consideration of stability of preferences: two preference relations - necessary and possible - are considered to take into account certain or conceivable preferences, respectively.
- Representation of incomparability between actions: the necessary preference is not complete, in general.
- Robust conclusions: the necessary and possible preference relations are based on all compatible value functions, rather than on only one among the many possible functions, as it is usual in MCDA.
- Interaction with the DM: the DM can modify the preference information verifying its impact on the preference relations in the set of considered actions.

5.2 The ordinal regression *via* linear programming

The preference information is given by the DM in the form of a partial preorder $\succsim$ on the set of reference actions $A^R \subseteq A$.

A value function is called *compatible* if it is able to restore partial preorder $\succsim$ on A^R. Each compatible value function induces, moreover, a complete preorder on the whole set A.

In particular, for any two actions $x, y \in A$, a compatible value function orders x and y in one of the following ways: $x \succ y$, $y \succ x$, $x \sim y$. With respect to $x, y \in A$, it is thus reasonable to ask the following two questions:

- Are x and y ordered in the same way by *all* compatible value functions?
- Is there *at least one* compatible value function ordering x at least as good as y (or y at least as good as x)?

Having answers to these questions for all pairs of actions $(x, y) \in A \times A$, one gets a *necessary weak preference relation* $\succsim^N$, whose semantics is $U(x) \geq U(y)$ for all compatible value functions, and a *possible weak preference relation* $\succsim^P$ in A, whose semantics is $U(x) \geq U(y)$ for at least one compatible value function.

Let us remark that preference relations $\succsim^N$ and $\succsim^P$ are meaningful only if there exists at least one compatible value function. Observe also that in this case, for any $a, b \in A^R$,

$$a \succsim b \Rightarrow a \succsim^N b$$

and

$$a \succ b \Rightarrow \text{ not } (b \succsim^P a).$$

In fact, if $a \succsim b$, then for any compatible value function, $U(a) \geq U(b)$ and, therefore, $a \succsim^N b$. Moreover, if $a \succ b$, then for any compatible value function, $U(a) > U(b)$ and, consequently, there is no compatible value function such that $U(b) \geq U(a)$, which means not$(b \succsim^P a)$.

Formally, a general additive compatible value function is an additive value function $U(x) = \sum_{i=1}^n u_i(x_i)$ satisfying the following set of constraints:

$$\left.\begin{array}{l} \left.\begin{array}{l} U(a) > U(b) \;\Leftrightarrow\; a \succ b \\ U(a) = U(b) \;\Leftrightarrow\; a \sim b \end{array}\right\} \forall a, b \in A^R \\ u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0,\; i = 1, ..., n,\; j = 2, ..., m \\ u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i),\; i = 1, ..., n \\ u_i(\alpha_i) = 0,\; i = 1, ..., n \\ \sum_{i=1}^n u_i(\beta_i) = 1 \end{array}\right\} (E^{A^R})$$

where τ_i is the permutation on the set of indices of actions from A^R that reorders them according to the increasing evaluation on criterion g_i, i.e.

$$g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \ldots \leq g_i(a_{\tau_i(m-1)}) \leq g_i(a_{\tau_i(m)})$$

Remark that, due to this formulation of the ordinal regression problem, no linear interpolation is required to express the marginal value of any reference action. Thus, one cannot expect that increasing the number of characteristic points will bring some "new" compatible additive value functions. In consequence, UTA$^{\text{GMS}}$ considers all compatible additive value functions while classical UTA ordinal regression (4) deals with a subset of the whole set of compatible additive value functions, more precisely the piecewise linear additive value functions relative to the considered characteristic points.

5.3 Computation of the relations $\succsim^N$ and $\succsim^P$

In order to compute binary relations $\succsim^P$ and $\succsim^N$, UTA$^{\text{GMS}}$ proceeds as follows. For all actions $x, y \in A$, let π_i be a permutation of the indices of actions from set $A^R \cup \{x, y\}$ that reorders them according to increasing evaluation on criterion g_i, i.e.

$$g_i(a_{\pi_i(1)}) \leq g_i(a_{\pi_i(2)}) \leq ... \leq g_i(a_{\pi_i(\omega-1)}) \leq g_i(a_{\pi_i(\omega)})$$

where,

- if $A^R \cap \{x, y\} = \emptyset$, then $\omega = m + 2$

- if $A^R \cap \{x,y\} = \{x\}$ or $A^R \cap \{x,y\} = \{y\}$, then $\omega = m+1$
- if $A^R \cap \{x,y\} = \{x,y\}$, then $\omega = m$.

Then, we can fix the characteristic points of $u_i(g_i)$, $i = 1, ..., n$, in

$$g_i^0 = \alpha_i, \quad g_i^j = g_i(a_{\pi_i(j)}) \;\text{ for }\; j = 1, ..., \omega, \quad g_i^{\omega+1} = \beta_i$$

Let us consider the following set $E(x,y)$ of ordinal regression constraints, with $u_i(g_i^j)$, $i = 1, ..., n$, $j = 1, ..., \omega+1$, as variables:

$$\left.\begin{array}{l} \left.\begin{array}{l} U(a) \geq U(b) + \varepsilon \;\Leftrightarrow\; a \succ b \\ U(a) = U(b) \;\Leftrightarrow\; a \sim b \end{array}\right\} \forall a, b \in A^R \\ u_i(g_i^j) - u_i(g_i^{j-1}) \geq 0, \; i = 1, ..., n, \; j = 1, ..., \omega+1 \\ u_i(g_i^0) = 0, \; i = 1, ..., n \\ \sum_{i=1}^n u_i(g_i^{\omega+1}) = 1 \end{array}\right\} (E(x,y))$$

where, ε is an arbitrarily small positive value, as in (4).

The above set of constraints depends on the pair of actions $x, y \in A$ because their evaluations $g_i(x)$ and $g_i(y)$ give coordinates for two of $(\omega+1)$ characteristic points of marginal value function $u_i(x_i)$, for each $i = 1, \ldots, n$. Note that for all $x, y \in A$, $E(x,y) = E(y,x)$.

Let us suppose that the polyhedron defined by the set of constraints $E(x,y)$ is not empty. In this case we have that:

$$x \succsim^N y \;\Leftrightarrow\; d(x,y) \geq 0$$

where

$$\begin{array}{rcl} d(x,y) & = & \min\{U(x) - U(y)\} \\ & & \text{s.t. set } E(x,y) \text{ of constraints} \end{array} \tag{6}$$

and

$$x \succsim^P y \;\Leftrightarrow\; D(x,y) \geq 0$$

where

$$\begin{array}{rcl} D(x,y) & = & \max\{U(x) - U(y)\} \\ & & \text{s.t. set } E(x,y) \text{ of constraints} \end{array} \tag{7}$$

6 GRIP methodology

In this section we present a comprehensive description of the proposed GRIP methodology, including its main features, the preference information provided by the DM, the constraints and the fundamental properties, the definition of the linear programming problem, and a theoretical comparison with AHP and MACBETH methodologies.

6.1 Main features of GRIP

GRIP generalizes both UTA and UTA$^{\text{GMS}}$ methods by adopting all features of UTA$^{\text{GMS}}$ and taking into account additional preference information in form of comparisons of intensities of preference between some pairs of reference actions. For actions $x, y, w, z \in A$, these comparisons are expressed in two possible ways (not exclusive):

1) Comprehensively, on all criteria, like "x is preferred to y at least as much as w is preferred to z".

2) Partially, on each criterion, like "x is preferred to y at least as much as w is preferred to z, on criterion $g_i \in F$ ".

6.2 The preference information provided by the decision maker

DM is expected to provide the following preference information,

- A partial preorder $\succsim$ on A^R whose meaning is: for $x, y \in A^R$

$$x \succsim y \Leftrightarrow x \text{ is at least as good as } y.$$

  Moreover, $\succ$ (preference) is the asymmetric part of $\succsim$ and $\sim$ (indifference) is the symmetric part given by $\succsim \cap \succsim^{-1}$ ($\succsim^{-1}$ is the inverse of $\succsim$, i.e. for all $x, y \in A^R$, $x \succsim^{-1} y \Leftrightarrow y \succsim x$).

- A partial preorder $\succsim^*$ on $A^R \times A^R$, whose meaning is: for $x, y, w, z \in A^R$,

$$(x, y) \succsim^* (w, z) \Leftrightarrow \ x \text{ is preferred to } y \text{ at least as much as } w \text{ is preferred to } z.$$

  Also in this case, $\succ^*$ is the asymmetric part of $\succsim^*$ and $\sim^*$ is the symmetric part given by $\succsim^* \cap \succsim^{*^{-1}}$ ($\succsim^{*^{-1}}$ is the inverse of $\succsim^*$, i.e. for all $x, y, w, z \in A^R$, $(x, y) \succsim^{*^{-1}} (w, z) \Leftrightarrow (w, z) \succsim^* (x, y)$).

- A partial preorder $\succsim_i^*$ on $A^R \times A^R$, whose meaning is: for $x, y, w, z \in A^R$, $(x, y) \succsim_i^* (w, z) \Leftrightarrow x$ is preferred to y at least as much as w is preferred to z on criterion g_i, $i \in I$.

The intensities of preferences can be handled by using a MACBETH-like procedure.

In the following, we also consider the weak preference relation $\succsim_i$ being a complete preorder whose meaning is: for all $x, y \in A$,

$$x \succsim_i y \quad \Leftrightarrow \quad x \text{ is at least as good as } y \text{ on criterion } g_i, \ \ i \in I.$$

Weak preference relations $\succsim_i$, $i \in I$, is not provided by the DM, but it is obtained directly from the evaluation of actions x and y on criterion g_i, i.e., $x \succsim_i y \Leftrightarrow g_i(x) \geq g_i(y)$.

6.3 Constraints and properties

In this subsection, we present a set of constraints that interprets the preference information in terms of conditions on the compatible value functions. After that, we give fundamental properties of six ordering relations resulting from the set of compatible value functions. These relations are core relations of the method proposed in this paper.

6.3.1 Constraints

The value function $U : A \rightarrow [0,1]$ should satisfy the following constraints corresponding to DM's preference information,

a) $U(w) > U(z)$ if $w \succ z$

b) $U(w) = U(z)$ if $w \sim z$

c) $U(w) - U(z) > U(r) - U(s)$ if $(w,z) \succ^* (r,s)$

d) $U(w) - U(z) = U(r) - U(s)$ if $(w,z) \sim^* (r,s)$

e) $u_i(w) \geq u_i(z)$ if $w \succsim_i z$, $i \in I$

f) $u_i(w) - u_i(z) > u_i(r) - u_i(s)$ if $(w,z) \succ_i^* (r,s)$, $i \in I$

g) $u_i(w) - u_i(z) = u_i(r) - u_i(s)$ if $(w,z) \sim_i^* (r,s)$, $i \in I$

Let us remark that within UTA-like methods, constraint *a)* is written as $U(w) \geq U(z) + \varepsilon$, where $\varepsilon > 0$ is a threshold exogenously introduced. Analogously, constraints *c)* and *f)* should be written as,

$$U(w) - U(z) \geq U(r) - U(s) + \varepsilon$$

and

$$u_i(w) - u_i(z) \geq u_i(r) - u_i(s) + \varepsilon.$$

However, we would like to avoid the use of any exogenous parameter and, therefore, instead of setting an arbitrary value of ε, we consider it as an auxiliary variable, and we test the feasibility of constraints *a)*, *c)*, and *f)* (see 5.4). In this way, we take into account all possible value functions, even those having a very small preference threshold ε. This way is also safer from the viewpoint of "objectivity" of the whole methodology. In fact, the value of ε is not meaningful in itself and it is useful only because it permits to discriminate preference from indifference.

