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A note on the hardness results for the labeled perfect matching problems in bipartite graphs

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Abstract

In this note, we strengthen the inapproximation bound of $O(\log n)$ for the labeled perfect matching problem established in J. Monnot, The Labeled perfect matching in bipartite graphs, *Information Processing Letters* **96** (2005) 81-88, using a self improving operation in some hard instances. It is interesting to note that this self improving operation does not work for all instances. Moreover, based on this approach we deduce that the problem does not admit constant approximation algorithms for connected planar cubic bipartite graphs.

Keywords: labeled matching; bipartite graphs; Approximation and complexity; inapproximation bounds.

1 Introduction

A matching M on a graph $G = (V, E)$ is a subset of edges that are pairwise non adjacent; M is said perfect if it covers the vertex set V of G. In the labeled perfect matching problem (Labeled *Min PM* in short), we are given a simple graph $G = (V, E)$ on $|V| = 2n$ vertices which contains a perfect matching together with a color (or label) function $\mathcal{L} : E \to \{c_0, \ldots, c_q\}$ on the edge set of G. For $i = 0, \ldots, q$, we denote by $\mathcal{L}_i \subseteq E$ the set of edges of color c_i. The goal of Labeled *Min PM* is to find a perfect matching on G that uses a minimum number of colors. Alternatively, if $G[\mathcal{L}']$ denotes the subgraph induced by the edges of colors $\mathcal{L}' \subseteq \{c_0, \ldots, c_q\}$, then Labeled *Min PM* aims at finding a subset $\mathcal{L}'$ of minimum size such that $G[\mathcal{L}']$ contains a perfect matching. Very recently, some approximation results are obtained for Labeled *Min PM* when the graphs are bipartite 2-regular or complete bipartite $K_{n,n}$, [6]. In particular, it is shown that the 2-regular bipartite case is equivalent to the minimum satisfiability problem, and that a greedy algorithm picking at each iteration a monocolored matching of maximum size provides a $\frac{r+H_r}{2}$-approximation in complete bipartite graphs where r is the maximum of times that a color appears in the graph and H_r is the r-th harmonic number. Moreover, it is proved that Labeled *Min PM* is not $O(\log n)$-approximable in bipartite complete graphs. In [5], this problem is motivated by some applications in timetable problems. Several related works concerning some matching problems on colored graphs can be found in [2, 3, 4]

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In this note, we prove first that LABELED $Min\ PM$ is not in **APX** whenever the bipartite graphs have a maximum degree of 3. Hence, there is a gap of approximability between graphs of maximum degree 2 and 3 since we can easily deduce from [6] that LABELED $Min\ PM$ is 2-approximable in bipartite graphs of maximum degree 2. Using a weaker complexity hypothesis, we can even obtain that LABELED $Min\ PM$ is not $2^{O(log^{1-\varepsilon} n)}$-approximable in bipartite graphs of maximum degree 3 on n vertices, unless **NP** $\subseteq$ **DTIME** $\left(2^{O(log^{1/\varepsilon} n)}\right)$. Dealing with the unbounded degree case, this yields to the fact that LABELED $Min\ PM$ is not in **polyLog-APX**, unless **P** = **NP**.

In the following, we denote by $opt(I)$ and $apx(I)$ the value of an optimal and an approximate solution, respectively for LABELED $Min\ PM$. We say that an algorithm $\mathcal{A}$ is a ρ-approximation (with $\rho \geq 1$) if $apx(I) \leq \rho \times opt(I)$ for any instance I.

