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Abstract

In this paper, we study the approximability properties of symmetric TSP under an approximation measure called the *differential ratio*. More precisely, we improve up to $3/4 - \varepsilon$ (for any $\varepsilon > 0$) the best differential ratio of 2/3 known so far, given in Hassin and Khuller, "z-Approximations", J. of Algorithms, 41(2), 429-442, 2001.

Keywords: Approximation algorithm; Differential ratio; Traveling salesman problem.

1 Introduction

Due to both its practical and theoretical interests, symmetric TSP is one of the most famous combinatorial optimization problems. Given a complete edge-weighted graph, one seeks a tour (Hamiltonian cycle) either of minimum length (MinTSP) or maximum length (MaxTSP). Shown to be NP-hard in the very early development of the complexity theory ([24]), it has been widely studied since then from an approximate point of view. A polynomial algorithm $\mathtt{A}$ is said to be ρ-approximate if for any instance I, $m(\mathtt{A}(I)) \leq \rho\, opt(I)$ for a minimization problem (resp. $m(\mathtt{A}(I)) \geq \rho\, opt(I)$ for a maximization problem), where $m(x)$ denotes the value of the solution x of I, and $opt(I)$ the optimum value of I.

While MinTSP is not $2^{p(n)}$-approximable where $n = |V|$, for any polynomial p, if $P \neq NP$, MaxTSP is in APX : the well known 3/4-approximation algorithm by Serdyukov [29] has recently been slightly improved up to 61/81 in [8], or even 25/33 using a randomized algorithm [22]. Many classical subcases have been studied, the most famous being the so-called metric case, restriction where the weights satisfy the triangle inequality. Using this assumption, Christofides devised in [9] a 3/2-approximation algorithm for MinMetricTSP, and this is up to now the best ratio obtained. Dealing with MaxMetricTSP, the 3/4-ratio that holds for the general case can be improved up to 17/20 [8], or even 7/8 using a randomized algorithm [23]. Note that all these problems do not admit approximation schemes if $P \neq NP$ [28].

In this article, we further study the approximation properties of symmetric TSP, but using another measure of the quality of a solution called the *differential ratio*. The differential ratio of a solution x of value $m(x)$ is defined as $\delta(x) = \frac{m(x) - wor(I)}{opt(I) - wor(I)}$, where $opt(I)$ is the value of an optimum solution, and $wor(I)$ is the value of a worst solution. For instance, if one studies MaxTSP, then a worst solution is a minimum length tour. In other words,

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this ratio measures the relative position of $m(x)$ in the interval $[wor(I), opt(I)]$ containing all feasible values (the definition can be rephrased for a maximization problem as : the solution x is δ-approximate if $m(x) \geq \delta\, opt(I) + (1-\delta)wor(I)$). Of course, $\delta \in [0,1]$ (0 for $wor(I)$ and 1 for $opt(I)$), and the closer to 1 the better the solution. The main property of this ratio is to be *stable under affine transformation of the objective function* (see [14] for a mathematical and operational justification of the ratio). Introduced in [2, 3], this ratio has been first used for studying mathematical programming problems, where the standard ratio is not suitable when very common operations such as "removing a constant" are performed, see for instance [31]. Afterwards, this approach has been considered for the main combinatorial optimization problems, leading to the development of new techniques and interesting results (see for instance [5] for vehicle routing, [20] for several results on graph problems, [10, 15, 17] for MinColoring, [21] for several weighted versions of graph partitioning, [12, 13] for Bin Packing, [7, 16] for satisfiability, [11, 6] for Set Cover, and very recently [18] for weighted Set Cover, etc.). A survey of many results about differential approximation can be found in the book chapter [4].

Dealing with symmetric TSP, we shall point out two major differences when using the differential ratio instead of the standard one. First, the dissymmetry between maximizing and minimizing completely disappears. More precisely, using an affine transformation of weights ($w(e) \rightarrow w'(e) = M - w(e)$, for a sufficiently large M, as for instance the heaviest weight plus 1), one can easily see that solving MinTSP (resp. MaxTSP) with the initial weights is equivalent to solve MaxTSP (resp. MinTSP) on the transformed weights. Indeed, the value of any tour T verifies $w'(T) = nM - w(T)$. Since the differential ratio is stable under affine transformation, this means that a δ-approximation algorithm for MinTSP (resp. MaxTSP) can be immediately derived from a δ-approximation algorithm for MaxTSP (resp. MinTSP).

The other difference, maybe rather surprising, is the equivalence between the metric case and the general case. While considering a metric distance is a rather important assumption when using the standard ratio, *TSP and MetricTSP are equivalent* when using the differential ratio. Indeed, again, one only has to affinely transform weights $w(e) \rightarrow w_M + w(e)$, where w_M is the weight of an heaviest edge, to get an equivalent metric instance of symmetric TSP.

