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A NEW HEURISTIC METHOD FOR SOLVING GLOBAL OPTIMIZATION PROBLEM

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Une nouvelle méthode heuristique pour l'optimisation globale

Résumé :

On propose une méthode heuristique pour calculer l'optimum global d'un problème de programmation non linéaire mono-objectif dans lequel la fonction économique ainsi que les contraintes ne sont pas nécessairement convexes. La méthode utilise les concepts de base de l'analyse multicritère et décompose le problème initial en un nombre fini de sous-problèmes. La solution optimale du problème original est contenue dans l'ensemble des solutions réalisables du nouveau problème.

Mots-clés : Programmation non linéaire, optimisation globale, analyse multi-objectif.

A new heuristic method for solving global optimization problem

Abstract :

Most of the mathematical methods in nonlinear programming problems generally yield local solutions. In the following, we propose a heuristic method (GMC or Global optimization with MultiCriteria analysis) to compute the *global solution* to optimization problems which are not necessarily convex. This method uses multicriteria analysis concepts. The original problem is decomposed into a finite number of subproblems generated by a certain combination of constraints and the objective function. The properties of the initial problem are not modified by this solution technique. The set of solutions of the new problem obtained from this decomposition contains the global solution to the original problem, which can finally be found by a selection routine.

Keywords : Nonlinear programming, global optimization, multicriteria analysis.

1. Introduction :

The present work is devoted to the investigation of the global solution of the following mono-objective optimization problem : find $x_g^* \in X$ such that :

$$\begin{aligned} f(x_g^*) &= \text{Global Max } f(x) \\ \text{s.t. } x &\in X \end{aligned} \quad (1)$$

We propose a heuristic method based on the concepts of the Multiobjective Mathematical Programming (MMP) in order to solve problem (1). MMP allows us to consider, explicitly and simultaneously, multiple objectives in a mathematical programming framework. The basic idea consists in transforming problem (1) into a standard multiobjective formulation (2) :

$$\begin{aligned} &\text{Max } f_1(x) \\ &\text{Max } f_2(x) \\ &\quad \dots \quad \quad \quad \text{or} \quad \quad \quad \text{Max } F(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ &\text{Max } f_p(x) \quad \quad \quad \text{s.t. } F(x) \in S \\ \text{s.t. } x &\in X \quad \quad \quad \text{where :} \end{aligned} \quad (2)$$

- $f(x) = f_1(x) + f_2(x) + \dots + f_p(x)$; p is the number of objectives.
- $x \in R^n$, $x = (x_1, x_2, \dots, x_n)^t$ is the vector of decision variables;
- $F(x)$ is the vector of objectives to be maximized;
- X is the feasible region in the decision variables space defined by the following inequalities $X = \{ x \in R^n / g_j(x) \leq 0 ; j = 1, \dots, m \}$;
- X is a compact set included in the hyperrectangle defined by the following lower/upper bounds $\alpha_i \leq x_i \leq \beta_i ; i = 1, \dots, n$;
- $S = F(X) = \{ F(x) / x \in X \}$; S is the image of X by F in the criterion space;
- all the nonlinear functions f_k ($k = 1, \dots, p$) and g_j ($j = 1, \dots, m$) are defined over R^n and differentiable everywhere ; $f, g : X \subset R^n \rightarrow R$, f_i and g_j are of polynomial type;
- each objective f and f_i is bounded above X ;
- the sets X and S are not necessarily convex ;
- the functions f_i and g_j are not necessarily convex.

Then, we use the concept of "*globally efficient solution*" so as to compute the "*best compromise solution*" of problem (2) which corresponds to the global solution of problem (1). This strategy provides a mean to find "good" starting points for the "local optimizer" to be used in the global optimization phase (GRG, Generalized Reduced Gradient, ABADIE, (1978)) and to accelerate convergence to the final solution of (1). This paper is organized as follows. In section 2, we define the problem and we discuss some specific questions and possible heuristics to resolve it. In section 3, we outline the method to be used. A brief report on computational experiments is given for small problems. Section 4 is devoted to conclusion and further developments.