Moreover, the following normalization constraints should also be taken into account:

h) $u_i(x_i^*) = 0$, where x_i^* is such that $x_i^* = \min\{g_i(x) \ : \ x \in A\}$

i) $\sum_{i \in I} u_i(y_i^*) = 1$, where y_i^* is such that $y_i^* = \max\{g_i(x) \ : \ x \in A\}$

If the constraints from *a)* to *i)* are fulfilled, then the partial preorders $\succsim$ and $\succsim^*$ on A^R can be extended on A in two different ways as follows:

1) Through the choice of one value function U considered the "best" among all the compatible value functions U satisfying constraints from *a)* to *i)* and setting, for all $x, y, w, z \in X$,

   - $x \succsim y \Leftrightarrow U(x) \geq U(y)$, and
   - $(x,y) \succsim^* (w,z) \Leftrightarrow U(x) - U(y) \geq U(w) - U(z)$

   (for a survey about different methodologies to choose the "best" value function consistent with preference information, see Jacquet-Lagrèze and Siskos 1982; Siskos et al. 2005).

2) Through the identification of two weak preference relations $\succsim^N$ and $\succsim^P$ and two binary relations comparing intensity of preference $\succsim^{*N}$ and $\succsim^{*P}$ as follows (see Greco et al. 2003, 2005), called necessary (N) and possible (P), respectively:

a) For each, $x, y \in A$, $x \succsim^N y$ ("x is necessarily at least as good as y") means that $U(x) \geq U(y)$ for any compatible value function U. The following holds,

$$x \succsim^N y \Leftrightarrow \inf\{U(x) - U(y)\} \geq 0,$$

where, the infimum is calculated on the set of value functions satisfying constraints from a) to i).

b) For each, $x, y \in A$, $x \succsim^P y$ ("x is possibly at least as good as y") means that $U(x) \geq U(y)$ for at least one compatible value function U. The following holds,

$$x \succsim^P y \Leftrightarrow \inf\{U(y) - U(x)\} \leq 0,$$

where, the infimum is calculated on the set of value functions satisfying constraints from a) to i).

Remark: Observe that $\inf\{U(y) - U(x)\} \leq 0$ means that it is false that for all U, $U(y) - U(x) > 0$, i.e. there exists at least one U for which $U(y) - U(x) \leq 0$ or, equivalently, $U(x) - U(y) \geq 0$. Observe also that, writing $\sup\{U(x) - U(y)\} \geq 0$ is not equivalent to write $\inf\{U(y) - U(x)\} \leq 0$. In fact, we can have $\sup\{U(x) - U(y)\} \geq 0$ also in case where for all U, $U(x) - U(y) < 0$. For example, if the value of $U(x) - U(y) \in [a, 0[$, where a is any negative number, we have that $U(x) - U(y) < 0$ always and, nevertheless, $\sup\{U(x) - U(y)\} \geq 0$.

c) For each, $x, y, w, z \in A$, $(x, y) \succsim^N (w, z)$ ("x is preferred to y necessarily at least as much as w is preferred to z") means that $U(x) - U(y) \geq U(w) - U(z)$ for any compatible value function U. The following holds,

$$(x, y) \succsim^{*N} (w, z) \Leftrightarrow \inf\left\{\Big(U(x) - U(y)\Big) - \Big(U(w) - U(z)\Big)\right\} \geq 0,$$

where, the infimum is calculated on the set of value functions satisfying constraints from a) to i).

d) For each, $x, y, w, z \in A$, $(x, y) \succsim^P (w, z)$ ("x is preferred to y possibly at least as much as w is preferred to z") means that $U(x) - U(y) \geq U(w) - U(z)$ for at least one compatible value function U. The following holds,

$$(x, y) \succsim^{*P} (w, z) \Leftrightarrow \inf\left\{\Big(U(w) - U(z)\Big) - \Big(U(x) - U(y)\Big)\right\} \leq 0,$$

where, the infimum is calculated on the set of value functions satisfying constraints from a) to i).

e) For each, $x, y, w, z \in A$, $(x, y) \succsim_i^{*N} (w, z)$ ("with respect to $i \in I$, x is preferred to y necessarily at least as much as w is preferred to z") means that $u_i(x) - u_i(y) \geq u_i(w) - u_i(z)$ for all compatible value functions. The following holds,

$$(x, y) \succsim_i^{*N} (w, z) \Leftrightarrow \inf\left\{\Big(u_i(x) - u_i(y)\Big) - \Big(u_i(w) - u_i(z)\Big)\right\} \geq 0,$$

where, the infimum is calculated on the set of value functions satisfying constraints from a) to i).

f) For each, $x, y, w, z \in A$, $(x, y) \succsim_i^{*P} (w, z)$ ("with respect to $i \in I$, x is preferred to y possibly at least as much as w is preferred to z") means that $u_i(x) - u_i(y) \geq u_i(w) - u_i(z)$ for at last one compatible value function. The following holds,

$$(x, y) \succsim_i^{*P} (w, z) \Leftrightarrow \inf\left\{\Big(u_i(w) - u_i(z)\Big) - \Big(u_i(x) - u_i(y)\Big)\right\} \leq 0,$$

where, the infimum is calculated on the set of value functions satisfying constraints from *a)* to *i)*.

6.3.2 Fundamental properties of necessary and possible binary relations

The following theorem presents some basic properties concerning binary relations $\succsim^N$, $\succsim^P$, $\succsim^{*N}$, and $\succsim^{*P}$.

Theorem 6.1. *If constraints a) to i) are satisfied, then the following properties hold:*

1) *for all $x, y \in A$, $x \succsim^N y \Rightarrow x \succsim^P y$, i.e. if the weak preference of x over y is necessary, then it is possible too (Greco et al. 2003, 2005);*

2) *for all $x, y \in A^R$, $x \succsim y \Rightarrow x \succsim^N y$, i.e. if the weak preference of x over y is specified by the DM, then it is necessary too (it should be noticed that for* 1. *it must be possible too) (Greco et al. 2003, 2005);*

3) *$\succsim^N$ is a partial preorder (i.e. the relation is transitive and reflexive) and $\succsim^P$ is strongly complete (i.e. for each $x, y \in A$ at least one of the following two relations is true, $x \succsim^P y$ or $y \succsim^P x$) and negatively transitive (i.e. for each $x, y, z \in A$, if not $x \succsim^P y$ and not $y \succsim^P z$, then not $x \succsim^P z$) (Greco et al. 2003, 2005);*

4) *for all $x, y \in A$, $x \succsim^N y$ or $y \succsim^P x$ (Greco et al. 2003, 2005);*

5) *for all $x, y, z \in A$, $[x \succsim^N y$ and $y \succsim^P z] \Rightarrow x \succsim^P z$;*

6) *for all $x, y, z \in A$, $[x \succsim^P y$ and $y \succsim^N z] \Rightarrow x \succsim^P z$;*

7) *for all $x, y, w, z \in A$, $(x, y) \succsim^{*N} (w, z) \Rightarrow (x, y) \succsim^{*P} (w, z)$;*

8) *for all $x, y, w, z \in A^R$, $(x, y) \succsim^{*} (w, z) \Rightarrow (x, y) \succsim^{*N} (w, z)$;*

9) *$\succsim^{*N}$ is a partial preorder and $\succsim^{*P}$ is strongly complete and negatively transitive;*

10) *for all $x, y \in A$, $(x, y) \succsim^N (w, z)$ or $(w, z) \succsim^P (x, y)$;*

11) *for all $x, y, z, w, r, s \in A$, $[(x, y) \succsim^{*N} (w, z)$ and $(w, z) \succsim^{*P} (r, s)] \Rightarrow (x, y) \succsim^{*P} (r, s)$;*

12) *for all $x, y, z, w, r, s \in A$, $[(x, y) \succsim^{*P} (w, z)$ and $(w, z) \succsim^{*N} (r, s)] \Rightarrow (x, y) \succsim^{*P} (r, s)$;*

13) *for all $x, x', y, z, w \in A$, $[x' \succsim^N x$ and $(x, y) \succsim^{*N} (w, z)] \Rightarrow (x', y) \succsim^{*N} (w, z)$;*

14) *for all $x, x', y, z, w \in A$, $[x' \succsim^N x$ and $(x, y) \succsim^{*P} (w, z)] \Rightarrow (x', y) \succsim^{*P} (w, z)$;*

15) *for all $x, x', y, z, w \in A$, $[x' \succsim^P x$ and $(x, y) \succsim^{*N} (w, z)] \Rightarrow (x', y) \succsim^{*P} (w, z)$;*

16) *for all $x, y, y', z, w \in A$, $[y \succsim^N y'$ and $(x, y) \succsim^{*N} (w, z)] \Rightarrow (x, y') \succsim^{*N} (w, z)$;*

17) *for all $x, y, y', z, w \in A$, $[y \succsim^N y'$ and $(x, y) \succsim^{*P} (w, z)] \Rightarrow (x, y') \succsim^{*P} (w, z)$;*

18) *for all* $x, y, y', z, w \in A$, $[y \succsim^P y'$ *and* $(x,y) \succsim^{*^N} (w,z)] \Rightarrow (x,y') \succsim^{*^P} (w,z)$;

19) *for all* $x, y, z, w, w' \in A$, $[w \succsim^N w'$ *and* $(x,y) \succsim^{*^N} (w,z)] \Rightarrow (x,y) \succsim^{*^N} (w',z)$;

20) *for all* $x, y, z, w, w' \in A$, $[w \succsim^N w'$ *and* $(x,y) \succsim^{*^P} (w,z)] \Rightarrow (x,y) \succsim^{*^P} (w',z)$;

21) *for all* $x, y, z, w, w' \in A$, $[w \succsim^P w'$ *and* $(x,y) \succsim^{*^N} (w,z)] \Rightarrow (x,y) \succsim^{*^P} (w',z)$;

22) *for all* $x, y, z, z', w \in A$, $[z' \succsim^N z$ *and* $(x,y) \succsim^{*^N} (w,z)] \Rightarrow (x,y) \succsim^{*^N} (w,z')$;

23) *for all* $x, y, z, z', w \in A$, $[z' \succsim^N z$ *and* $(x,y) \succsim^{*^P} (w,z)] \Rightarrow (x,y) \succsim^{*^P} (w,z')$;

24) *for all* $x, y, z, z', w \in A$, $[z' \succsim^P z$ *and* $(x,y) \succsim^{*^N} (w,z)] \Rightarrow (x,y) \succsim^{*^P} (w,z')$;

25) *for all* $x, x', y \in A$, $(x',y) \succsim^{*^N} (x,y) \Leftrightarrow x' \succsim^N x$;

26) *for all* $x, x', y \in A$, $(x',y) \succsim^{*^P} (x,y) \Leftrightarrow x' \succsim^P x$;