To sum up, dealing with differential approximation ratios, MinTSP, MaxTSP, MinMetricTSP and MaxMetricTSP are all equivalent. These problems have been tackled several time from a differential approximation point of view. The best ratio obtained so far is 2/3 ([20, 25]), which can be improved up to 3/4 when the weights are restricted to be 1 or 2, [27] (note that in this case the best ratio known for the standard ratio is 7/6, see [28]). Let us also mention that classical optimization strategies have been studied, such as the well known local 2-opt which has been shown in [26] to be a 1/2-differential approximation (while not being a constant standard approximation algorithm even for MinMetricTSP). Note that, as well as in the standard approximation framework, these problems do not admit differential approximation schemes. Finally, dealing with asymmetric TSP, the best differential ratio obtained so far is 1/2 [20].

In this article, we improve these results by showing that symmetric TSP (*i.e.* MinTSP, MaxTSP, MinMetricTSP and MaxMetricTSP) is differential approximable within an asymptotic ratio of 3/4 (more precisely within a ratio of $3/4 - O(1/n)$). Note that this is a noticeable improvement respect to 2/3 also because this is very close to the best ratio known for

MaxTSP (61/81). Since for a maximization problem the differential ratio is smaller than the standard one ($m(x) \geq \delta\, opt(x) + (1-\delta)wor(x)$ implies $m(x) \geq \delta\, opt(x)$, when solution values are nonnegative), the gap is now almost as small as it can be.

Let us already mention that, carrying on with this line of research, the study of symmetric TSP in the geometric case seems to be of particular interest. Indeed, when vertices are points in the plane (and the weight is the Euclidean distance), then it has been shown that both MaxTSP and MinTSP admit an approximation scheme (see resp. [30] and [1]). The existence of a differential approximation scheme is undoubtedly a very interesting and challenging question that would deserve further research.

In the following, we denote by $opt(I)$, $apx(I)$ and $wor(I)$ the value of an optimal, an approximate and a worst solution respectively for an instance I. Due to the equivalence between MaxTSP and MinTSP, the results, only proven for MaxTSP, are obviously also valid for MinTSP. The proof of the result of the paper consists of two parts : in Section 2 we devise a 3/4-differential algorithm when the number of vertices is even. In Section 3, we show that the general case reduces to the previous subcase obtaining asymptotically the same ratio ($3/4 - O(1/n)$ in our case).

2 Approximation for even instances

In this section, we assume that the number of vertices is even (ie $|V| = 2n$), and provide a 3/4-differential approximation for symmetric TSP.

The method used is based on the computation of a maximum weight 2-matching $E_2 = \{C_1, \ldots, C_p\}$ of $I = (K_{2n}, w)$, which can be done in polynomial time, [19]. We separate two cases depending on the existence of a cycle of size 3 in E_2.

Case 1: There exists $j \in \{1, \ldots, p\}$ such that $|C_j| = 3$. Wlog, assume that $j = p$ and $C_p = \{v_1, v_2, v_3\}$.

We present a heuristic which is an adaptation of the Serdyukov's algorithm working for MaxTSP, [29].

Let us first remind this algorithm and the result that can be derived from it (Lemma 2.1). This method consists in computing a maximum weight perfect matching E_1 of I and moving one edge of each cycle C_i of E_2 to E_1 in such a way that we do not create any cycle (see Figure 1 for an illustration). At the end of the process, we obtain two collections of paths $\mathcal{P}_1$ (containing E_1) and $\mathcal{P}_2$ such that:

Lemma 2.1 *The collection of paths $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy the following properties:*

- (i) *$\mathcal{P}_1$ and $\mathcal{P}_2$ are two collections of vertex disjoint paths such that the vertex sets of these collections of paths are exactly $V(K_{2n})$.*
- (ii) *If V_i are the endpoints of the paths of $\mathcal{P}_i$ for $i = 1, 2$, then $V_1 \cup V_2 = V(K_{2n})$ and $V_1 \cap V_2 = \emptyset$.*
- (iii) *Each path P of $\mathcal{P}_1$ alternates between edge of E_1 and E_2 and the end edges of P are in E_1.*

Proof: By construction, (i) is true for $\mathcal{P}_2$.

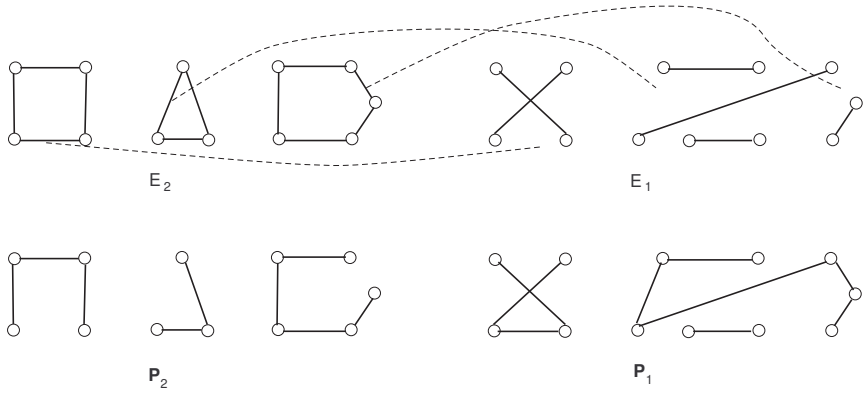

Figure 1: The two partition into paths $\mathcal{P}_1$ and $\mathcal{P}_2$.