2. Global optimization and efficiency :

Generally, the algorithms which solve problem (1) converge to a local solution as opposed to global solution (Fig. 1) , i.e. a point $x^* \in X$ such that : $f(x^*) \geq f(x) \quad \forall x \in V(x^*) \cap X$, where $V(x^*)$ is a vicinity of x^* ; x^* is the "global solution" of the problem:

$$\begin{aligned} & \text{Max } f(x) \\ & \text{s.t. } x \in V(x^*) \cap X \end{aligned}$$

A solution $x_g^* \in X$ is said to be a "global maximum" of (1) if and only if x_g^* maximizes f over X ; i.e. $f(x_g^*) \geq f(x) \quad \forall x \in X$.

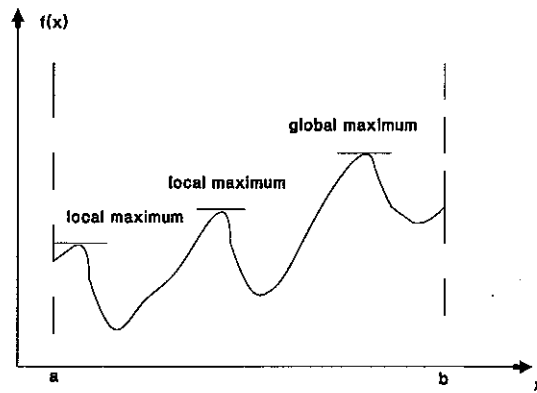


Fig.1: Global and local solutions ; $X = [a, b]$

We can resume our methodology by the following scheme :

- a) (Recursive) decomposition of f ; $f(x) = f_1(x) + f_2(x)$;
- b) Definition of the MMP problem :

$$\begin{aligned} &\text{Max } f_1(x) \\ &\text{Max } f_2(x) \\ &\text{s.t. } x \in X \end{aligned}$$
- c) Multicriteria analysis ; transformation of the MMP problem (more precisely transformation of X) ;
- d) Determination of the representative subset of the set of "globally efficient solutions" of problem (2) ;
- e) Global optimization based on the "globally efficient solutions" as starting points for the "local optimizer" ; improvement of the objective value function by filtering technique in order to eliminate "local solutions".
- f) Determination of the "global solution" of the initial problem (1) with a fixed probability p_g (TÖRN, (1983)).

This approach reveals the following characteristics :

- i) the first point is the data structure inherent to the decomposition procedure (ABADIE and M'SILTI, (1988); M'SILTI and TOLLA, (1992))¹;
- ii) the second point is the computation of the bounds of the objective functions f and f_i , and the transformation of the feasible region X ;
 $X' = \{ x \in X / f_1(x) + f_2(x) \geq K - \varepsilon ; f_i(x) \geq K_i - \varepsilon_i , i = 1, \dots, p \}$; we use the interval analysis to compute these bounds.

Fundamental to the MMP is the Pareto optimal solution (so-called efficient solution). Let $x \in X$ and $y \in X$ be two feasible solutions to the problem (1) in the decision variables space, and $u = F(x)$, $v = F(y)$ the corresponding points in the criterion space (Fig.2).

¹ Classification of variables ; relation existing between f_1 and f_2 , $f_1/f_2 = \lambda$ with $\lambda \geq 0$ except in the set of measure null. It seems clear that this condition of proportionality does not maintain the differentiability and continuity properties of the objective functions.

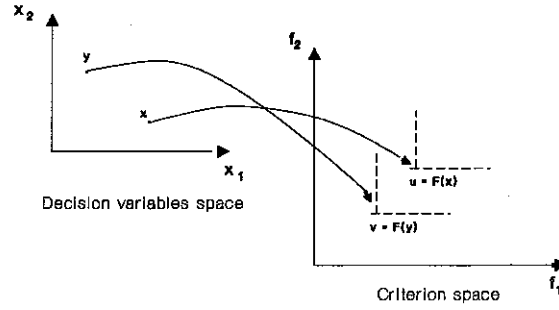


Fig.2 : notion of non dominance

The notion of efficiency refers to feasible points in the decision variables space whereas that of non-dominance concerns vectors in the criterion space. It follows from the definition that v is strongly dominated by u ; x is said "Pareto optimal solution" if there does not exist another point w in the cone of vertex u which dominates u . Mathematically, there exist different definitions of the Pareto optimality (WIGHT,1982). Here, we focus on the concept of a strongly efficient solution (Fig.3).