27) *for all* $x, y, y' \in A$, $(x,y) \succsim^{*^N} (x,y') \Leftrightarrow y' \succsim^N y$;

28) *for all* $x, y, y' \in A$, $(x,y) \succsim^{*^P} (x,y') \Leftrightarrow y' \succsim^P y$;

29) $\succsim_i^{*^N}$ *is a partial preorder and* $\succsim_i^{*^P}$ *is strongly complete and negatively transitive, for all* $i \in I$;

30) *for all* $x, y, w, z \in A$, $(x,y) \succsim_i^N (w,z)$ *or* $(w,z) \succsim_i^P (x,y)$, *for all* $i \in I$;

31) *for all* $x, y, w, z, r, s \in A$, $[(x,y) \succsim_i^{*^N} (w,z)$ *and* $(w,z) \succsim_i^{*^P} (r,s)] \Rightarrow (x,y) \succsim_i^{*^P} (r,s)$, *for all* $i \in I$;

32) *for all* $x, y, w, z, r, s \in A$, $[(x,y) \succsim_i^{*^P} (w,z)$ *and* $(w,z) \succsim_i^{*^N} (r,s)] \Rightarrow (x,y) \succsim_i^{*^P} (r,s)$, *for all* $i \in I$;

33) *for all* $x, x', y, w, z \in A$, $[g_i(x') \geq g_i(x)$ *and* $(x,y) \succsim_i^{*^N} (w,z)] \Rightarrow (x',y) \succsim_i^{*^N} (w,z)$, *for all* $i \in I$;

34) *for all* $x, x', y, w, z \in A$, $[g_i(x') \geq g_i(x)$ *and* $(x,y) \succsim_i^{*^P} (w,z)] \Rightarrow (x',y) \succsim_i^{*^P} (w,z)$, *for all* $i \in I$;

35) *for all* $x, y, y', w, z \in A$, $[g_i(y) \geq g_i(y')$ *and* $(x,y) \succsim_i^{*^N} (w,z)] \Rightarrow (x,y') \succsim_i^{*^N} (w,z)$, *for all* $i \in I$;

36) *for all* $x, y, y', w, z \in A$, $[g_i(y) \geq g_i(y')$ *and* $(x,y) \succsim_i^{*^P} (w,z)] \Rightarrow (x,y') \succsim_i^{*^P} (w,z)$, *for all* $i \in I$;

37) *for all* $x, y, w, w', z \in A$, $[g_i(w) \geq g_i(w')$ *and* $(x,y) \succsim_i^{*^N} (w,z)] \Rightarrow (x,y) \succsim_i^{*^N} (w',z)$, *for all* $i \in I$;

38) *for all* $x, y, w, w', z \in A$, $[g_i(w) \geq g_i(w')$ *and* $(x,y) \succsim_i^{*^P} (w,z)] \Rightarrow (x,y) \succsim_i^{*^P} (w',z)$, *for all* $i \in I$;

39) *for all* $x, y, w, z, z' \in A$, $[g_i(z') \geq g_i(z)$ *and* $(x,y) \succsim_i^{*^N} (w,z)] \Rightarrow (x,y) \succsim_i^{*^N} (w,z')$, *for all* $i \in I$;

40) *for all* $x, y, w, z, z' \in A$, $[g_i(z') \geq g_i(z)$ *and* $(x,y) \succsim_i^{*^P} (w,z)] \Rightarrow (x,y) \succsim_i^{*^P} (w,z')$, *for all* $i \in I$;

41) *for all* $x, x', y \in A$, $g_i(x') \geq g_i(x) \Rightarrow (x', y) \succsim_i^{*N} (x, y)$, *for all* $i \in I$;

42) *for all* $x, x', y \in A$, $(x', y) \succ_i^{*P} (x, y) \Rightarrow g_i(x') > g_i(x)$, *for all* $i \in I$;

43) *for all* $x, y, y' \in A$, $g_i(y) \geq g_i(y') \Rightarrow (x, y) \succsim_i^{*N} (x, y')$, *for all* $i \in I$;

44) *for all* $x, y, y' \in A$, $(x, y) \succ_i^{*P} (x, y') \Rightarrow g_i(y) > g_i(y')$, *for all* $i \in I$.

□

The proof of this theorem is provided in the Appendix. Now, let us explain the contents of the theorem with respect to their relevance from the point of view of MCDA.

1) corresponds to the general idea that if something is necessary, then it must be also possible: within MCDA this implies that if a weak preference is necessary, then it is possible too (Greco et al. 2003, 2005);

2) since we are considering a set of value functions compatible with the preferences expressed by the DM, then if the weak preference of x over y is specified by the DM, then each compatible value function must represent the weak preference of x over y, which is thus necessary (and, for 1), possible too) (Greco et al. 2003, 2005);

3) expresses the preference structure of $\succsim^N$ and $\succsim^P$, that is $\succsim^N$ is a partial preorder (i.e. the relation is transitive and reflexive), and $\succsim^P$ is strongly complete and negatively transitive (Greco et al. 2003, 2005);

4) expresses a specific completeness condition for $\succsim^N$ and $\succsim^P$, that is for all $x, y \in A$, $x \succsim^N y$ or $y \succsim^P x$ (Greco et al. 2003, 2005);

5) expresses a specific transitivity condition for $\succsim^N$ and $\succsim^P$, that is for all $x, y, z \in A$, $[x \succsim^N y$ and $y \succsim^P z] \Rightarrow x \succsim^P z$;

6) expresses a specific transitivity condition for $\succsim^N$ and $\succsim^P$, that is for all $x, y, z \in A$, $[x \succsim^P y$ and $y \succsim^N z] \Rightarrow x \succsim^P z$;

7) analogously to 1), it expresses the general idea that if something is necessary, then it must be possible too; differently from 1), it predicates this principle for $\succsim^{*N}$ and $\succsim^{*P}$, that is, if it is necessary that the intensity of preference of x over y is not smaller than the intensity of preference of w over z, then it is possible too;

8) analogously to 2), since we are considering a set of value functions compatible with the preferences expressed by the DM, then if the fact that the intensity of preference of x over y is not smaller than the intensity of preference of w over z is specified by the DM, then each compatible value function must represent this fact, which is thus necessary (and, for 7), possible too);

9) analogously to 3), it expresses the preference structure of $\succsim^{*N}$ and $\succsim^{*P}$, that is $\succsim^{*N}$ is a partial preorder and $\succsim^{*P}$ is strongly complete and negatively transitive;

10) analogously to 4), it expresses a specific completeness condition; differently from 4), it predicates this condition for $\succsim^{*N}$ and $\succsim^{*N}$, that is for all $x, y, w, z \in A$, $(x, y) \succsim^N (w, z)$ or $(w, z) \succsim^P (x, y)$;

11) analogously to 5), it expresses a specific transitivity condition; differently from 5) it expresses this condition for $\succsim^{*^N}$ and $\succsim^{*^P}$, that is for all $x, y, z, w, r, s \in A$, $[(x, y) \succsim^{*^N} (w, z)$ and $(w, z) \succsim^{*^P} (r, s)] \Rightarrow (x, y) \succsim^{*^P} (r, s)$;

12) analogously to 6), it expresses a specific transitivity condition; differently from 6) it expresses this condition for $\succsim^{*^N}$ and $\succsim^{*^P}$, that is for all $x, y, z, w, r, s \in A$, $[(x, y) \succsim^{*^P} (w, z)$ and $(w, z) \succsim^{*^N} (r, s)] \Rightarrow (x, y) \succsim^{*^P} (r, s)$;

13) states a property relating $\succsim^N$ and $\succsim^{*^N}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^N} (w, z)$) and we replace x by x', with x' necessarily at least as good as x (i.e. $x' \succsim^N x$), then the intensity of preference of x' over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x', y) \succsim^{*^N} (w, z)$);

14) states a property relating $\succsim^N$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is possibly not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^P} (w, z)$) and we replace x by x', with x' necessarily at least as good as x (i.e. $x' \succsim^N x$), then the intensity of preference of x' over y is possibly not smaller than the intensity of preference of w over z (i.e. $(x', y) \succsim^{*^P} (w, z)$);

15) states a property relating $\succsim^P$, $\succsim^{*^N}$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^N} (w, z)$) and we replace x by x', with x' possibly at least as good as x (i.e. $x' \succsim^N x$), then the intensity of preference of x' over y is possibly not smaller than the intensity of preference of w over z (i.e. $(x', y) \succsim^{*^P} (w, z)$);

16) analogously to 13), it states a property relating $\succsim^N$ and $\succsim^{*^N}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^N} (w, z)$) and, differently from 13), we replace y by y', with y necessarily at least as good as y' (i.e. $y \succsim^N y'$), then the intensity of preference of x over y' is necessarily not smaller than the intensity of preference of w over z (i.e. $(x, y') \succsim^{*^N} (w, z)$);

17) analogously to 14), it states a property relating $\succsim^N$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is possibly not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^P} (w, z)$) and, differently from 14), we replace y by y', with y necessarily at least as good as y' (i.e. $y \succsim^N y'$), then the intensity of preference of x over y' is possibly not smaller than the intensity of preference of w over z (i.e. $(x, y') \succsim^{*^P} (w, z)$);

18) analogously to 15), it states a property relating $\succsim^P$, $\succsim^{*^N}$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^N} (w, z)$) and, differently from 15), we replace y by y', with y possibly at least as good as y' (i.e. $y \succsim^N y'$), then the intensity of preference of x over y' is possibly not smaller than the intensity of preference of w over z (i.e. $(x, y') \succsim^{*^P} (w, z)$);

19) analogously to 13) and 16), it states a property relating $\succsim^N$ and $\succsim^{*^N}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x, y) \succsim^{*^N} (w, z)$) and, differently from 13) and 16), we replace w by w', with w necessarily at least as good as w' (i.e. $w \succsim^N w'$), then the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w' over z (i.e. $(x, y) \succsim^{*^N} (w', z)$);

20) analogously to 14) and 17), it states a property relating $\succsim^N$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is possibly not smaller than the intensity of preference of w

over z (i.e. $(x,y) \succsim^{*^P} (w,z)$) and, differently from 14) and 17), we replace w by w', with w necessarily at least as good as w' (i.e. $w \succsim^N w'$), then the intensity of preference of x over y is possibly not smaller than the intensity of preference of w' over z (i.e. $(x,y) \succsim^{*^P} (w',z)$);

21) analogously to 15) and 18), it states a property relating $\succsim^P$, $\succsim^{*^N}$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x,y) \succsim^{*^N} (w,z)$) and, differently from 15) and 18), we replace w by w', with w possibly at least as good as w' (i.e. $w \succsim^P w'$), then the intensity of preference of x over y is possibly not smaller than the intensity of preference of w' over z (i.e. $(x,y) \succsim^{*^P} (w',z)$);

22) analogously to 13), 16) and 19), it states a property relating $\succsim^N$ and $\succsim^{*^N}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x,y) \succsim^{*^N} (w,z)$) and, differently from 13), 16) and 19), we replace z by z', with z' necessarily at least as good as z (i.e. $z' \succsim^N z$), then the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z' (i.e. $(x,y) \succsim^{*^N} (w,z')$);