Finally, in order to simplify the proofs exposed in the rest of the paper, the results concern a variation of LABELED $Min\ PM$, where the value of each perfect matching M is given by $val_1(M) = val(M) - 1$. This problem is denoted LABELED $Min\ PM_1$ and we have for any instance I, $apx_1(I) = apx(I) - 1$ and $opt_1(I) = opt(I) - 1$. It is important to note that a $\rho(n)$-approximation of LABELED $Min\ PM$ becomes a $2\rho(n)$-approximation of LABELED $Min\ PM_1$, and conversely a $\rho(n)$-approximation of LABELED $Min\ PM_1$ remains a $\rho(n)$-approximation of LABELED $Min\ PM$. Actually, since LABELED $Min\ PM$ is simple, [7] (i.e., the restriction to $opt(I) \leq k$ is polynomial), we can see that LABELED $Min\ PM$ and LABELED $Min\ PM_1$ are asymptotically equivalent to approximate. Hence, the proposed results for LABELED $Min\ PM_1$ also hold LABELED $Min\ PM$.

2 A self improving operation on some classes of graphs

We now propose a self improving operation for some classes of instances $\mathcal{P}_k$ described as follows. $I = (H, \mathcal{L}) \in \mathcal{P}_k$ where $H = (V, E)$ if and only if the following properties are satisfy:

- (i) H is planar of maximum degree k and connected.
- (ii) $\exists u, v \in V$ such that $[u, u_1]$ and $[v, v_1]$ for some $u_1, v_1 \in V$ are the only edges incident to u and v. Moreover, these two edges have color c_0, i.e., $\mathcal{L}([u, u_1]) = \mathcal{L}([v, v_1]) = c_0$.
- (iii) H is bipartite and admits a perfect matching.
- (iv) $H[\{c_0\}]$, the subgraph induced by edges of color c_0 does not have any perfect matching and the subgraph $H[\mathcal{L}(E) \setminus \{c_0\}]$ induced by edges of colors different from c_0 is acyclic.
- (v) if $H' = H \setminus \{u, v\}$ denotes the subgraph induced by $V \setminus \{u, v\}$, then $H'[\{c_0\}]$ has a perfect matching denoted by M_{c_0}.

We have $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2$ is the set of odd paths from u to v alternating matchings M and M_{c_0} where M_{c_0} is only colored by color c_0. Finally, we define the class $\mathcal{P}$ by $\mathcal{P} = \cup_k \mathcal{P}_k$.

Restricted label squaring operation. Given an instance $I = (H, \mathcal{L}) \in \mathcal{P}_k$ of LABELED $Min\ PM$, its *label squaring* instance is $I^2 = (H^2, \mathcal{L}^2)$ with $H^2 = (V^2, E^2)$, where

1. The graph H^2 is created by removing each edge $e = [x, y]$ of H with color different from c_0 and placing instead of it a copy $H(e)$ of H, such that x and y are now identified with u and v of H, respectively.

2. For each copy $H(e)$ of H and for an edge e' in $H(e)$ with color different from c_0, the new color of e' is $\mathcal{L}^2(e') = (\mathcal{L}(e), \mathcal{L}(e'))$. The remaining edges of copy $H(e)$ keep their color c_0, that is if $\mathcal{L}(e') = c_0$, then $\mathcal{L}^2(e') = c_0$.

Let us prove that classes $\mathcal{P}_k$ are closed under restricted label squaring operation.

Lemma 2.1 *If $I \in \mathcal{P}_k$, then $I^2 \in \mathcal{P}_k$.*

Proof: Let $I \in \mathcal{P}_k$. The proofs of (i) and (ii) are obvious.

For (iii), since H and $H \setminus \{u, v\}$ admit a perfect matching, we deduce that $u \in L$ and $v \in R$ where (L, R) is the bipartition of H. Thus, we can extend the bipartition to H^2 by taking for each $H(e)$ a copy of the bipartition. Finally, it is easy to verify that H^2 admits a perfect matching if H does.

For (iv) assume the reverse, that is $H^2[\{c_0\}]$ admits a perfect matching M and $H[\{c_0\}]$ not. By hypothesis, in each copy $H([x, y])$, the vertices x and y are not saturated by M and then the edges of M which do not traverse copies $H(e)$ form a perfect matching of $H[\{c_0\}]$, contradiction. Moreover, using property (ii), it is easy to verify that the subgraph $H^2[\mathcal{L}^2(E^2) \setminus \{c_0\}]$ is acyclic whenever $H[\mathcal{L}(E) \setminus \{c_0\}]$ is acyclic.