Definition 1 : Globally and locally efficient solutions

Given the set X of feasible solutions in the decision variables space, a solution $x \in X$ is said to be "*locally efficient*" if there exists a vicinity $V(x)$ of x , such that the following conditions cannot be verified for any $y \in V(x) \cap X$:

$$\begin{aligned} \exists r \in \{ 1, 2, \dots, p \} \\ f_r(y) > f_r(x) \\ f_k(y) \geq f_k(x) \quad \forall k \neq r \end{aligned}$$

The solution x is said to be "*globally efficient*" if $V(x) \equiv \mathbb{R}^n$.

Definition 2 : Globally or locally non dominated solutions

A solution $u \in S = F(X)$, in the criterion space, is said to be "*locally non dominated*" if there exists no point v , $v \in V(u) \cap S$, where $V(u)$ is a vicinity of u , and $v \geq u$ with $v \neq u$. The solution u is said to be "*globally non dominated*" if $V(u) \equiv \mathbb{R}^p$.

Note that, in the convex case (X and S convex), any locally efficient solution (respectively locally non dominated) is globally efficient (respectively globally non dominated). A very important property of the set of non dominated solutions (respectively efficient solutions) is the following : all the efficient (respectively non dominated) solutions are localized in the frontier of X (respectively S). Conceptually, an efficient solution of MMPs is the one where any improvement of one objective function value can be achieved only at the expense of another criterion (Fig. 3). Note that generally :

- i) the set of efficient (respectively non dominated) solutions is infinite ;
- ii) there is no point $x^* \in X$ at which all objectives are simultaneously maximized.

In such a case, this point, called a "superior solution", is the solution of problem (1), i.e $x^* \in X$ and $F(x^*) \geq F(x) \forall x \in X$.

Definition 3 : Ideal point

The ideal point for the MMP problem (2) is a point in the criterion space denoted by F_G^*

$$F_G^* = (f_{G1}^*, f_{G2}^*, \dots, f_{Gp}^*)^t \in R^p$$

and such that the i^{th} component of F_G^* is the global optimum for the following problem (Pi):

$$\begin{aligned} f_{Gi}^* &= \text{Global Max } f_i(x) \\ \text{s.t. } &x \in X \end{aligned} \quad (Pi)$$

Assuming that $f_i(X)$ is closed in R , we have :

$$f_{Gi}^* = f_i(x_{Gi}^*)$$

where

$$x_{Gi}^* \in X$$

is the global solution of problem (Pi). Generally, the ideal point is not feasible, i.e it is out of S in the criterion space (Fig. 3).

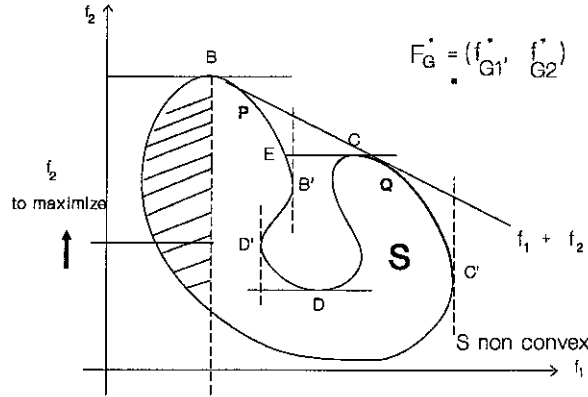


Fig.3 : Non dominated solutions in the criterion space

- the points B, C, D have horizontal tangents;
- the points B', C', D' have vertical tangents;
- the points P, Q have a common tangent.

The arcs BE and CC' are "*globally non dominated*". The arcs EB' and DD' are "*locally non dominated*". The arcs B'D' and DC are over the frontier of S, but they are neither locally nor globally non dominated. The major problem is that it is impossible, except in the particular case of convex mathematical programs, to characterize the global solution of problem (1). However, we can see that the set of solutions of problem (2) contains the set of solutions of problem (1). Indeed, let $x \in X$ be a globally efficient solution for (2), then x is a global optimum for (1). The demonstration is immediate. It follows from the definition 1 that there exists no $y \in V(x) \equiv X$ such that the following conditions are verified :

$$\begin{aligned} f_r(y) &> f_r(x) \\ f_k(y) &\geq f_k(x) \quad \forall k \neq r \end{aligned}$$