23) analogously to 14), 17) and 20), it states a property relating $\succsim^N$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is possibly not smaller than the intensity of preference of w over z (i.e. $(x,y) \succsim^{*^P} (w,z)$) and, differently from 14), 17) and 20), we replace z by z', with z' necessarily at least as good as z (i.e. $z' \succsim^N z$), then the intensity of preference of x over y is possibly not smaller than the intensity of preference of w over z' (i.e. $(x,y) \succsim^{*^P} (w,z')$);

24) analogously to 15), 18) and 21), it states a property relating $\succsim^P$, $\succsim^{*^N}$ and $\succsim^{*^P}$; more precisely, if the intensity of preference of x over y is necessarily not smaller than the intensity of preference of w over z (i.e. $(x,y) \succsim^{*^N} (w,z)$) and, differently from 15), 18) and 21), we replace z by z', with z' possibly at least as good as z (i.e. $z' \succsim^P z$), then the intensity of preference of x over y is possibly not smaller than the intensity of preference of w over z' (i.e. $(x,y) \succsim^{*^P} (w,z')$);

25) relates $\succsim^{*^N}$ and $\succsim^N$; if we replace x by x', the intensity of preference of x over x' is necessarily not smaller than the intensity of preference of x over y (i.e. $(x',y) \succsim^{*^N} (x,y)$), if and only if x' is necessarily at least as good as x (i.e. $x' \succsim^N x$);

26) relates $\succsim^{*^P}$ and $\succsim^P$ in a way analogous to the one in which 25) relates $\succsim^{*^N}$ and $\succsim^N$; if we replace x by x', the intensity of preference of x over x' is possibily not smaller than the intensity of preference of x over y (i.e. $(x',y) \succsim^{*^P} (x,y)$), if and only if x' is possibly at least as good as x (i.e. $x' \succsim^P x$);

27) similarly to 25), it relates $\succsim^{*^N}$ and $\succsim^N$; if, differently form 25), we replace y by y', the intensity of preference of x over y is necessarily not smaller than the intensity of preference of x over y' (i.e. $(x,y) \succsim^{*^N} (x,y')$), if and only if y' is necessarily at least as good as y (i.e. $y' \succsim^N y$);

28) relates $\succsim^{*^P}$ and $\succsim^P$ in a way analogous to the one in which 27) relates $\succsim^{*^N}$ and $\succsim^N$; if we replace y by y', the intensity of preference of x over y is possibly not smaller than the intensity of preference of x over y' (i.e. $(x,y) \succsim^{*^P} (x,y')$), if and only if y' is possibly at least as good as y (i.e. $y' \succsim^P y$);

29) expresses the preference structures of $\succsim_i^{*^N}$ and $\succsim_i^{*^P}$, $i \in I$; more precisely, $\succsim_i^{*^N}$ is a partial preorder and $\succsim_i^{*^P}$ is strongly complete and negatively transitive;

30) expresses a specific completeness condition for $\succsim_i^{*^N}$ and $\succsim_i^{*^P}$, that is for all $x,y,w,z \in A$, $(x,y) \succsim_i^{*^N} (w,z)$ or $(w,z) \succsim_i^{*^P} (x,y)$, for all $i \in I$;

31) expresses a specific transitivity condition for $\succsim_i^{*^N}$ and $\succsim_i^{*^P}$, for all $i \in I$, that is, for all $x, y, w, z, r, s \in A$, $\quad [(x, y) \succsim_i^{*^N} (w, z)$ and $(w, z) \succsim_i^{*^P} (r, s)] \Rightarrow (x, y) \succsim_i^{*^P} (r, s)$;

32) expresses a specific transitivity condition for $\succsim_i^{*^N}$ and $\succsim_i^{*^P}$, for all $i \in I$, that is, for all $x, y, w, z, r, s \in A$, $\quad [(x, y) \succsim_i^{*^P} (w, z)$ and $(w, z) \succsim_i^{*^N} (r, s)] \Rightarrow (x, y) \succsim_i^{*^P} (r, s)$;

33) relates the order between evaluations on criterion g_i and $\succsim_i^{*^N}$, for all $i \in I$; more precisely, for all $x, x', y, w, z \in A$, if the intensity of preference of x over y with respect to criterion g_i is necessarily not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace x by x' with $g_i(x') \geq g_i(x)$, then also the intensity of preference of x' over y with respect to criterion g_i is necessarily not smaller than that of w over z (i.e. $(x', y) \succsim_i^{*^N} (w, z)$);

34) relates the order between evaluations on criterion g_i and $\succsim_i^{*^P}$, for all $i \in I$; more precisely, for all $x, x', y, w, z \in A$, if the intensity of preference of x over y with respect to criterion g_i is possibly not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^P} (w, z)$), and we replace x by x' with $g_i(x') \geq g_i(x)$, then also the intensity of preference of x' over y with respect to criterion g_i is possibly not smaller than that of w over z (i.e. $(x', y) \succsim_i^{*^P} (w, z)$);

35) relates the order between evaluations on criterion g_i and $\succsim_i^{*^N}$, for all $i \in I$; more precisely, for all $x, y, y', w, z \in A$, if the intensity of preference of x over y with respect to criterion g_i is necessarily not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace y by y' with $g_i(y) \geq g_i(y')$, then also the intensity of preference of x over y' with respect to criterion g_i is necessarily not smaller than that of w over z (i.e. $(x, y') \succsim_i^{*^N} (w, z)$);

36) relates the order between evaluations on criterion g_i and $\succsim_i^{*^P}$, for all $i \in I$; more precisely, for all $x, y, y', w, z \in A$, if the intensity of preference of x over y with respect to criterion g_i is possibly not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace y by y' with $g_i(y) \geq g_i(y')$, then also the intensity of preference of x over y' with respect to criterion g_i is possibly not smaller than that of w over z (i.e. $(x, y') \succsim_i^{*^P} (w, z)$);

37) relates the order between evaluations on criterion g_i and $\succsim_i^{*^N}$, for all $i \in I$; more precisely, for all $x, y, y, w', z \in A$, if the intensity of preference of x over y with respect to criterion g_i is necessarily not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace w by w' with $g_i(w) \geq g_i(w')$, then also the intensity of preference of x over y with respect to criterion g_i is necessarily not smaller than that of w' over z (i.e. $(x, y) \succsim_i^{*^N} (w', z)$);

38) relates the order between evaluations on criterion g_i and $\succsim_i^{*^P}$, for all $i \in I$; more precisely, for all $x, y, y, w', z \in A$, if the intensity of preference of x over y with respect to criterion g_i is possibly smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace w by w' with $g_i(w) \geq g_i(w')$, then also the intensity of preference of x over y with respect to criterion g_i is possibly not smaller than that of w' over z (i.e. $(x, y) \succsim_i^{*^P} (w', z)$);

39) relates the order between evaluations on criterion g_i and $\succsim_i^{*^N}$, for all $i \in I$; more precisely, for all $x, y, y, w, z' \in A$, if the intensity of preference of x over y with respect to criterion g_i is necessarily not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace z by z' with $g_i(z') \geq g_i(z)$, then also the intensity of preference of x over y with respect to criterion g_i is necessarily not smaller than that of w over z' (i.e. $(x, y) \succsim_i^{*^N} (w, z')$);

40) relates the order between evaluations on criterion g_i and $\succsim_i^{*^P}$, for all $i \in I$; more precisely, for all $x, y, y, w, z' \in A$, if the intensity of preference of x over y with respect to criterion g_i is possibly not smaller than that of w over z (i.e. $(x, y) \succsim_i^{*^N} (w, z)$), and we replace z by z' with $g_i(z') \geq g_i(z)$, then also the intensity of preference of x over y with respect to criterion g_i is possibly not smaller than that of w over z' (i.e. $(x, y) \succsim_i^{*^P} (w, z')$);

41) states a property relating the order between evaluations on criterion g_i and $\succsim_i^{*N}$; more precisely, for all $x, x', y \in A$, if $g_i(x') \geq g_i(x)$, then, with respect to criterion g_i, the intensity of preference of x' over y is necessarily not smaller than the intensity of preference of x over y (i.e. $(x', y) \succsim_i^{*N} (x, y)$);

42) states a property relating the order between evaluations on criterion g_i and $\succsim_i^{*P}$; more precisely, for all $x, x', y \in A$, if $g_i(x') \geq g_i(x)$, then, with respect to criterion g_i, the intensity of preference of x' over y is possibly not smaller than the intensity of preference of x over y (i.e. $(x', y) \succsim_i^{*P} (x, y)$);

43) states a property relating the order between evaluations on criterion g_i and $\succsim_i^{*N}$; more precisely, for all $x, x, y' \in A$, if $g_i(y) \geq g_i(y')$, then, with respect to criterion g_i, the intensity of preference of x' over y is necessarily not smaller than the intensity of preference of x over y (i.e. $(x, y) \succsim_i^{*N} (x, y')$);

44) states a property relating the order between evaluations on criterion g_i and $\succsim_i^{*P}$; more precisely, for all $x, x, y' \in A$, if $g_i(y) \geq g_i(y')$, then, with respect to criterion g_i, the intensity of preference of x' over y is possibly not smaller than the intensity of preference of x over y (i.e. $(x, y) \succsim_i^{*P} (x, y')$).

6.4 Computational issues

In order to conclude the truth or falsity of binary relations $\succsim^N$, $\succsim^P$, $\succsim^{*^N}$, $\succsim^{*^P}$, $\succsim_i^{*N}$ and $\succsim_i^{*P}$, we have to take into account that, for all $x, y, w, z \in A$ and $i \in I$:

1) $x \succsim^N y \Leftrightarrow \inf\Big\{U(x) - U(y)\Big\} \geq 0,$

2) $x \succsim^P y \Leftrightarrow \inf\Big\{U(y) - U(x)\Big\} \leq 0,$

3) $(x, y) \succsim^{*^N} (w, z) \Leftrightarrow \inf\Big\{\Big(U(x) - U(y)\Big) - \Big(U(w) - U(z)\Big)\Big\} \geq 0,$

4) $(x, y) \succsim^{*^P} (w, z) \Leftrightarrow \inf\Big\{\Big(U(w) - U(z)\Big) - \Big(U(x) - U(y)\Big)\Big\} \leq 0,$

5) $(x, y) \succsim_i^{*N} (w, z) \Leftrightarrow \inf\Big\{\Big(u_i(x_i) - u_i(y_i)\Big) - \Big(u_i(w_i) - u_i(z_i)\Big)\Big\} \geq 0,$

6) $(x, y) \succsim_i^{*P} (w, z) \Leftrightarrow \inf\Big\{\Big(u_i(w_i) - u_i(z_i)\Big) - \Big(u_i(x_i) - u_i(y_i)\Big)\Big\} \leq 0.$

with the infimum calculated on the set of value functions satisfying constraints from $a)$ to $i)$. Let us remark, however, that the linear programming is not able to handle strict inequalities such as the above $a)$, $c)$, and $f)$. Moreover, linear programming permits to calculate the minimum or the maximum of an objective function and not an infimum. Therefore, to use linear programming for testing the truth of binary relations $\succsim^N$, $\succsim^P$, $\succsim^{*^N}$, $\succsim^{*^P}$, $\succsim_i^{*N}$ and $\succsim_i^{*P}$, we need to reformulate properly the above properties from 1) to 6). With this aim, the following result (see Marichal and Roubens 2000) can be taken into account (the adopted notation is used only locally in this proposition and does not inherit the meaning from other parts of the paper).