For (v) let M_{c_0} be a perfect matching of $H' = H \setminus \{u, v\}$ only using color c_0. We complete M_{c_0} by taking for each copy $H(e)$ a copy of M_{c_0}. In this way, we obtain a perfect matching of $H^2 \setminus \{u, v\}$ that uses only color c_0.

■

We now propose an approximation preserving reduction using the label squaring operation on $\mathcal{P}_k$.

Theorem 2.2 *Let $I = (H, \mathcal{L}) \in \mathcal{P}_k$. If there exists a (polynomial) ρ-approximation of I^2 for* Labeled *Min PM$_1$, then there exists a $\sqrt{\rho}$-approximation of I for* Labeled *Min PM$_1$.*

Proof: Let M^* be an optimal perfect matching of $I \in \mathcal{P}_k$ using $opt(I)$ colors and let $e_1, \cdots, e_p$ be the edges of H using colors distinct of c_0. For each copy $H(e_i)$ we take a copy of M^* using colors $(\mathcal{L}(e_i), \mathcal{L}(e_j))$ for $j = 1, \cdots, p$ and color c_0. For the remaining copies, we take a copy of M_{c_0} (a perfect matching on $H \setminus \{u, v\}[\{c_0\}]$) and we complete this matching into a perfect matching of H^2 using the remaining edges of M^*. This matching uses $(opt(I) - 1)^2 + 1$ colors and thus

$$opt_1(I^2) \le opt_1^2(I) \tag{1}$$

Now, consider an approximate perfect matching M^2 of H^2 with value $apx(I^2)$ and let $H(e_1), \cdots, H(e_p)$ be the copies of H such that the restriction of M^2 to $H(e_i)$ is a perfect matching. Hence, we may always assume that $M^2 \setminus (\cup_{i=1}^{p} H(e_i))$ only uses color c_0. Therefore, if we denote $\mathcal{L}' = \{\mathcal{L}(e_i) : i = 1, \cdots, p\}$, then for any $c_j \in \mathcal{L}'$ there exists a perfect matching $M_{c_j,k} \subseteq M^2$ in copy $H(e_k)$ such that edge e_k has color c_j. Let M_{c_j} be a matching of H minimizing $|\mathcal{L}(M_{c_j,k})|$ for any $c_j \in \mathcal{L}'$ and let M_0 be a perfect matching of H containing edges $\{e_1, \ldots, e_p\}$ and some other edges of color c_0.

The approximate perfect matching M of I will be given by one of the matchings M_{c_j} or M_0 with value $apx(I) = \min\{|\mathcal{L}(M_0)|, |\mathcal{L}(M_{c_j})| : c_j \in \mathcal{L}'\}$. Thus, we deduce that $apx_1(I) = apx(I) - 1 = \min\{|\mathcal{L}(M_0)| - 1, |\mathcal{L}(M_{c_j})| - 1 : c_j \in \mathcal{L}'\}$ and hence:

$$apx_1^2(I) \le (|\mathcal{L}(M_0)|-1)\min\{|\mathcal{L}(M_{c_j})|-1 : c_j \in \mathcal{L}'\} \le \sum_{c_j\in\mathcal{L}'} \big(|\mathcal{L}(M_{c_j})|-1\big) \le apx_1(I^2) \quad (2)$$

Applying inequality (2) with an optimal perfect matching M^2 of H^2, we obtain $opt_1^2(I) \le opt_1(I^2)$. Using inequality (1), we deduce $opt_1^2(I) = opt_1(I^2)$ and the expected result follows. ∎

3 Inapproximability results

In [6], an inapproximability bound of $O(\log n)$ is obtained for Labeled *Min PM* in complete bipartite graphs via a reduction from Set Cover. A slight modification of this reduction allow us to obtain the same result for instances in $\mathcal{P}$.