Let $\mathcal{P}_2^j$ be the set of the j paths built from $C_1, \ldots, C_j$, *i.e.* after iteration j (and similarly $\mathcal{P}_1^j$ the collection of paths built from E_1 after iteration j). In particular, $\mathcal{P}_k^p = \mathcal{P}_k$, $k = 1, 2$. We will prove the result by induction.

At the beginning (before moving any edge), (ii) and (iii) are true for $\mathcal{P}_1^0$ and $\mathcal{P}_2^0$, and (i) is true for $\mathcal{P}_1^0$. Suppose this is true after iteration $j-1$, and proceed the j^{th} iteration as follows: choose any vertex v in C_j, and consider the two edges $e_1 = [v, a]$ and $e_2 = [v, b]$ incident to v in C_j. v cannot be an internal vertex of a path of $\mathcal{P}_1^{j-1}$ since otherwise v would be incident to 3 edges of E_2 (using (iii)), contradiction. Using (i), we obtain that v is an endpoint of a path P of $\mathcal{P}_1^{j-1}$. Thus, since $a \neq b$, at least one of these two vertices (assume that it is a) is not the other endpoint of P. For the same reason, a is also the endpoint of another path P' of $\mathcal{P}_1$. When we move e_1:

- properties (i) (for $\mathcal{P}_1^j$) and (iii) still hold;
- property (ii) also, since now v and a are new endpoints of a path in $\mathcal{P}_2^j$, but are no more endpoints of paths in $\mathcal{P}_1^j$.

■

Now, we describe our method. It uses the fact that we assume the existence of a triangle $C_p = \{v_1, v_2, v_3\}$ in E_2 in order to apply two times the previous construction, thus producing 4 collections of paths $\mathcal{P}_1$, $\mathcal{P}_2$,$\mathcal{P}'_1$,$\mathcal{P}'_2$, as follows.

We apply once the construction and get a first couple $\mathcal{P}_1$, $\mathcal{P}_2$. Then remark that, at each iteration in the construction, we can choose the vertex v incident to the two edges candidate to move from E_2 to E_1. Then, wlog., assume that when applying the first construction the edge $[v_1, v_2]$ has moved from $C_p \in E_2$ to E_1. To get $\mathcal{P}'_1$,$\mathcal{P}'_2$, we apply exactly the same construction as previously, except for the last cycle. For C_p, we choose to move one of the two edges $[v_1, v_3]$ or $[v_2, v_3]$ incident to v_3 (instead of $[v_1, v_2]$). Wlog, assume that it is $[v_1, v_3]$; using arguments of the proof of Lemma 2.1, we obtain two other collections of paths $\mathcal{P}'_1$ (containing E_1) and $\mathcal{P}'_2$ satisfying properties $(i), (ii)$ and (iii).

Moreover, if V'_i denotes the endpoints of $\mathcal{P}'_i$ for $i = 1, 2$, then we can observe that $v_3 \in V_1$, $v_2 \in V'_1$ and $V_1 \setminus \{v_3\} = V'_1 \setminus \{v_2\}$. Thus, it is possible to complete $\mathcal{P}_1$ into a tour T_1

and $\mathcal{P}'_1$ into a tour T'_1 such that the added edges form an hamiltonian path HP_1 on vertices $V_1 \cup \{v_2\}$ and with endpoints v_2, v_3.

Similarly, we have $v_2 \in V_2$, $v_3 \in V'_2$ and $V_2 \setminus \{v_2\} = V'_2 \setminus \{v_3\}$. Thus, we can also add some edges to $\mathcal{P}_2$ (resp., $\mathcal{P}'_2$) in order to obtain a tour T_2 (resp., T'_2) in such a way that the added edges form an hamiltonian path HP_2 on vertices $V_2 \cup \{v_3\}$ and with endpoints v_2, v_3.

An illustration of this construction is given in Figure 2. For completeness, let us also give a formal proof of this claim for HP_2. Let $V_2 = \{a_i, b_i : i = 1, \dots p\}$ be the endpoints of the paths of $\mathcal{P}_2$, where $[a_i, b_i]$ is the edge that has moved from E_2 to E_1 at iteration i. In particular, we have $a_p = v_1$ and $b_p = v_2$. Similarly, let $V'_2 = \{a'_i, b'_i : i = 1, \dots p\}$ be the endpoints of the paths of $\mathcal{P}'_2$ with $a'_p = v_1 = a_p$, $b'_p = v_3$ and $a'_i = a_i$, $b'_i = b_i$ for $i = 1, \dots, p-1$. The two tours T_2 and T'_2 depend on the parity of p and can be described as follow.