Proposition 6.1. *$x \in \mathbb{R}$ is a solution of the linear system,*

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \geq b_i, & i = 1, \dots, p \\ \sum_{j=1}^{n} c_{ij}x_j > d_i, & i = 1, \dots, q \end{cases}$$

iff there exists $\varepsilon > 0$, such that

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \geq b_i, & i = 1, \dots, p \\ \sum_{j=1}^{n} c_{ij}x_j \geq d_i + \varepsilon, & i = 1, \dots, q \end{cases}$$

In particular, a solution exists iff the following linear programming:

$$\max \varepsilon \quad \textit{subject to:} \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \geq b_i, & i = 1, \dots, p \\ \sum_{j=1}^{n} c_{ij}x_j \geq d_i + \varepsilon, & i = 1, \dots, q \end{cases}$$

has an optimal solution (x^*, ε^*) with an optimal value $\varepsilon^* > 0$. In this case x^* is a solution of the first system.

In order to use Proposition 5.1, first the constraints $a)$, $c)$ and $f)$ have to be reformulated as follows:

- $a')$ $U(x) \geq U(y) + \varepsilon$ if $x \succ y$;
- $c')$ $U(x) - U(y) \geq U(w) - U(z) + \varepsilon$ if $(x, y) \succ^* (w, z)$;
- $f')$ $u_i(x) - u_i(y) \geq u_i(w) - u_i(z) + \varepsilon$ if $(x, y) \succ_i^* (w, z)$.

with $\varepsilon > 0$.

After properties $1) - 6)$ have to be reformulated such that the search of the infimum is replaced by the calculation of the minimum value of ε on the set of value functions satisfying constraints from $a)$ to $i)$, with constraints $a)$, $c)$, and $f)$ transformed to $a')$, $c')$, and $f')$, plus constraints specific for each point:

1') $x \succsim^N y \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \min\ \varepsilon$, subject to the constraints $a')$, $b)$, $c')$, $d)$, $e)$, $f')$, plus the constraint $U(x) \geq U(y)$;

2') $x \succsim^P y \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \min\ \varepsilon$, subject to the constraints $a')$, $b)$, $c')$, $d)$, $e)$, $f')$, plus the constraint $U(y) \geq U(x) + \varepsilon$;

3') $(x, y) \succsim^{*N} (w, z) \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \min\ \varepsilon$, subject to the constraints $a')$, $b)$, $c')$, $d)$, $e)$, $f')$, plus the constraint $\Big((U(x) - U(y)\Big) - \Big(U(w) - U(z))\Big) \geq 0$;

4′) $(x, y) \succsim^{*P} (w, z) \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \min\ \varepsilon$, subject to the constraints $a')$, $b)$, $c')$, $d)$, $e)$, $f')$, plus the constraint $\Big((U(w) - U(z)\Big) - \Big(U(x) - U(y))\Big) \geq \varepsilon$;

5′) $(x, y) \succsim_i^{*N} (w, z) \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \min\ \varepsilon$, subject to the constraints $a')$, $b)$, $c')$, $d)$, $e)$, $f')$, plus the constraint $\Big(u_i(x_i) - u_i(y_i)\Big) - \Big(u_i(w_i) - u_i(z_i)\Big) \geq 0$;

6′) $(x, y) \succsim_i^{*P} (w, z) \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \min\ \varepsilon$, subject to the constraints $a')$, $b)$, $c')$, $d)$, $e)$, $f')$, plus the constraint $\Big((u_i(w_i) - u_i(z_i)\Big) - \Big(u_i(x_i) - u_i(y_i)\Big) \geq \varepsilon$.

6.5 A theoretical comparison of GRIP with the Analytical Hierarchy Process

In AHP (Saaty 1980 and 2005), criteria should be compared pairwise with respect to their importance. Actions are also compared pairwise on particular criteria with respect to intensity of preference. The following nine point scale is used:

1 - equal importance-preference,

3 - moderate importance-preference,

5 - strong importance-preference,

7 - very strong or demonstrated importance-preference,

9 - extreme importance-preference.

2, 4, 6 and 8 are intermediate values between the two adjacent judgements. The difference of importance of criterion g_i over criterion g_j is the inverse of the difference of importance of g_j over g_i. Analogously, the intensity of preference of action x over action y is the inverse of the intensity of preference of y over x. The above scale is a ratio scale. Therefore, the difference of importance is read as the ratio of weights w_i and w_j corresponding to criteria g_i and g_j, and the intensity of preference is read as the ratio of the attractiveness of x and the attractiveness of y, with respect to the considered criterion g_i. In terms of value functions, the intensity of preference can be interpreted as the ratio $\frac{u_i(g_i(x))}{u_i(g_i(y))}$. Thus, the problem is how to obtain values of w_i and w_j from ratio $\frac{w_i}{w_j}$, and values of $u_i(g_i(x))$ and $u_i(g_i(y))$ from ratio $\frac{u_i(g_i(x))}{u_i(g_i(y))}$.

In AHP it is proposed that these values are supplied by principal eigenvectors of matrices composed of the ratios $\frac{w_i}{w_j}$ and $\frac{u_i(g_i(x))}{u_i(g_i(y))}$. The marginal value functions $u_i(g_i(x))$ are then aggregated by means of a weighted-sum using the weights w_i.

Comparing AHP with GRIP, we can say that with respect to single criteria the type of questions addressed to the DM is the same: express intensity of preference in qualitative-ordinal terms (equal, moderate, strong, very strong, extreme). However, differently from GRIP, this intensity of preference is translated into quantitative terms (the scale from 1 to 9) in a quite arbitrary way. In GRIP, instead, the marginal value functions are just a numerical representation of the original qualitative-ordinal information, and no intermediate transformation in quantitative terms is exogenously imposed.

Other differences between AHP and GRIP are related to the following aspects.

1) In GRIP the value functions $u_i(g_i(x))$ depend mainly on comprehensive preferences involving jointly all the criteria, while this is not the case in AHP.

2) In AHP the weights w_i of criteria g_i are calculated on the basis of pairwise comparisons of criteria with respect to their importance; in GRIP, this is not the case, because the value functions $u_i(g_i(x))$, are expressed on the same scale and thus they can be summed up without any further weighting.

3) In AHP all non-ordered pairs of actions must be compared from the viewpoint of the intensity of preference with respect to each particular criterion. Therefore, if n is the number of actions, and m the number of criteria, then the DM has to answer $m \times \frac{n \times (n-1)}{2}$ questions. Moreover, the DM has to answer questions relative to $\frac{m \times (m-1)}{2}$ pairwise comparisons of considered criteria with respect to their importance. This is not the case in GRIP, which accepts partial information about preferences in terms of pairwise comparison of some reference actions. Finally, in GRIP there is no question about comparison of relative importance of criteria.

As far as point 2) is concerned, observe that the weights w_i used in AHP represent tradeoffs between evaluations on different criteria. For this reason it is doubtful if they could be inferred from answers to questions concerning comparison of importance. Therefore, AHP has a problem with meaningfulness of its output with respect to its input, and this is not the case of GRIP.

6.6 A theoretical comparison with MACBETH

MACBETH (Measuring Attractiveness by a Categorical Based Evaluation TecHnique) is a method for multiple criteria decision analysis that appeared in the early nineties. This approach only requires qualitative judgements from DMs about differences of value to quantify the relative attractiveness of actions or criteria.

When using MACBETH, the DM is asked to provide preference information about two actions of A at a time (see Bana e Costa and Vansnick 1994, and Bana e Costa et al. 2005),

- firstly, through an (ordinal) judgement on their relative attractiveness, and
- secondly, (if the two actions are not considered to be equally attractive), through a qualitative judgement about the difference of attractiveness between the most attractive of the two actions and a third one.

Seven semantic categories of difference of attractiveness are considered in MACBETH,

1) "null",

2) "very weak",

3) "weak",

4) "moderate",

5) "strong",

6) "very strong",

7) "extreme".

The main idea of MACBETH is to build an interval scale from the preference information provided by the DM. It is, however, necessary that the above categories correspond to disjoint intervals (represented in terms of the real numbers). The bounds for such intervals should not be arbitrarily fixed *a priori*, they should be compatible simultaneously with the numerical values of all particular actions from A, so as to ensure compatibility between these values (see Bana e Costa et al. 2005). Linear programming models are used for these calculations. In case of inconsistent judgments, MACBETH provides the DM with information in order to eliminate such inconsistency.

When comparing MACBETH with GRIP, the following aspects should be considered:

- both deal with qualitative judgements;
- both need a set of comparisons of actions or pairs of actions to work out a numerical representation of preferences. However, MACBETH depends on the definition of two characteristic levels on the original scale, "neutral" and "good", to obtain the numerical representation of preferences;
- GRIP adopts the disaggregation-aggregation approach and, therefore, it considers also comprehensive preferences relative to comparisons involving jointly all the criteria, which is not the case of MACBETH;
- GRIP is, however, more general than MACBETH since it can take into account the same kind of qualitative judgments as MACBETH (the difference of attractiveness between pairs of actions) and the intensity of preferences of the type "x is preferred to y at least as much as z is preferred to w".

As for the last item, it should be noticed that the intensity of preference considered in MACBETH and the intensity coming from comparisons of the type "x is preferred to y at least as strongly as w is preferred to z" (i.e., the quaternary relation $\succsim^*$) are substantially the same. In fact, the intensities of preference are equivalence classes of the preorder generated by $\succsim^*$. This means that all the pairs (x, y) and (w, z), such that x is preferred to y with the same intensity as w is preferred to z, belong to the same semantic category of difference of attractiveness considered in MACBETH. To be more precise, the structure of intensity of preference considered in MACBETH is a particular case of the structure of intensity of preference represented by $\succsim^*$ in GRIP. Still more precisely, GRIP has the same structure of intensity as MACBETH when $\succsim^*$ is a complete preorder. When this does not occur, MACBETH cannot be used while GRIP can naturally deal with this situation.

A detailed comparison of GRIP and MACBETH should take into account the following features,

1) Preference information:

   (a) GRIP

      i. Ordinal comprehensive preference information on pairwise comparison of some reference actions, $x \succsim y$.
      ii. Absolute qualitative judgements of intensity of preference for some pairs of reference actions partial and/or comprehensive (e.g. very weak, weak, moderate,..., extreme intensity of preferences for (x, y)).

or

iii. Comparison of intensities of preference for some pairs of reference actions partial and/or comprehensive, $(x, y) \succsim_i (w, z)$ and/or $(x, y) \succsim (w, z)$.

(b) MACBETH

i. Ordinal preference information with respect to each criterion for all not equally attractive pairs of actions: $x \succ_i y$ or $y \succ_i x$.

ii. Absolute qualitative judgements of differences of attractiveness for all not equally attractive pairs of actions with respect to each criterion, including "good" and "neutral" reference points (e.g. very weak, weak, moderate,..., extreme intensity of preference).

iii. Ordinal preference information for all not equally attractive criteria: g_i is more important than g_j, or g_j is more important than g_i.

iv. Absolute qualitative judgements of differences of attractiveness for all not equally attractive pairs of criteria (e.g. very weak, weak, moderate,..., extreme difference of importance).