Theorem 3.1 Labeled *Min PM*$_1$ *is not* $c\log n$ *approximable for some constant* $c > 0$ *for instances in* $\mathcal{P}$ *having* $2n$ *vertices, unless* ***P*=*NP***.

Proof: See Appendix. ∎

Starting from the **APX**-completeness result for the vertex cover problem in cubic graphs, [1], we are able to obtain the following result.

Corollary 3.2 Labeled *Min PM*$_1$ *for instances in* $\mathcal{P}_3$ *is not in* ***PTAS***.

Proof: See Appendix. ∎

By applying the well known method of self improving, we obtain the two following results:

Theorem 3.3 Labeled *Min PM*$_1$ *for instances in* $\mathcal{P}_3$ *is not in* ***APX***, *unless* ***P*** = ***NP***.

Proof: Assume the reverse and let A be a polynomial algorithm solving Labeled *Min PM*$_1$ within a constant performance ratio ρ. Let $\varepsilon > 0$ (with $\varepsilon < \rho - 1$) and choose the smallest integer q such that:

$$q \ge \log\log\rho - \log\log(1+\varepsilon) \quad (3)$$

Consider now an instance $I = (H, \mathcal{L}) \in \mathcal{P}_3$ and use the restricted label squaring operation on I. We produce the instance $I^2 = (H^2, \mathcal{L}^2)$ and by repeating q times this operation on I^2, we obtain thanks to Lemma 2.1 the instance $I^{2^q} = (H^{2^q}, \mathcal{L}^{2^q}) \in \mathcal{P}_3$, in time $P(|I|)$ for some polynomial P since on the one hand, I^2 is obtained from I in time $O(|I|^2)$ (we have $|V(H^2)| = O(|V(H)|^2)$ and $|\mathcal{L}^2(E(H^2))| = O(|\mathcal{L}(E(H))|^2)$) and on the other hand, we repeat this operation a constant number of times. Using Theorem 2.2, from the ρ-approximation on I^{2^q} given by A , we obtain a $\rho^{2^{-q}}$-approximation on I. Thanks to inequality (3), we deduce $\rho^{2^{-q}} \le 1+\varepsilon$. Hence, we obtain a polynomial time approximation scheme for instances in $\mathcal{P}_3$, contradiction with Corollary 3.2. ∎

Theorem 3.4 *For any* $\varepsilon > 0$ Labeled *Min PM*$_1$ *is not* $2^{O(log^{1-\varepsilon} n)}$*-approximable for instances in* $\mathcal{P}_3$ *on* n *vertices, unless* ***NP*** $\subseteq$ ***DTIME*** $\left(2^{O(log^{1/\varepsilon} n)}\right)$.

Proof: Let $\varepsilon > 0$ and $I = (H, \mathcal{L}) \in \mathcal{P}_3$ where H has n vertices. Choose the smallest integer p such that $n^{2^p} \geq 2^{\log^{1/\varepsilon} n}$. Thus, $2^{2^p \times \log n} \geq 2^{\log^{1/\varepsilon} n}$ and then,

$$2^{p \times \varepsilon} \geq \log^{1-\varepsilon} n \tag{4}$$

Using the restricted label squaring operation on I, we produce the instance $I^2 = (H^2, \mathcal{L}^2)$. By repeating p times this operation on I^2, we obtain the instance $I^{2^p} = (H^{2^p}, \mathcal{L}^{2^p}) \in \mathcal{P}_3$. Since, H has n vertices, we derive from property (iv) of Lemma 2.1 that the number n' of vertices of H^{2^p} and the number $|\mathcal{L}^{2^p}(E(H^{2^p}))|$ of colors of H^{2^p} satisfy:

$$n' \leq n^{2^p} \text{ and } |\mathcal{L}^{2^p}(E(H^{2^p}))| \leq |\mathcal{L}(E(H))|^{2^p} \tag{5}$$