- Assume first that p is odd. We build $T_2 = \mathcal{P}_2 \cup \{[a_i, b_{i+1}] : i = 1, \dots, p\}$ and $T'_2 = \mathcal{P}'_2 \cup \{[b'_i, a'_{i+1}] : i = 1, \dots, p\}$, with $b_{p+1} = b_1$ and $a'_{p+1} = a'_1$. Since $a_p = a'_p$, $b_p = v_2$, $b'_p = v_3$, and $a'_i = a_i$, $b'_i = b_i$ for $i = 1, \dots, p-1$, we deduce that the added edges $HP_2 = \{[a_i, b_{i+1}], [b'_i, a'_{i+1}] : i = 1, \dots, p\}$ form an hamiltonian path from b'_p to b_p described by the sequence $HP_2 = (b'_p, a_1, b_2, \dots, a_p, b_1, a_2, b_3, \dots, b_p)$.

- Now, if p is even, then we only modify T'_2 and define $T'_2 = \mathcal{P}'_2 \cup \{[b'_i, a'_{i+1}] : i = 1, \dots, p-1\} \cup \{[a'_p, a'_1], [b'_p, b'_{p-1}]\}$. As previously, one can easily check that the added edges $HP_2 = \{[a_i, b_{i+1}] : i = 1, \dots, p\} \cup \{[b'_i, a'_{i+1}] : i = 1, \dots, p-1\} \cup \{[a'_1, a'_p], [b'_p, b'_1]\}$ form an hamiltonian path from b'_1 to b_1 described by the sequence $HP_2 = (b'_p, b_1, a_2, b_3, \dots, a_p, a_1, b_2, a_3, \dots, a_{p-1}, b_p)$.

In conclusion, we get 4 tours T_1, T'_1, T_2 and T'_2. By taking the solution of maximum weight with cost $apx(I)$, we obtain:

$$4apx(I) \geq \sum_{i=1}^{2} (w(T_i) + w(T'_i)) = 2(w(E_1) + w(E_2)) + (w(HP_1) + w(HP_2)) \qquad (1)$$

On the one hand, we have $w(E_2) \geq opt(I)$ and $w(E_1) \geq opt(I)/2$, and on the other hand since $HP_1 \cup HP_2$ is a tour on K_{2n}, we get $w(HP_1) + w(HP_2) \geq wor(I)$. Plugging these inequalities with inequality (1), we deduce:

$$apx(I) \geq \frac{3}{4} opt(I) + \frac{1}{4} wor(I) \qquad (2)$$

Case 2: For all $j \in \{1, \dots, p\}$ we have $|C_j| \geq 4$. In this case, we extend the method proposed in [20, 25]. We study 2 subcases depending on the parity of p.

Case 2.1: If p is odd. Obviously, we can assume $p > 1$. For each cycle C_i of the 2 matching $E_2 = \{C_1, \dots, C_p\}$, we consider 4 consecutive edges $A_i = \{[a_i, b_i], [b_i, c_i], [c_i, d_i], [d_i, f_i]\}$ [1](with eventually $f_i = a_i$ if $|C_j| = 4$), and we produce 4 solutions by starting from E_2: the first solution T_a deletes the edge $[a_i, b_i]$ for each cycle C_i with $i = 1, \dots, p$, and for each

[1]we denote the fourth vertex by f in order to avoid confusion with edges, denoted e in this article