2) Preference model and final results:

(a) GRIP

i. Uses linear programming to identify a set of comprehensive additive value functions with interval scales, compatible with preference information.

ii. Builds necessary and possible weak preference relations on set A,
- $\succsim^N$ (partial preorder);
- $\succsim^P$ (strongly complete and negatively transitive).

iii. Builds necessary and possible comprehensive intensity of preference relations on set $A \times A$,
- $\succsim^{*N}$ (partial preorder);
- $\succsim^{*P}$ (strongly complete and negatively transitive).

iv. Builds necessary and possible partial intensity of preference relations on set $A \times A$,
- $\succsim_i^{*N}$ (partial preorder);
- $\succsim_i^{*P}$ (strongly complete and negatively transitive).

(b) MACBETH

i. Uses linear programming to build a single interval scale for each criterion, compatible with preference information and computes a numerical marginal value for each action on each criterion.

ii. Computes a weight for each criterion.

iii. Builds a weighted-sum model on marginal values which is additive piecewise linear or discrete.

iv. Uses the model to set up a complete preorder on set A.

3) Summary of the crucial differences between the two methodologies:

(a) GRIP is using preference information relative to: 1) comprehensive preference on a subset of reference actions with respect to all criteria, 2) partial intensity of preference on some single criteria and 3) comprehensive intensity of preference with respect to all criteria, while MACBETH requires preference information on all pairs of actions with respect to each one of the considered criteria.

(b) Information about partial intensity of preference is of the same nature in GRIP and MACBETH (equivalence classes of relation $\succsim_i^*$ correspond to qualitative judgements of MACBETH), but in GRIP it may not be complete.

(c) GRIP is a “disaggregation-aggregation” approach while MACBETH makes use of the “aggregation” approach and, therefore, it needs weights to aggregate evaluations on the criteria.

(d) GRIP works with all compatible value functions, while MACBETH builds a single interval scale for each criterion, even if many such scales would be compatible with preference information.

(e) Distinguishing necessary and possible consequences of using all value functions compatible with preference information, GRIP includes a kind of robustness analysis instead of using a single “best-fit” value function.

(f) The necessary and possible preference relations considered in GRIP have several properties of general interest for MCDA.

6.7 Other characteristics of GRIP

It is interesting to note the following characteristics of GRIP:

1) In the absence of any pairwise comparison of reference actions, the preference relation $\succsim^N$ boils down to the weak dominance relation Δ on A ($x\Delta y$ iff $x_i \geq y_i$, $i \in I$) (Greco et al. 2005).

2) Each pairwise comparison of the type $x \succsim y$, $x, y \in A$, provided by the DM contributes to enrich $\succsim^N$, i.e., if $\succsim^N (x,y)^+$ and $\succsim^N (x,y)^-$ a re preference relations $\succsim^N$ with and without information $x \succsim y$, respectively, we have that $\succsim^N (x,y)^+ \supseteq \succsim^N (x,y)^-$ (Greco et al. 2005).

3) In the absence of any pairwise comparison of pairs of reference actions, the preference relation $\succsim^{*^N}$ boils down to the weak dominance relation with respect to difference of preferences Δ^* on A ($(x,y)\Delta^*(w,x)$ iff $x_{\geq} w_i$ and $y_i \leq z_i$, $i \in I$).

4) Each pairwise comparison of the type $(x,y) \succsim^* (w,z)$, $x, y, w, z \in A$, provided by the DM contributes to enrich $\succsim^{*^N}$, i.e., if $\succsim^{*^N} (x,y,w,z)^+$ and $\succsim^{*^N} (x,y,w,z)^-$ are preference relations $\succsim^{*^N}$ with and without information $(x,y) \succsim^* (w,z)$, respectively, we have that

$$\succsim^{*^N} (x,y,w,z)^+ \supseteq \succsim^{*^N} (x,y,w,z)^-.$$

5) In the absence of any pairwise comparison of reference actions, the preference relation $\succsim^P$ is a complete relation such that for any pair $x, y \in A$,

$$(x \succsim^P y \text{ and } y \succsim^P x) \Leftrightarrow \{(\text{not } x\Delta y \text{ and not } y\Delta x) \text{ or } (x\Delta y \text{ and } y\Delta x)\},$$
$$(x \succsim^P y \text{ and not } y \succsim^P x) \Leftrightarrow (x\Delta y \text{ and not } y\Delta x)$$

(Greco et al. 2005).

6) Each pairwise comparison $x \succsim y$, $x, y \in A$, provided by the DM contributes to impoverish $\succsim^P$, i.e., if $\succsim^P (x,y)^+$ and $\succsim^P (x,y)^-$ are preference relations $\succsim^P$ with and without information $x \succsim y$, respectively, w e have that $\succsim^P (x,y)^+ \subseteq \succsim^P (x,y)^-$ (Greco et al. 2005).

7) In the absence of any pairwise comparison of pairs of reference actions, the preference relation $\succsim^{*P}$ is a complete relation such that for any pair $x, y, w, z \in A$,

$$((x,y) \succsim^{*P} (w,z) \text{ and } (w,z) \succsim^{*P} (x,y)) \Leftrightarrow$$
$$\{(\text{not } (x,y)\Delta^*(w,z) \text{ and not } (w,z)\Delta^*(x,y)) \text{ or } ((w,z)\Delta^*(w,z) \text{ and } (w,z)\Delta^*(x,y))\},$$

$$((x,y) \succsim^{*P} (c,d) \text{ and not } (w,z) \succsim^{*P} (x,y)) \Leftrightarrow ((x,y)\Delta^*(w,z) \text{ and not } (w,z)\Delta^*(x,y)).$$

8) Each pairwise comparison of the type $(x,y) \succsim^* (w,z)$, $x, y, w, z \in A$, provided by the DM contributes to impoverish $\succsim^{*P}$, i.e., if $\succsim^{*P} (x,y,w,z)^+$ and $\succsim^{*P} (x,y,w,z)^-$ are preference relations $\succsim^{*P}$ with and without information $(x,y) \succsim^* (w,z)$, respectively, we have that

$$\succsim^{*P} (x,y,w,z)^+ \subseteq \succsim^{*P} (x,y,w,z)^-.$$

7 Conclusion and directions for future research

In this paper, we proposed the GRIP methodology for building a set of additive value functions compatible with the following preference information provided by the DM: a partial preorder in the set of reference actions and/or partial and comprehensive comparisons of intensities of preference between some pairs of reference actions. This preference information is used within a specially designed ordinal regression approach.

Considering all the compatible value functions, GRIP permits to achieve the following results:

- a necessary weak preference relation $\succsim^N$ on A, being a partial preorder,
- a possible weak preference relation $\succsim^P$ on A, being a strongly complete and negatively transitive relation,
- a comprehensive necessary intensity of preference relation $\succsim^{*N}$, being a partial preorder on $A \times A$,
- a comprehensive possible intensity of preference relation $\succsim^{*P}$, being a strongly complete and negatively transitive relation on $A \times A$,
- a partial necessary intensity of preference relation $\succsim_i^{*N}$, being a partial preorder on $A \times A$,
- a partial possible intensity of preference relation $\succsim_i^{*P}$, being a strongly complete and negatively transitive relation on $A \times A$.

There are several topics which constitute interesting directions for future research, in particular,

1) Handling preference information with gradual credibility.
2) Building a decision support system based on the GRIP methodology.
3) Extending the method to group decision making situations.
4) Applying the method to real-world problems.

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Appendix

Proof of Theorem 5.1.

In what follows $\mathcal{U}$ represents the set of compatible value functions satisfying constraints from $a)$ to $i)$.

1) For all $x, y \in A$,

$$x \succsim^N y \Leftrightarrow \forall\, U \in \mathcal{U},\, U(x) \geq U(y) \Rightarrow \exists U \in \mathcal{U} \text{ such that } U(x) \geq U(y) \Leftrightarrow x \succsim^{*^P} y.$$

2) For all $x, y \in A^R$,

$$x \succsim y \Rightarrow \forall\, U \in \mathcal{U},\, U(x) \geq U(y) \Leftrightarrow x \succsim^N y.$$

3) Let us recall that a partial preorder is a transitive and reflexive binary relation. Thus, in order to show that $\succsim^N$ is a partial preorder, we have to show that it is transitive and reflexive,

- $\succsim^N$ is transitive: for all $x, y, z \in A$,

$$\begin{aligned} &x \succsim^N y \text{ and } y \succsim^N z \Leftrightarrow \forall\, U \in \mathcal{U},\, U(x) \geq U(y) \text{ and } U(y) \geq U(z) \Leftrightarrow \\ &\Leftrightarrow \forall\, U \in \mathcal{U}, U(x) \geq U(y) \geq U(z) \Rightarrow \forall\, U \in \mathcal{U}, U(x) \geq U(z) \Rightarrow x \succsim^N z. \end{aligned}$$

- $\succsim^N$ is reflexive: for all $x \in A$,

$$U(x) = U(x),\ \forall\, U \in \mathcal{U} \Leftrightarrow U(x) \geq U(x),\ \forall\, U \in \mathcal{U} \Leftrightarrow x \succsim^N x.$$

  Thus, we proved that $\succsim^N$ is a partial preorder.

$\succsim^P$ is strongly complete: for all $x, y \in A$,

$$U(x) \geq U(y) \text{ or } U(y) \geq U(x),\ \forall\, U \in \mathcal{U} \Leftrightarrow x \succsim^P y \text{ or } y \succsim^P x.$$

$\succsim^P$ is negatively transitive: for all $x, y, z \in A$,

$$\begin{aligned} &\textit{not } x \succsim^P y \text{ and } \textit{not } y \succsim^P z \Leftrightarrow \\ &\Leftrightarrow\ \nexists\, U \in \mathcal{U}, \text{ such that } U(x) \geq U(y), \text{ and } \nexists\, U \in \mathcal{U}, \text{ such that } U(y) \geq U(z)\ \Leftrightarrow \\ &\Leftrightarrow\ \forall\, U \in \mathcal{U},\ U(x) < U(y) \text{ and } U(y) < U(z)\ \Rightarrow\ \forall\, U \in \mathcal{U},\ U(x) < U(z)\ \Leftrightarrow \\ &\Leftrightarrow\ \nexists\, U \in \mathcal{U}, \text{ such that } U(x) \geq U(z)\ \Leftrightarrow\ \textit{not } x \succsim^P z. \end{aligned}$$

4) For all $x, y \in A$,

$$U(x) \geq U(y)\ \forall U \in \mathcal{U} \text{ or } \exists U \in \mathcal{U} \text{ such that } U(y) > U(x) \Rightarrow x \succsim^N y \text{ or } y \succsim^P x.$$

5) For all $x, y, z \in A$,

$$\begin{aligned} &x \succsim^N y \text{ and } y \succsim^P z \Leftrightarrow U(x) \geq U(y)\ \forall\, U \in \mathcal{U} \text{ and } U(y) \geq U(z) \text{ for at least one } U \in \mathcal{U} \Rightarrow \\ &\Rightarrow U(x) \geq U(z) \text{ for at least one } U \in \mathcal{U} \Leftrightarrow x \succsim^P z. \end{aligned}$$