Now, assume that we have a $f(n')$-approximation on I^{2^p} where $f(n') \leq 2^{c \times \log^{1-\varepsilon} n'}$ for some $c > 0$. Using Theorem 2.2, we obtain a $f(n')^{2^{-p}}$-approximation on I. Using inequalities (4) and (5), we deduce:

$$\begin{aligned} apx_1(I) &\leq f(n')^{2^{-p}} opt_1(I) \\ &\leq 2^{c \times \frac{\log^{1-\varepsilon} n'}{2^p}} opt_1(I) \\ &\leq 2^{c \times \frac{\log^{1-\varepsilon} n}{2^{\varepsilon \times p}}} opt_1(I) \\ &\leq 2^c opt_1(I) \ , \end{aligned}$$

Thus, using inequality (5), we obtain a constant approximation in time $poly(n') = 2^{O(\log^{1/\varepsilon} n)}$, and thus, a contradiction with Theorem 3.3. ∎

It is natural to ask the question whether the problem is easier in cubic bipartite graphs. Here, we prove that the answer is negative.

Theorem 3.5 Labeled *Min* PM_1 *is not in* $\boldsymbol{APX}$ *in connected planar cubic bipartite graphs, unless* $\boldsymbol{P} = \boldsymbol{NP}$.

Proof: The proof consists of two steps. First, using a quite similar reduction to the one of Corollary 3.2, we prove that Theorem 3.4 also holds for the sub-family $\mathcal{P}'_3$ of $\mathcal{P}_3$ where each vertex has a degree 3, except u and v. Then, we transform any instance of $\mathcal{P}'_3$ into a connected planar cubic bipartite graph.

Let $G = (V, E)$ with $V = \{v_1, \cdots, v_n\}$ and $E = \{e_1, \cdots, e_n\}$ be an instance of vertex cover. We transform any edge $e_j = [x, y]$ into gadget $H(e_j)$ described in Figure 1. All edges of $H(e_j)$, except $[v_{3,j}, l_{j,x}]$ and $[v_{3,j}, l_{j,y}]$ have color c_0. We have $\mathcal{L}([v_{3,j}, l_{j,x}]) = c_x$ and $\mathcal{L}([v_{3,j}, l_{j,y}]) = c_y$. Finally, $H(e_j)$ is linked to $H(e_{j+1})$ using the graph depicted in Figure 2 where each edge is colored with c_0.

Clearly, Labeled *Min* PM_1 is **APX**-hard in class $\mathcal{P}'_3$. Since the restricted label squaring operation also preserves the membership in $\mathcal{P}'_3$, we deduce that Labeled *Min* PM_1 is not in **APX** when the instances are restricted to $\mathcal{P}'_3$. Finally, given $I \in \mathcal{P}'_3$ with $I = (G, \mathcal{L})$, we consider the instance I' where G is duplicated 3 times into G_1, G_2, G_3. If u_i, v_i denote the extreme vertices of G_i, we shrink vertices u_1, u_2, u_3 into u and v_1, v_2, v_3 into v. Clearly, this new graph G' is connected bipartite, planar and cubic. Finally, since we can restrict ourselves to perfect matchings M' of G' that use only color c_0 for exactly two copies of G, the result follows. ∎