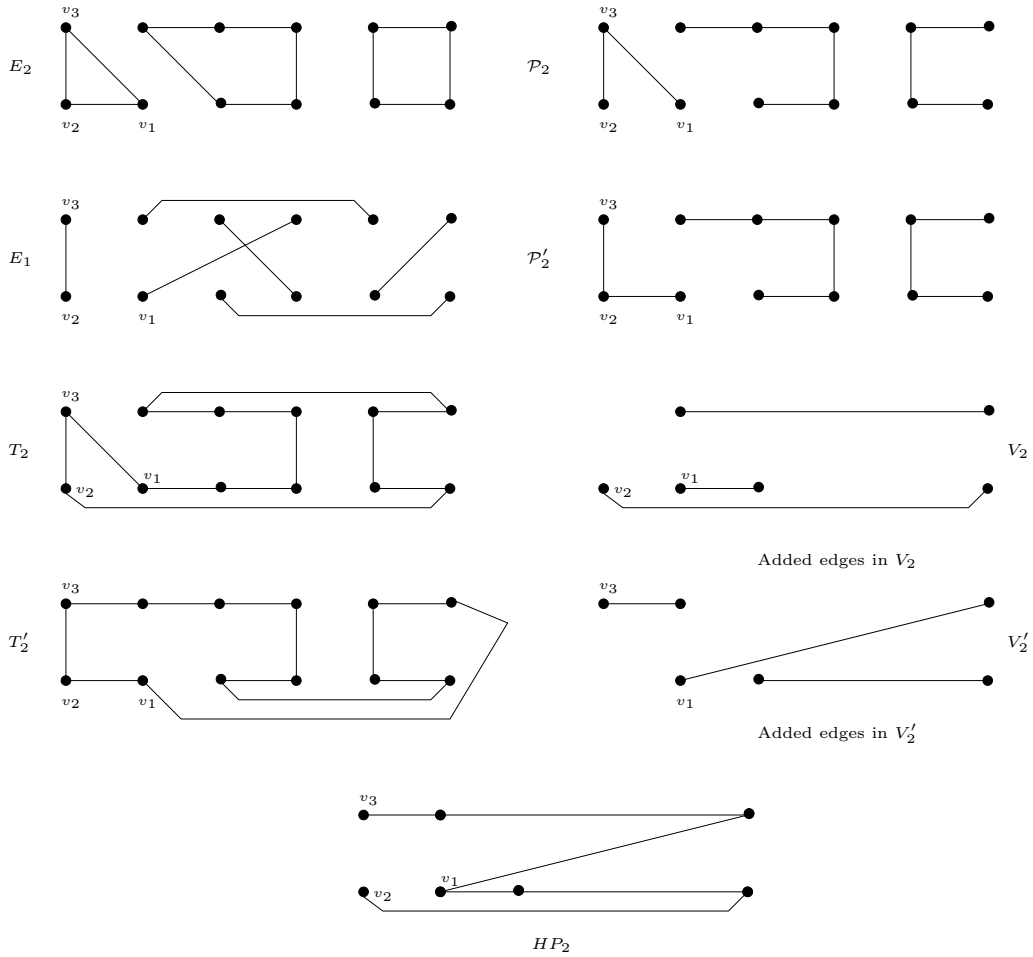

Figure 2: The construction of tours T_2 and T_2' and the hamiltonian path HP_2.

$i \in \{1, \ldots, p-1\}$ adds the edge $[a_i, a_{i+1}]$ if i is odd, adds the edge $[b_i, b_{i+1}]$ if i is even and finally adds the edge $[a_p, b_1]$. The 3 other solutions T_b, T_c and T_d are described similarly by deleting edges $[b_i, c_i], [c_i, d_i]$ and $[d_i, f_i]$ respectively. In particular, for the last solution T_d, we have added for $i \in \{1, \ldots, p-1\}$ the edges $[d_i, d_{i+1}]$ if i is odd, $[f_i, f_{i+1}]$ if i is even and finally edge $[d_p, f_1]$.

In the multigraph of these 4 solutions, that is $(V, T_a + T_b + T_c + T_d)$, each edge e of $A = \cup_{i=1}^{p} A_i$ appears exactly 3 times whereas the other edges of E_2 appears exactly 4 times. On the other hand, the edges of $B = (T_a \cup T_b \cup T_c \cup T_d) - E_2$ appears one time.

Thus, by considering the best of the 4 solutions we produced, we get: $4apx(I) \geq 3w(A) + 4w(E_2 - A) + w(B) = 3w(E_2) + w(B + E_2 - A) \geq 3opt(I) + w(B + E_2 - A)$.

Now, remark that $B + E_2 - A$ is a tour of I. Indeed, $E_2 - A$ contains paths $(f_i, g_i, \ldots, a_i)$ for each $i = 1, \ldots, p$. In B, we get a path $P = (a_p, b_1, b_2, \ldots b_p, c_1, \ldots, c_p, d_1, \ldots, d_p, f_1)$, and edges $[a_i, a_{i+1}]$ (i odd) or $[f_i, f_{i+1}]$ (i even). These edges and the paths of $E_2 - A$ create a path from f_1 to a_p, which constitutes together with P a tour. Hence, $w(B + E_2 - A) \geq wor(I)$. We get:

$$apx(I) \geq \frac{3}{4} opt(I) + \frac{1}{4} wor(I) \tag{3}$$

Case 2.2: If p is even, then the previous construction does not produce a tour. We adapt it as follows. As previously, we consider 4 consecutive edges $A_i = \{[a_i, b_i], [b_i, c_i], [c_i, d_i], [d_i, f_i]\}$ in cycle C_i of the 2-matching E_2, except for the last cycle C_p where we replaced edge $[d_p, f_p]$ by $[z_p, a_p]$ with z_p is the other neighbor of a_p in C_p (eventually, $z_p = d_p$ if $|C_p| = 4$). Thus, $A_p = \{[z_p, a_p], [a_p, b_p], [b_p, c_p], [c_p, d_p]\}$.