Thus, there exists no $y \in X$ such that :

$$f_1(y) + f_2(y) + \dots + f_p(y) > f_1(x) + f_2(x) + \dots + f_p(x)$$

or equivalently, there exists no $y \in X$ such that : $f(y) > f(x)$. In other words, if x is a globally efficient solution of (2), we have $f(x) > f(y) \quad \forall y \in X$, i.e x is a global maximum of (1). The reciprocal is false. The difficulty is the following. If x is a global maximum of (1), then the corresponding point $u = F(x)$, in the criterion space, must

dominate all the points in S so as to be globally non dominated. Now, such a point is generally not feasible in the criterion space, i.e $u \notin S$; the reason is that we have the following inequality :

$$\text{Max } f_1 + \text{Max } f_2 \geq \text{Max } (f_1 + f_2)$$

However, this argument allows us to search for heuristic methods based on the MMP formulation (2) in order to solve (1). More precisely, we use this idea so as to compute the global solution of (1) by "geometrical considerations" in the set S in the criterion space. The problem :

$$\begin{array}{ll} \text{Max } (f_1(x) + f_2(x)) & \text{Max } (\lambda_1 \cdot f_1(x) + \lambda_2 \cdot f_2(x)) \\ \text{s.t. } x \in X & \text{or s.t. } x \in X \\ & \lambda_1, \lambda_2 > 0 ; \lambda_1/\lambda_2 \approx 1 \end{array}$$

is solved by a special routine (local optimizer, GRG) using a globally efficient solution as starting point. This globally efficient solution is computed from a "*reference point*" in the criterion space, itself obtained as a by-product of the interval analysis, by an appropriate algorithm based on the augmented weighted Tchebycheff metric. After this stage, a new strategy (filtering technique) allows us to improve the value of the objective function f (if it is possible) by exploding in details the efficient frontier in the vicinity of the current solution x^0 and to decide if x^0 is a global maximum of (1) or not. If the answer is positive, the process is stopped; if not, it is repeated.

In most cases, there will be many efficient solutions. One of these ones will be preferred in terms of its outcome in the criterion space. It corresponds to the global solution of (1). However, determining what this solution is requires further information about its vicinity and about the characteristics of the objective function f and the type of decomposition selected. Several computational methods have been proposed so as to characterize Pareto optimal solutions (STEUER, 1986). We consider among them the parametric method (so called Lagrange multipliers method), ϵ -constraint method and the augmented weighted Tchebycheff metric. Figure 3 illustrates the application of the parametric and ϵ -constraint methods in the case where S is not convex and f_2 is the principal criteria (for the ϵ -constraint method).

a) Parametric method :

A global parametric method (PM), i.e with global maximization, will find the arcs BP and QC' (Fig.3) :

$$\begin{aligned} & \text{Max } \{ \lambda_1.f_1(x) + \lambda_2.f_2(x) + \dots + \lambda_p.f_p(x) \} \\ \text{s.t. } & x \in X \\ & \lambda_k > 0, k = 1, \dots, p \end{aligned} \quad (\text{PM})$$

A parametric method with local maximization will find perhaps, beyond BP and QC', the arcs PB' and QC ; therefore, it allows us to find BB' and CC'. Note that the solution of (PM) always gives an efficient solution : if $x^0 \in X$ maximizes $\sum_1^p \lambda_k.f_k(x)$ with $\lambda_k > 0$, then x^0 is efficient. The reciprocal is true under certain convexity assumptions (S convex).

b) ϵ -constraint method :

Suppose that f_j is the principal criteria. Then, if $x^0 \in X$ is the solution of problem (CT), x^0 is efficient and reciprocally :

$$\begin{aligned} & \text{Max } f_j(x) \\ \text{s.t. } & x \in X \\ & f_k(x) \geq \Delta_k \quad k \neq j, k = 1, \dots, p \end{aligned} \quad (\text{CT})$$

We can see that with the global maximization, the method will find BE and CC' and that, with local maximization, it will find, perhaps, beyond BE and CC', the arcs EB' and DD'. Thus, with this method, we obtain the arcs BB', CC' and DD'. Consequently, by solving (CT), all the Pareto optimal solutions (locally or globally efficient) to the MMPs involving non convexities can be found. In practice, it is not desirable to obtain the set of locally non dominated solutions. Moreover, in (CT), the objective f_j is specially treated. For this reason, we propose a hybrid program based on the augmented weighted Tchebycheff metric in order to compute a representative set of efficient solutions.

c) Augmented weighted Tchebycheff metric :

One way to characterize the globally non dominated solutions in the criterion space is to consider the points in $S \subset \mathbb{R}^p$, closest to FD_G^* (which dominates the ideal point), according to the Tchebycheff norm. To find such points, we use augmented weighted Tchebycheff metric defined by:

$$||| FD_G^* - F ||| = || FD_G^* - F ||_\infty + \delta \cdot \sum_{i=1}^p | FD_{Gi}^* - f_i | \quad (3)$$

where :

$$|| FD_G^* - F || = \text{Max}_{i=1, \dots, p} (\theta_i \cdot | FD_{Gi}^* - f_i |)$$

The weights $\theta_i \geq 0$ are used to define different metrics (see, for more details, STEUER, (1986), chap.14).