6) For all $x, y, z \in A$,

$$\begin{aligned} &x \succsim^P y \text{ and } y \succsim^N z \Leftrightarrow U(x) \geq U(y) \text{ for at least one } U \in \mathcal{U} \text{ and } U(y) \geq U(z)\ \forall\, U \in \mathcal{U} \Rightarrow \\ &\Rightarrow U(x) \geq U(z) \text{ for at least one } U \in \mathcal{U} \Leftrightarrow x \succsim^P z. \end{aligned}$$

7) For all $x, y, w, z \in A$,

$$
\begin{aligned}
&(x, y) \succsim^{*^N} (w, z) \Leftrightarrow U(x) - U(y) \geq U(w) - U(z) \; \forall \, U \in \mathcal{U} \Rightarrow \\
&\Rightarrow U(x) - U(y) \geq U(w) - U(z) \text{ for at least one } U \in \mathcal{U} \Leftrightarrow (x, y) \succsim^{*^P} (w, z).
\end{aligned}
$$

8) For all $x, y, w, z \in A^R$

$$
(x, y) \succsim^{*} (w, z) \Rightarrow U(x) - U(y) \geq U(w) - U(z) \; \forall \, U \in \mathcal{U} \Leftrightarrow (x, y) \succsim^{N} (w, z).
$$

9) To show that $\succsim^{*^N}$ is a partial preorder we have to show that it is transitive and reflexive,

- $\succsim^{*^N}$ is transitive: for all $x, y, w, z, r, s \in A$,

$$
\begin{aligned}
&(x, y) \succsim^{*^N} (w, z) \text{ and } (w, z) \succsim^{*^N} (r, s) \Leftrightarrow \\
&\Leftrightarrow U(x) - U(y) \geq U(w) - U(z) \text{ and} \\
&U(w) - U(z) \geq U(r) - U(s) \; \forall \, U \in \mathcal{U} \Rightarrow \\
&\Rightarrow U(x) - U(y) \geq U(r) - U(s) \; \forall \, U \in \mathcal{U} \Leftrightarrow (x, y) \succsim^{*^N} (r, s).
\end{aligned}
$$

- $\succsim^{*^N}$ is reflexive: for all $x, y \in A$,

$$
\begin{aligned}
&U(x) - U(y) = U(x) - U(y) \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow U(x) - U(y) \geq U(x) - U(y) \; \forall \, U \in \mathcal{U} \Leftrightarrow (x, y) \succsim^{*^N} (x, y).
\end{aligned}
$$

Thus we proved that $\succsim^{*^N}$ is a partial preorder.

$\succsim^{*^P}$ is strongly complete: for all $x, y, w, z \in A$,

$$
\begin{aligned}
&U(x) - U(y) \geq U(w) - U(z) \text{ or } U(w) - U(z) \geq U(x) - U(y) \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x, y) \succsim^{*^P} (w, z) \text{ or } (w, z) \succsim^{*^P} (x, y).
\end{aligned}
$$

$\succsim^{*^P}$ is negatively transitive: for all $x, y, w, z, r, s \in A$,

$$
\begin{aligned}
&\text{not } (x, y) \succsim^{*^P} (w, z) \text{ and not } (w, z) \succsim^{*^P} (r, s) \;\; \Leftrightarrow \\
&\Leftrightarrow \;\; \nexists \, U \in \mathcal{U} \;\text{ such that } [U(x) - U(y)] \geq [U(w) - U(z)] \text{ and} \\
&\nexists \, U \in \mathcal{U} \;\text{ such that } [U(w) - U(z)] \geq [U(r) - U(s)] \;\; \Leftrightarrow \\
&\Leftrightarrow \;\; \forall \, U \in \mathcal{U}, \; [U(x) - U(y)] < [U(w) - U(z)] \text{ and } [U(w) - U(z)] < [U(r) - U(s)] \;\; \Rightarrow \\
&\Rightarrow \;\; \forall \, U \in \mathcal{U}, \; [U(x) - U(y)] < [U(r) - U(s)] \;\; \Leftrightarrow \\
&\Leftrightarrow \;\; \nexists \, U \in \mathcal{U} \;\text{ such that } [U(x) - U(y)] \geq [U(r) - U(s)] \;\; \Leftrightarrow \\
&\Leftrightarrow \;\; \text{not } (x, y) \succsim^{*^P} (r, s).
\end{aligned}
$$

10) For all $x, y, w, z \in A$,

$$
\begin{aligned}
&U(x) - U(y) \geq U(w) - U(z) \; \forall U \in \mathcal{U} \text{ or } \exists U \in \mathcal{U} \text{ such that } U(w) - U(z) > U(x) - U(y) \Rightarrow \\
&\Rightarrow (x, y) \succsim^{*^N} (w, z) \text{ or } (w, z) \succsim^{*^P} (x, y).
\end{aligned}
$$

11) For all $x, y, w, z, r, s \in A$,

$$
\begin{aligned}
&(x, y) \succsim^{*^N} (w, z) \text{ and } (w, z) \succsim^{*^P} (r, s) \Leftrightarrow \\
&\Leftrightarrow U(x) - U(y) \geq U(w) - U(z) \; \forall U \in \mathcal{U} \text{ and} \\
&U(w) - U(z) \geq U(r) - U(s) \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow U(x) - U(y) \geq U(r) - U(s) \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x, y) \succsim^{*^P} (r, s).
\end{aligned}
$$

12) For all $x, y, w, z, r, s \in A$,

$$\begin{aligned}
&(x,y) \succsim^{*^P} (w,z) \text{ and } (w,z) \succsim^N (r,s) \Leftrightarrow \\
&\Leftrightarrow U(x) - U(y) \geq U(w) - U(z) \text{ for at least one } U \in \mathcal{U} \text{ and} \\
&U(w) - U(z) \geq U(r) - U(s)\ \forall U\ \in \mathcal{U} \Rightarrow \\
&\Rightarrow U(x) - U(y) \geq U(r) - U(s) \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y) \succsim^{*^P} (r,s).
\end{aligned}$$

13) For all $x, x', y, w, z \in A$,

$$\begin{aligned}
&x' \succsim^N x \text{ and } (x,y) \succsim^{*^N} (w,z) \Leftrightarrow \\
&\Leftrightarrow U(x') \geq U(x) \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)]\ \forall\ U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [U(x') - U(y)] \geq [U(w) - U(z)]\ \forall\ U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x',y) \succsim^{*^N} (w,z).
\end{aligned}$$

14) For all $x, x', y, w, z \in A$,

$$\begin{aligned}
&x' \succsim^N x \text{ and } (x,y) \succsim^{*^P} (w,z) \Leftrightarrow \\
&\Leftrightarrow U(x') \geq U(x)\ \forall\ U \in \mathcal{U} \text{ and} \\
&[U(x) - U(y)] \geq [U(w) - U(z)] \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow U(x') \geq U(x) \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \\
&\text{for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [U(x') \geq U(y)] \geq [U(w) - U(z)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x',y) \succsim^{*^P} (w,z).
\end{aligned}$$

15) For all $x, x', y, w, z \in A$,

$$\begin{aligned}
&x' \succsim^P x \text{ and } (x,y) \succsim^{*^N} (w,z) \Leftrightarrow \\
&\Leftrightarrow U(x') \geq U(x) \text{ for at least one } U \in \mathcal{U} \text{ and} \\
&[U(x) - U(y)] \geq [U(w) - U(z)]\ \forall\ U\ \in \mathcal{U} \Rightarrow \\
&\Rightarrow [U(x') - U(y)] \geq [U(w) - U(z)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x',y) \succsim^{*^P} (w,z).
\end{aligned}$$

16) For all $x, y, y', w, z \in A$,

$$\begin{aligned}
&y \succsim^N y' \text{ and } (x,y) \succsim^{*^N} (w,z) \Leftrightarrow \\
&\Leftrightarrow U(y) \geq U(y') \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)]\ \forall\ U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [U(x) - U(y')] \geq [U(w) - U(z)]\ \forall\ U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim^{*^N} (w,z).
\end{aligned}$$

17) For all $x, y, y', w, z \in A$,

$$\begin{aligned}
&y \succsim^N y' \text{ and } (x,y) \succsim^{*^P} (w,z) \Leftrightarrow \\
&\Leftrightarrow U(y) \geq U(y')\ \forall\ U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \\
&\text{for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [U(x) - U(y')] \geq [U(w) - U(z)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim^{*^P} (w,z).
\end{aligned}$$

18) For all $x, y, y', w, z \in A$,

$$\begin{aligned}
& y \succsim^P y' \text{ and } (x,y) \succsim^{*N} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(y) \geq U(y') \text{ for at least one } U \in \mathcal{U} \text{ and} \\
& \ [U(x) - U(y)] \geq [U(w) - U(z)] \ \forall\, U \ \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y')] \geq [U(w) - U(z)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y') \succsim^{*P} (w, z).
\end{aligned}$$

19) For all $x, y, w, w', z \in A$,

$$\begin{aligned}
& w \succsim^N w' \text{ and } (x,y) \succsim^{*N} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(w) \geq U(w') \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \ \forall\, U \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y)] \geq [U(w') - U(z)] \ \forall\, U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y) \succsim^{*N} (w', z).
\end{aligned}$$

20) For all $x, y, w, w', z \in A$,

$$\begin{aligned}
& w \succsim^N w' \text{ and } (x,y) \succsim^{*P} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(w) \geq U(w') \forall\, U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \\
& \text{for at least one } U \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y)] \geq [U(w') - U(z)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y) \succsim^{*P} (w', z).
\end{aligned}$$

21) For all $x, y, w, w', z \in A$,

$$\begin{aligned}
& w \succsim^P w' \text{ and } (x,y) \succsim^{*N} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(w) \geq U(w') \text{ for at least one } U \in \mathcal{U} \text{ and} \\
& \ [U(x) - U(y)] \geq [U(w) - U(z)] \ \forall\, U \ \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y)] \geq [U(w') - U(z)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y) \succsim^{*P} (w', z).
\end{aligned}$$

22) For all $x, y, w, z, z' \in A$,

$$\begin{aligned}
& z \succsim^N z' \text{ and } (x,y) \succsim^{*N} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(z) \geq U(z') \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \ \forall\, U \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y)] \geq [U(w) - U(z')] \ \forall\, U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y) \succsim^{*N} (w, z').
\end{aligned}$$

23) For all $x, y, w, z, z' \in A$,

$$\begin{aligned}
& z \succsim^N z' \text{ and } (x,y) \succsim^{*P} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(z) \geq U(z') \ \forall\, U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \\
& \text{for at least one } U \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y)] \geq [U(w) - U(z')] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y) \succsim^{*P} (w, z').
\end{aligned}$$

24) For all $x, y, w, z, z' \in A$,

$$\begin{aligned}
& z \succsim^P z' \text{ and } (x,y) \succsim^{*N} (w,z) \Leftrightarrow \\
& \Leftrightarrow U(z) \geq U(z') \text{ for at least one } U \in \mathcal{U} \text{ and} \\
& \ [U(x) - U(y)] \geq [U(w) - U(z)] \ \forall\, U \ \in \mathcal{U} \Rightarrow \\
& \Rightarrow [U(x) - U(y)] \geq [U(w) - U(z')] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y) \succsim^{*P} (w, z').
\end{aligned}$$