Moreover, for C_2, we do not choose consecutive vertices a_2, b_2, c_2, d_2, f_2 at random. We choose them such that:

$$w([a_1, b_2]) + w([a_2, b_1]) \leq w([a_1, a_2]) + w([b_1, b_2]) \tag{4}$$

Actually, this is always possible since otherwise for all $e = [x, y] \in C_2$ we would get $w([a_1, y]) + w([a_2, x]) < w([a_1, x]) + w([b_1, y])$ (here, we assume that each edge $e = [x, y]$ is considered as a directed edge where the orientation is given when one walks around C_2). Summing up the previous inequality for each edge $e \in C_2$, we obtain the inequality $\sum_{v \in V(C_2)} (w([a_1, v]) + w([b_1, v])) > \sum_{v \in V(C_2)} (w([a_1, v]) + w([b_1, v]))$, contradiction.

Then, we produce 4 tours T_a, T_b, T_c and T_d as follows: first, T_a deletes from E_2 edges $[a_i, b_{i+1}]$ if $i < p$ and $[z_p, a_p]$; then, solution T_a adds edges $[a_i, a_{i+1}]$ if $i < p$ is odd, adds the edge $[b_i, b_{i+1}]$ if $i < p$ is even and finally adds the edge $[z_p, b_1]$. The other tours T_b, T_c and T_d are constructed similarly. In particular, T_d deletes edges $[d_i, f_i]$ if $i < p$ and $[z_p, a_p]$, adds for $i \in \{1, \ldots, p-1\}$ edges $[d_i, d_{i+1}]$ if i is odd, $[f_i, f_{i+1}]$ if i is even and edge $[c_p, f_1]$.

As previously, in the multigraph $(V, T_a + T_b + T_c + T_d)$, each edge e of $A = \cup_{i=1}^{p} A_i$ appears exactly 3 times whereas the other edges of E_2 appears exactly 4 times. On the other hand, the edges of $B = (T_a \cup T_b \cup T_c \cup T_d) - E_2$ appears one time. Thus, by considering the best of these 4 solutions, we get:

$$4apx(I) \geq 3w(A) + 4w(E_2 - A) + w(B) = 3w(E_2) + w(B + E_2 - A) \tag{5}$$

However, now $B + E_2 - A$ is not a tour of I, but a 2-matching constituted by two cycles. The first one is $(b_1, b_2, \ldots, b_p, d_1, \ldots, d_p, g_1, \ldots, g_p, b_1)$, constituted of edges in B ; the second one is constituted by the path $(a_p, c_1, \ldots, c_p, f_1)$ of B, the paths $(f_i, g_i, \ldots, z_i, a_i)$ of $E_2 - A$, and edges $[a_i, a_{i+1}]$ (i odd) or $[f_i, f_{i+1}]$ (i even).

But using inequality (4), one can flip edges $[a_1, a_2], [b_1, b_2]$ by edges $[a_1, b_2], [a_2, b_1]$ without increasing the global weight and one obtain a tour T such that $wor(I) \leq w(T) \leq w(B + E_2 - A)$. In conclusion, using this inequality and inequality (5), we obtain:

$$apx(I) \geq \frac{3}{4} opt(I) + \frac{1}{4} wor(I) \tag{6}$$

Combining the results obtained in cases 1 (equation (2)) and 2 (equation (6)), we obtain the following result.

Theorem 2.2 *When the number of vertices is even, symmetric TSP is 3/4-differential approximable.*

3 General case

In the previous section, we dealt with even instances. Here, we show that one can solve symmetric TSP also on odd instances within an asymptotic differential ratio of 3/4.

Theorem 3.1 *In the general case, symmetric TSP can be differential approximated with ratio* $3/4 - O(1/n)$.

Proof: From the discussion above, we have to deal with instances the number of vertices of which is odd. In this case, we find a $(3/4 - O(1/n))$-approximate solution using the previous result on even instances. Let n odd, $I = (K_n, w)$ an instance of symmetric TSP and denote $V = \{v_1, \ldots, v_n\}$ the set of vertices.

We find an approximate solution as follows: for each $i \in \{1, \ldots, n\}$, we consider the sub-instance I_i on the subgraph induced by $V \setminus \{v_i\}$. On this instance, we apply our approximation method given above and get a tour T_i. Then, we insert v_i in the best position in T_i, thus producing a tour T_i' on I. Finally, we take the best tour T among these n tours T_i', *i.e.* $apx(I) = w(T) = \max_{i=1,\ldots,n} w(T_i')$.