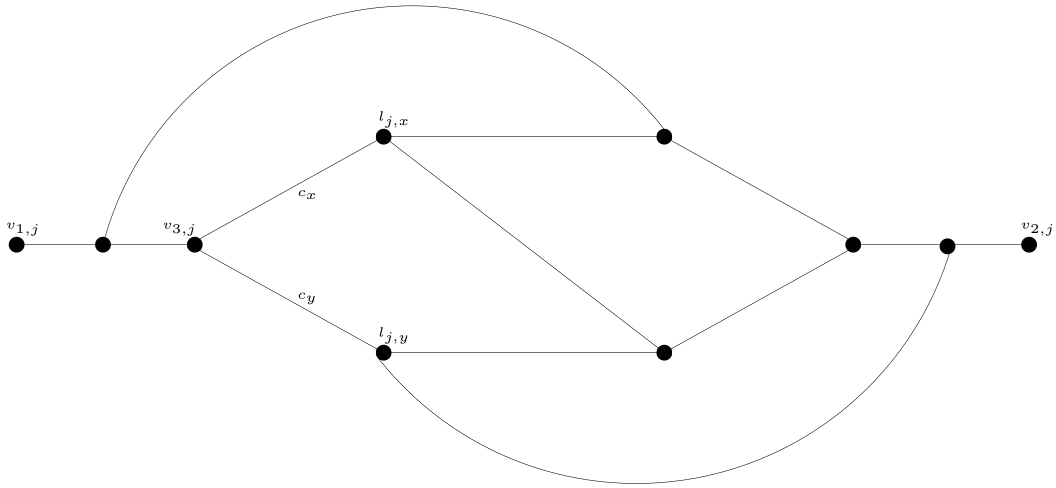

Figure 1: The gadget $H(e_j)$ for $e_j = [x, y]$.

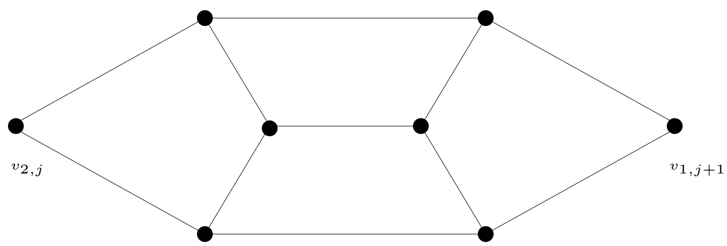

Figure 2: The gadget linking $H(e_j)$ to $H(e_{j+1})$.

Dealing with the unbounded degree case (that is instances of $\mathcal{P}$), we can deduce the following stronger result:

Theorem 3.6 LABELED *Min* PM_1 *for instances in* $\mathcal{P}$ *is not in* ***polyLog-APX*** *, unless* ***P*** $=$ ***NP***.

Proof: Assume the reverse, that is LABELED *Min* PM_1 is $f(n)$-approximable with $f(n) \leq c \log^k n$ for some constants $c > 0$ and $k \geq 1$. Let $I = (H, \mathcal{L}) \in \mathcal{P}$ where H has $2n$ vertices. Let $p = \lceil logk \rceil + 1$. Using as previously 2^p times the restricted label squaring operation on I, we produce in polynomial-time the instance $I^{2^p} = (H^{2^p}, \mathcal{L}^{2^p}) \in \mathcal{P}$. The same arguments as in Theorem 3.4 allow us to obtain a contradiction with Theorem 3.1. ■

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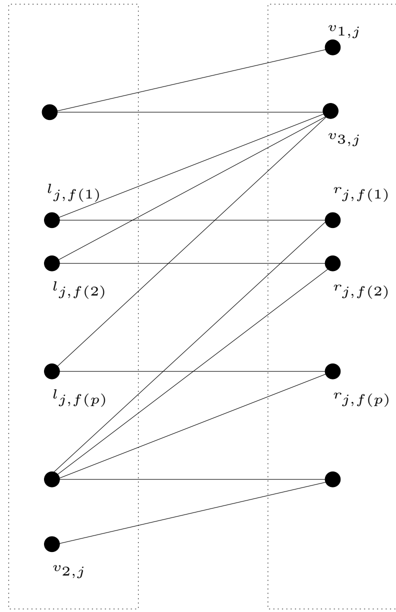

Figure 3: The gadget $H(x_j)$.