The program (TP) for finding the closest points to F_G^* according to relation (3) is the following :

$$\begin{aligned} \text{Min}_{\alpha, x} \quad & \alpha + \delta \sum_{i=1}^p (FD_{Gi}^* - f_i(x)) \\ \text{s.t.} \quad & \alpha \geq \theta_i (FD_{Gi}^* - f_i(x)) \quad i = 1, \dots, p \\ & f_i(x) \geq K_i - \varepsilon_i \quad i = 1, \dots, p \quad (TP) \\ & f_i(x) \leq \beta_i \mu_i, \quad \beta_i \in \{0, 1\} \\ & x \in X \end{aligned}$$

where :

- K_i is obtained from the interval analysis, i.e. $f_i \in [f_{iMin}, f_{iMax}]$; $K_i = f_{iMax}$;
- $\mu_i \geq f_{iMax} - \Delta_i$.

Note that we can drop the absolute sign in the definition of Tchebycheff norm because the quantity concerned will never be negative. The reason for the second term in (3) is to give the contour a "slight slope" (Fig.6-7). The parameter δ is selected sufficiently small (10^{-6} to 10^{-3}) in practice. By solving the Tchebycheff program (TP),

we search for "the closest point" to F_G^* within a restricted region in the criterion space. It should be emphasized here that strong global Pareto optimality for a particular solution is guaranteed by the following fundamental property : $x^0 \in X$ is a global efficient solution to the MMP (2) if and only if x^0 is a global solution to problem (TP), without a convexity assumption.

3. Outline of the GMC method : Global optimization with MultiCriteria analysis

Generally, the methods for solving non convex programming problems yield local solutions because there is no local criteria to decide if a local solution is global (HORST, (1988); RINNOY KAN and TIMMER (1989)). The deterministic method proposed here mainly uses the behaviour in the vicinity of the current local solution in order to compute the global solution. More precisely, we attempt to eliminate local solutions once there are obtained, by addition of specific constraints. This property has been studied in other works (M'SILTI and TOLLA, (1992)). We propose here a new strategy based on the filtering technique so as to guarantee that a local solution is really global. We proceed by successive reductions of interval concerning the objective value function, $[f_{iMin}, f_{iMax}]$, and the decision variables, $[\alpha_i, \beta_i]$, in order to force convergence near to the optimum. In other words, the solution x^0 of problem (TP) becomes an "initial" solution to the global analysis based on the filtering technique. The method involves the choice of a "reference point" in the criterion space and uses a Tchebycheff procedure to find a suitably defined "closest point" to FD_G^* , where FD_G^* is a point which dominates the ideal point F_G^* (Fig.4). Note that the determination of the ideal point is itself a global optimization problem, and is subject to the same difficulties as the initial problem. In practice, we determine the bounds of variation of the objective functions by interval analysis and we then proceed by parametric variation of K_i :

$$K_i \leftarrow K_i - \eta_i ,$$

$$K_i \leftarrow K_i - 2\eta_i , \text{ etc.}$$

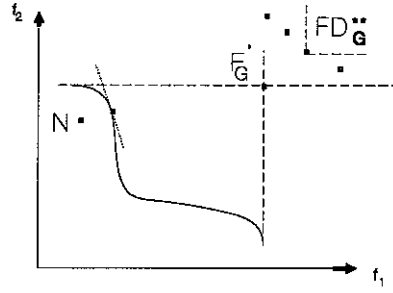


Fig.4 : FD_G^{**} and N are reference points

The technique of "reference points" allows us to generate points hopefully localized in the vicinity of the final solution to problem (1). This approach consists in searching for non dominated solutions by moving from one non dominated solution to another (Fig 5). We can thus control oscillations of the objective value function. The following program is built in order to generate a sequence of solutions, each of them being preferred to its predecessor.