Note that, when inserting vertex v_i in T_i between two vertices v_j and v_k (consecutive in T_i), we get a tour of value $w(T_i) + w([v_j, v_i]) + w([v_i, v_k]) - w([v_j, v_k])$. Since we take the best of these nodes, by considering the $n-1$ possible insertions, we get:

$$\begin{aligned}(n-1)w(T_i') &\geq (n-1)w(T_i) + 2\sum_{k,k\neq i} w([v_i, v_k]) - w(T_i) \\ &\geq (n-2)w(T_i) + 2\sum_{k,k\neq i} w([v_i, v_k])\end{aligned}$$

Since we take the best tour among the T_i''s, we get:

$$n(n-1)apx(I) \geq (n-2)\sum_{i=1}^{n} w(T_i) + 2S \tag{7}$$

where $S = \sum_{i=1}^{n}\sum_{k,k\neq i} w([v_i, v_k])$ is twice the total weight of all edges in the graph.

Similarly, by inserting v_i in any position in a worst tour on I_i, we get a tour on I. The worst solution on I is of course worse than each of these solutions, *i.e.*:

$$\begin{aligned}(n-1)wor(I) &\leq (n-1)wor(I_i) + 2\sum_{k,k\neq i} w([v_i, v_k]) - wor(I_i) \\ &\leq (n-2)wor(I_i) + 2\sum_{k,k\neq i} w([v_i, v_k])\end{aligned}$$

Hence:

$$n(n-1)wor(I) \leq (n-2)\sum_{i=1}^{n} wor(I_i) + 2S \tag{8}$$

Finally, consider an optimum solution $(v_1^*, v_2^*, \ldots, v_n^*)$ on I. By deleting v_i^* in this tour, we get a tour on I_i the value of which is $opt(I) - w([v_i^*, v_{i-1}^*]) - w([v_i^*, v_{i+1}^*]) + w([v_{i-1}^*, v_{i+1}^*]) \leq opt(I_i)$. By considering each of the possible deletion, we get (obviously v_0^* means v_n^* and v_{n+1}^* means v_1^*):

$$n \times opt(I) - 2\sum_{i=1}^{n} w([v_i^*, v_{i+1}^*]) + \sum_{i=1}^{n} w([v_{i-1}^*, v_{i+1}^*]) \leq \sum_{i=1}^{n} opt(I_i)$$

Since n is odd, $\sum_{i=1}^{n} w([v_{i-1}^*, v_{i+1}^*])$ is the value of a tour, hence at least $wor(I)$. Then:

$$(n-2) \times opt(I) + wor(I) \leq \sum_{i=1}^{n} opt(I_i) \tag{9}$$

Now, using equations (7), (8), (9) and the fact that $w(T_i) \geq (3opt(I_i) + wor(I_i))/4$, we get:

$$\begin{aligned} 4n(n-1)apx(I) &\geq 3(n-2)\sum_{i=1}^{n} opt(I_i) + (n-2)\sum_{i=1}^{n} wor(I_i) + 8S \\ &\geq 3(n-2)^2 opt(I) + 3(n-2)wor(I) + n(n-1)wor(I) + 6S \end{aligned}$$

Finally, recall that S is twice the total weight of the $n(n-1)/2$ edges of the graph. But by symmetry, the medium value of all the tours on the graph is equal to n times the medium value of the edges, *i.e.* $n \times S/(n(n-1))$. This medium value of the tours is of course greater than the worst value. Hence, $wor(I) \leq S/(n-1)$. This leads to:

$$4(n^2-n)apx(I) \geq 3(n^2-4n+4)opt(I) + \left(n^2+8n-12\right) wor(I)$$

This is $apx(I) = (3/4 - \alpha(n))opt(I) + (3/4 + \alpha(n))wor(I)$, where $\alpha(n) = (9n-12)/(4n^2 - 4n) = O(1/n)$ (remark that $4(n^2-n) = 3(n^2-4n+4) + (n^2+8n-12)$). ∎

Let us remark that Theorem 3.1 also holds for any ρ-differential approximation of symmetric TSP: any ρ-differential approximation algorithm of symmetric TSP on even instances can be polynomially converted in a $\rho(1-\alpha(n))$-differential approximation of symmetric TSP (working on any instance) where we recall that $\alpha(n) = (9n-12)/(4n^2-4n) = O(1/n)$.

An interesting open question is to know whether one can improve the differential ratio of 1/2 for asymmetric TSP given in [20] using similar ideas.

References

[1] S. Arora. Polynomial time approximation scheme for Euclidean TSP and other geometric problems. In *Proceedings of the 37th Ann. IEEE Symposium on Foundations of Computer Science*, pages 2–11. IEEE Computer Society, 1996.

[2] G. Ausiello, A. D'Atri, and M. Protasi. Structure preserving reductions among convex optimization problems. *J. Comput. Syst. Sci.*, 21(1):136–153, 1980.

[3] G. Ausiello, A. Marchetti-Spaccamela, and M. Protasi. Toward a unified approach for the classification of NP-complete optimization problems. *Theor. Comput. Sci.*, 12:83–96, 1980.