25) For all $x, x', y \in A$,

$$
\begin{aligned}
&(x', y) \succsim^{*^N} (x, y) \Leftrightarrow \\
&\Leftrightarrow [U(x') - U(y)] \geq [U(x) - U(y)] \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow U(x') \geq U(x) \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow x' \succsim^N x.
\end{aligned}
$$

26) For all $x, x', y \in A$,

$$
\begin{aligned}
&(x', y) \succsim^{*^P} (x, y) \Leftrightarrow \\
&\Leftrightarrow [U(x') - U(y)] \geq [U(x) - U(y)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow U(x') \geq U(x) \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow x' \succsim^P x.
\end{aligned}
$$

27) For all $x, y, y' \in A$,

$$
\begin{aligned}
&(x, y') \succsim^{*^N} (x, y) \Leftrightarrow \\
&\Leftrightarrow [U(x) - U(y')] \geq [U(x) - U(y)] \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow U(y) \geq U(y') \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow y \succsim^N y'.
\end{aligned}
$$

28) For all $x, y, y' \in A$,

$$
\begin{aligned}
&(x, y') \succsim^{*^P} (x, y) \Leftrightarrow \\
&\Leftrightarrow [U(x) - U(y')] \geq [U(x) - U(y)] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow U(y) \geq U(y') \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow y \succsim^P y'.
\end{aligned}
$$

29) To show that $\succsim_i^{*^N}$, $i \in I$, is a partial preorder we have to show that it is transitive and reflexive,

- $\succsim_i^{*^N}$ is transitive: for all $x, y, w, z, r, s \in A$,

$$
\begin{aligned}
&(x, y) \succsim_i^{*^N} (w, z) \text{ and } (w, z) \succsim_i^{*^N} (r, s) \Leftrightarrow \\
&\Leftrightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ and} \\
&[u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \; \forall \, U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x, y) \succsim_i^{*^N} (r, s).
\end{aligned}
$$

- $\succsim_i^{*^N}$ is reflexive: for all $x, y \in A$,

$$
\begin{aligned}
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x, y) \succsim_i^{*^N} (x, y).
\end{aligned}
$$

Thus we proved that $\succsim_i^{*^N}$, $i \in I$ is a partial preorder.

$\succsim_i^{*^P}$, $i \in I$, is strongly complete: for all $x, y, w, z \in A$,

$$
\begin{aligned}
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ or} \\
&[u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \; \forall \, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x, y) \succsim_i^{*^P} (w, z) \text{ or } (w, z) \succsim_i^{*^P} (x, y).
\end{aligned}
$$

$\succsim_i^{*^P}$, $i \in I$, is negatively transitive: for all $x, y, w, z, r, s \in A$,

$$\begin{aligned}
&\text{not } (x,y) \succsim_i^{*^P} (w,z) \text{ and not } (w,z) \succsim_i^{*^P} (r,s) \;\Leftrightarrow\\
&\Leftrightarrow\; \nexists\, U \in \mathcal{U} \;\text{ such that } [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ and}\\
&\nexists\, U \in \mathcal{U} \;\text{ such that } [u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \;\Leftrightarrow\\
&\Leftrightarrow\; \forall\, U \in \mathcal{U},\; [u_i(g_i(x)) - u_i(g_i(y))] < [u_i(g_i(w)) - u_i(g_i(z))] \text{ and}\\
&[u_i(g_i(w)) - u_i(g_i(z))] < [u_i(g_i(r)) - u_i(g_i(s))] \;\Rightarrow\\
&\Rightarrow\; \forall\, U \in \mathcal{U},\; [u_i(g_i(x)) - u_i(g_i(y))] < [u_i(g_i(r)) - u_i(g_i(s))] \;\Leftrightarrow\\
&\Leftrightarrow\; \nexists\, U \in \mathcal{U} \;\text{ such that } [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \;\Leftrightarrow\\
&\Leftrightarrow\; \text{not } (x,y) \succsim_i^{*^P} (r,s).
\end{aligned}$$

30) For all $x, y, w, z \in A$, $i \in I$,

$$\begin{aligned}
&u_i(x) - u_i(y) \geq u_i(w) - u_i(z)\ \forall U \in \mathcal{U} \text{ or } \exists U \in \mathcal{U} \text{ such that } u_i(w) - u_i(z) > u_i(x) - u_i(y) \Rightarrow\\
&\Rightarrow (x,y) \succsim_i^{*^N} (w,z) \text{ or } (w,z) \succsim_i^{*^P} (x,y).
\end{aligned}$$

31) For all $x, y, w, z, r, s \in A$, $i \in I$,

$$\begin{aligned}
&(x,y) \succsim_i^{*^N} (w,z) \text{ and } (w,z) \succsim_i^{*^P} (r,s)] \Leftrightarrow\\
&\Leftrightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]\ \forall\, U \in \mathcal{U} \text{ and}\\
&[u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \text{ for at least one } U \in \mathcal{U} \Rightarrow\\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow\\
&\Leftrightarrow (x,y) \succsim_i^{*^P} (r,s).
\end{aligned}$$

32) For all $x, y, w, z, r, s \in A$, $i \in I$

$$\begin{aligned}
&(x,y) \succsim_i^{*^P} (w,z) \text{ and } (w,z) \succsim_i^{*^N} (r,s) \Leftrightarrow\\
&\Leftrightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \;\text{ for at least one } U \in \mathcal{U} \text{ and}\\
&[u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(r)) - u_i(g_i(s))]\ \forall\, U \in \mathcal{U} \Rightarrow\\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow\\
&\Leftrightarrow (x,y) \succsim_i^{*^P} (r,s).
\end{aligned}$$

33) For all $x, x', y, w, z \in A$, $i \in I$,

$$\begin{aligned}
&g_i(x') \geq g_i(x) \text{ and } (x,y) \succsim_i^{*^N} (w,z) \Leftrightarrow\\
&\Leftrightarrow u_i(g_i(x')) \geq u_i(g_i(x)) \text{ and } [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]\ \forall\, U \in \mathcal{U} \Rightarrow\\
&\Rightarrow [u_i(g_i(x')) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]\ \forall\, U \in \mathcal{U} \Leftrightarrow\\
&\Leftrightarrow (x',y) \succsim_i^{*^N} (w,z).
\end{aligned}$$

34) For all $x, x', y, w, z \in A$, $i \in I$,

$$\begin{aligned}
&g_i(x') \geq g_i(x) \text{ and } (x,y) \succsim_i^{*^P} (w,z) \Leftrightarrow\\
&\Leftrightarrow u_i(g_i(x')) \geq u_i(g_i(x)) \text{ and}\\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Rightarrow\\
&\Rightarrow [u_i(g_i(x')) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow\\
&\Leftrightarrow (x',y) \succsim_i^{*^P} (w,z).
\end{aligned}$$

35) For all $x, y, y', w, z \in A$, $i \in I$,

$$\begin{aligned}
&g_i(y) \geq g_i(y') \text{ and } (x,y) \succsim_i^{*^N} (w,z) \Leftrightarrow \\
&\Leftrightarrow u_i(g_i(y)) \geq u_i(g_i(y')) \text{ and} \\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \; \forall\, U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y'))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \;\; \forall\, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim_i^{*^N} (w,z).
\end{aligned}$$

36) For all $x, y, y', w, z \in A$, $i \in I$,

$$\begin{aligned}
&g_i(y) \geq g_i(y') \text{ and } (x,y) \succsim_i^{*^P} (w,z) \Leftrightarrow \\
&\Leftrightarrow u_i(g_i(y)) \geq u_i(g_i(y')) \text{ and} \\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y'))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim_i^{*^P} (w,z).
\end{aligned}$$

37) For all $x, y, w, w', z \in A$, $i \in I$,

$$\begin{aligned}
&g_i(w) \geq g_i(w') \text{ and } (x,y) \succsim_i^{*^N} (w,z) \Leftrightarrow \\
&\Leftrightarrow u_i(g_i(w)) \geq u_i(g_i(w')) \text{ and} \\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \; \forall\, U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w')) - u_i(g_i(z))] \;\; \forall\, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim_i^{*^N} (w',z).
\end{aligned}$$

38) For all $x, y, w, w', z \in A$, $i \in I$,

$$\begin{aligned}
&g_i(w) \geq g_i(w') \text{ and } (x,y) \succsim_i^{*^P} (w,z) \Leftrightarrow \\
&\Leftrightarrow u_i(g_i(w)) \geq u_i(g_i(w')) \text{ and} \\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w')) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim_i^{*^P} (w',z).
\end{aligned}$$

39) For all $x, y, w, z, z' \in A$, $i \in I$,

$$\begin{aligned}
&g_i(z') \geq g_i(z) \text{ and } (x,y) \succsim_i^{*^N} (w,z) \Leftrightarrow \\
&\Leftrightarrow u_i(g_i(z')) \geq u_i(g_i(z)) \text{ and} \\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \; \forall\, U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z'))] \; \forall\, U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim_i^{*^N} (w,z').
\end{aligned}$$

40) For all $x, y, w, z, z' \in A$, $i \in I$,

$$\begin{aligned}
&g_i(z') \geq g_i(z) \text{ and } (x,y) \succsim_i^{*^P} (w,z) \Leftrightarrow \\
&\Leftrightarrow u_i(g_i(z')) \geq u_i(g_i(z)) \text{ and} \\
&[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
&\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z'))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
&\Leftrightarrow (x,y') \succsim_i^{*^P} (w,z').
\end{aligned}$$

41) For all $x, x', y \in A$, $i \in I$,

$$\begin{aligned}
& g_i(x') \geq g_i(x) \Rightarrow u_i(g_i(x')) \geq u_i(g_i(x)) \;\forall\, U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow [u_i(g_i(x')) - u_i(g_i(y))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \;\forall\, U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x', y) \succsim_i^{*^N} (x, y).
\end{aligned}$$

42) For all $x, x', y \in A$, $i \in I$,

$$\begin{aligned}
& (x', y) \succ_i^{*^P} (x, y) \Leftrightarrow \\
& \Leftrightarrow [u_i(g_i(x')) - u_i(g_i(y))] > [u_i(g_i(x)) - u_i(g_i(y))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow u_i(g_i(x')) > u_i(g_i(x)) \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
& \Rightarrow g_i(x') > g_i(x).
\end{aligned}$$

43) For all $x, y, y' \in A$, $i \in I$,

$$\begin{aligned}
& g_i(y) \geq g_i(y') \Rightarrow u_i(g_i(y)) \geq u_i(g_i(y')) \;\forall\, U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow [u_i(g_i(x)) - u_i(g_i(y'))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \;\forall\, U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow (x, y') \succsim_i^{*^N} (x, y).
\end{aligned}$$

44) For all $x, x', y \in A$, $i \in I$,

$$\begin{aligned}
& (x, y) \succ_i^{*^P} (x, y') \Leftrightarrow \\
& \Leftrightarrow [u_i(g_i(x)) - u_i(g_i(y'))] > [u_i(g_i(x)) - u_i(g_i(y))] \text{ for at least one } U \in \mathcal{U} \Leftrightarrow \\
& \Leftrightarrow u_i(g_i(y)) > u_i(g_i(y')) \text{ for at least one } U \in \mathcal{U} \Rightarrow \\
& \Rightarrow g_i(y) > g_i(y').
\end{aligned}$$

$\square$