Appendix

Proof of Theorem 3.1. Given a family $\mathcal{S} = \{S_1, \dots, S_{n_0}\}$ of subsets of a ground set $X = \{x_1, \dots, x_{m_0}\}$ (we assume that $\cup_{i=1}^{n_0} S_i = X$), a set cover of X is a sub-family $\mathcal{S}' = \{S_{f(1)}, \dots, S_{f(p)}\} \subseteq \mathcal{S}$ such that $\cup_{i=1}^{p} S_{f(i)} = X$; MinSC is the problem of determining a minimum-size set cover $\mathcal{S}^* = \{S_{f^*(1)}, \dots, S_{f^*(q)}\}$ of X. Given an instance $I_0 = (\mathcal{S}, X)$ of MinSC, its characteristic graph $G_{I_0} = (L_0, R_0; E_{I_0})$ is a bipartite graph with a left set $L_0 = \{l_1, \dots, l_{n_0}\}$ that represents the members of the family $\mathcal{S}$ and a right set $R_0 = \{r_1, \dots, r_{m_0}\}$ that represents the elements of the ground set X; the edge-set E_{I_0} of the characteristic graph is defined by $E_{I_0} = \{[l_i, r_j] : x_j \in S_i\}$.

From I_0, we construct the instance $I = (H, \mathcal{L})$ of Labeled $Min\ PM_1$ containing (n_0+1) colors $\{c_0, c_1, \cdots, c_{n_0}\}$, described as follows:

- For each element $x_j \in X_0$, we build a gadget $H(x_j)$ that consists of a bipartite graph of $2(d_{G_{I_0}}(r_j) + 3)$ vertices and $3d_{G_{I_0}}(r_j) + 4$ edges, where $d_{G_{I_0}}(r_j)$ denotes the degree of vertex $r_j \in R$ in G_{I_0}. The graph $H(x_j)$ is illustrated in Figure 3.

- Assume that vertices $\{l_{f(1)}, \dots, l_{f(p)}\}$ are the neighbors of r_j in G_{I_0}, then color $H(x_j)$ as follows: for any $k = 1, \dots, p$, $L(v_{3,j}, l_{j,f(k)}) = c_{f(k)}$ and the other edges receive color c_0.

- We complete $H = \cup_{x_j \in X} H(x_j)$ by adding edges $[v_{2,j}, v_{1,j+1}]$ with color c_0 for $j = 1, \cdots, m_0 - 1$.

- Finally, we set $u = v_{1,1}$ and $v = v_{2,m_0}$.

Clearly, $I \in \mathcal{P}$ and has $2n = 2\sum_{r_j \in R}(d_{G_{I_0}}(r_j) + 3) = 2|E_{I_0}| + 6m_0$ vertices.

Let $\mathcal{S}^*$ be an optimal set cover on I_0. From $\mathcal{S}^*$, we can easily construct a perfect matching M^* of $I = (H, \mathcal{L})$ that uses exactly $(|\mathcal{S}^*| + 1)$ colors. Conversely, let M be a perfect matching on I; by construction, the subset $\mathcal{S}' = \{S_k : c_k \in \mathcal{L}(M)\}$ of $\mathcal{S}$ is a set cover of X using $(|\mathcal{L}(M)| - 1)$ sets.

Now, it is well known that the set cover problem is **NP**-hard to approximate within factor $c \log n_0$ for some constant $c > 0$. This result also applies to instances $(X, \mathcal{S})$ when $|X|$ and $|\mathcal{S}|$ are polynomially related (i.e., $|X|^q \leq |\mathcal{S}| \leq |X|^p$ for some constants p, q).

Hence, given such an instance $I_0 = (X, \mathcal{S})$, from any algorithm A solving Labeled *Min PM*$_1$ within a performance ratio $\rho_{\mathrm{A}}(I) \leq \frac{c}{q+1} \times \log(n)$ for a bipartite graph on $2n$ vertices, we can deduce an algorithm for MinSC that guarantees the performance ratio $c\frac{1}{q+1}\log(n) \leq c\frac{1}{q+1}\log(n_0^{q+1}) = c\log(n_0)$, contradiction.

Proof of Corollary 3.2. Starting from the restriction of set cover where each element x_i is covered by exactly two sets (this case is usually called vertex cover), we apply the same proof as in Theorem 3.1. The instance I becomes an element of $\mathcal{P}_3$, and using for instance the hardness result of [1], the expected result follows.