[4] G. Ausiello and V. Th. Paschos. *Differential ratio approximation*, chapter 16 in Handbook of Approximation Algorithms and Metaheuristics, Teofilo F. Gonzalez (Ed). Taylor and Francis, 2007.

[5] C. Bazgan, R. Hassin, and J. Monnot. Approximation algorithms for some vehicle routing problems. *Discrete Applied Mathematics*, 146(1):27–42, 2005.

[6] C. Bazgan, J. Monnot, V. T. Paschos, and F. Serrière. On the differential approximation of min set cover. *Theor. Comput. Sci.*, 332(1-3):497–513, 2005.

[7] C. Bazgan and V. Th. Paschos. Differential approximation for optimal satisfiability and related problems. *European Journal of Operational Research*, 147(2):397–404, 2003.

[8] Z.-Z. Chen, Y. Okamoto, and L. Wang. Improved deterministic approximation algorithms for Max TSP. *Inf. Process. Lett.*, 95(2):333–342, 2005.

[9] N. Christofides. Worst-case analysis of a new heuristic for the traveling salesman problem. Technical report 338, Grad. School of Industrial Administration, CMU, 1976.

[10] M. Demange, P. Grisoni, and V. T. Paschos. Approximation results for the minimum graph coloring problem. *Inf. Process. Lett.*, 50(1), 1994.

[11] M. Demange, P. Grisoni, and V. Th.. Paschos. Differential approximation algorithms for some combinatorial optimization problems. *Theoretical Computer Science*, 209(1-2):107–122, 1998.

[12] M. Demange, J. Monnot, and V. Th.. Paschos. Bridging gap between standard and differential polynomial approximation : the case of bin-packing. *Applied Mathematics Letters*, 12:127–133, 1999.

[13] M. Demange, J. Monnot, and V. Th.. Paschos. Maximizing the number of unused bins. *Foundations of Computing and Decision Sciences*, 26(2):169–186, 2001.

[14] M. Demange and V. Th.. Paschos. On an approximation measure founded on the links between optimization and polynomial approximation theory. *Theoretical Computer Science*, 158(1-2):117–141, 1996.

[15] R. Duh and M. Fürer. Approximation of k-set cover by semi-local optimization. In *STOC*, pages 256–264, 1997.

[16] B. Escoffier and V. Th.. Paschos. Differential approximation of MIN SAT, MAX SAT and related problems. *European Journal of Operational Research*, 181(2):620–633, 2007.

[17] M. M. Halldórsson. Approximating discrete collections via local improvements. In *Proceedings of the Symposium on Discrete Algorithms*, pages 160–169, 1995.

[18] M. M. Halldórsson and E. Losievskaja. Independent sets in bounded-degree hypergraphs (to appear). In *Proceedings of the Algorithms and Data Structures*, 2007.

[19] D. Hartvigsen. *Extension of Matching Theory*. PhD thesis, Carnegie-Mellon University, 1984.

[20] R. Hassin and S. Khuller. z-approximations. *J. of Algorithms*, 41(2):429–442, 2001.

[21] R. Hassin and J. Monnot. The maximum saving partition problem. *Operations Research Letters*, 33:242–248, 2005.

[22] R. Hassin and S. Rubinstein. Better approximations for max TSP. *Inf. Process. Lett.*, 75(4):181–186, 2000.

[23] R. Hassin and S. Rubinstein. A 7/8-approximation algorithm for metric Max TSP. *Inf. Process. Lett.*, 81(5):247–251, 2002.

[24] R. M.. Karp. Reducibility among combinatorial problems. In R. E.. Miller and J. W.. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, 1972.

[25] J. Monnot. Differential approximation results for the traveling salesman and related problems. *Inf. Process. Lett.*, 82(5):229–235, 2002.

[26] J. Monnot, V. T. Paschos, and S. Toulouse. Approximation algorithms for the traveling salesman problem. *Mathematical Models of Operations Research*, 56:387–405, 2002.

[27] J. Monnot, V. T. Paschos, and S. Toulouse. Differential approximation results for traveling salesman problem with distance 1 and 2. *European Journal of Operational Research*, 145(3):537–548, 2003.

[28] C. H. Papadimitriou and M. Yannakakis. The traveling salesman problem with distances one and two. *Mathematics of Operations Research*, 18:1–11, 1993.

[29] A. I. Serdyukov. An algorithm with an estimate for the traveling salesman problem of the maximum. *Upravlyaemye Sistemy (in Russian)*, 25:80–86, 1984.

[30] A. I.. Serdyukov. An asymptotically exact algorith for the traveling salesman problem for a maximum in euclidian space. *Uppravlyaemye Sistemy (in Russian)*, 27:79–87, 1987.

[31] S. A. Vavasis. Approximation algorithms for indefinite quadratic programming. *Math. Program.*, 57:279–311, 1992.