$$\begin{aligned}
 & \text{Max } \lambda_1 f_1(x) + \lambda_2 f_2(x) \\
 \text{s.t. } & x \in X \\
 & \lambda_1 > 0 ; \lambda_1/\lambda_2 \approx 1.
 \end{aligned}
 \tag{GB}$$

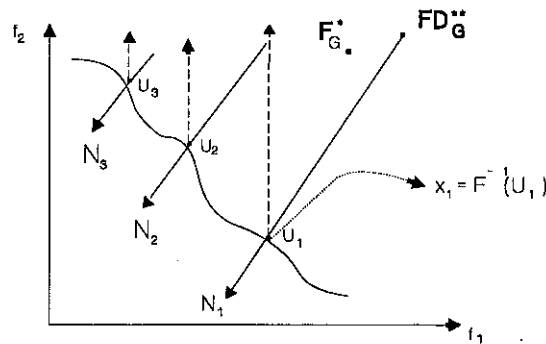


Fig.5 : The reference points are variables

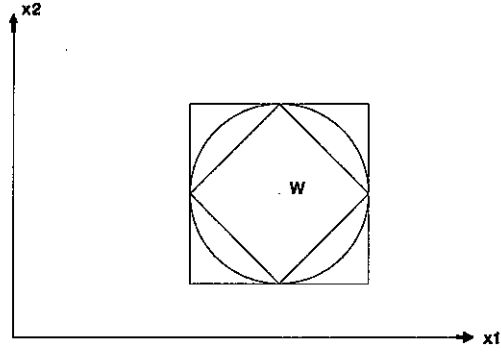


Fig. 6 : Set of equidistant points of w.

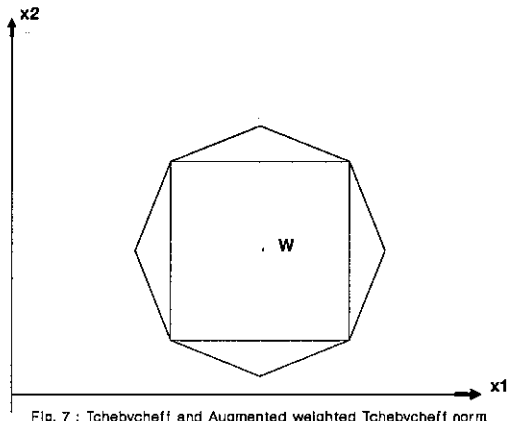


Fig. 7 : Tchebycheff and Augmented weighted Tchebycheff norm

The "distances" are a family of scalarizing functions. In practice, three norms are generally used D_1 , D_2 and D_∞ , where :

$$D_t(u, v) = \left(\sum_{i=1}^p |u_i - v_i|^t \right)^{1/t}$$

In order to avoid certain cases of degeneracy in the nonlinear programming problems, we use an Augmented Tchebycheff norm : $D_\infty^+ = D_\infty + \theta.D_1$, where D_∞ is the Tchebycheff norm. The square corresponds to D_∞ , the lozenge to D_1 , the circle to D_2 and the polygon to the augmented Tchebycheff norm.

The method relies on three steps :

- Step 1 : initialisation of the paramaters, determination of the first reference points, FD_G^* and N (Nadir point) : $K_i = 0$, $\epsilon_i = 0$, etc. The algorithm is typically initiated through the finding of the first reference point FD_G^* and the corresponding Nadir point (Fig. 4-5) by using interval analysis.
- Step 2 : determination of a first "feasible solution" for initializing the process : at this stage of the procedure, we ask the system to determine the minimum and maximum values for each objective function at each iteration. A "feasible solution" for the process is obtained by minimizing the distance from the reference point (Fig.5) according to the augmented weighted Tchebycheff norm, following a fixed direction, $(FD_G^* - N)$. Note that each time the system selects new bounds, the new ones are immediately incorporated into the program and the process is reiterated. Iterations continue until the computer program decides that the current solution is "closed enough" to a best compromise, i.e to the global solution of (1).
- Step 3 : Global optimality analysis : Using the filtering technique, we define a globally efficient solution x^0 as the starting point for the programm (GB).

The GMC method is based on a combination of GRG code which is the "local optimizer", Tchebycheff norm and parametric method in order to filter "good" globally efficient solutions. The latter must be sufficiently spaced so as to avoid generating the same solution more than once (clustering problem; TÖRN, (1983)). The optimality analysis is based on multiple starting points and reduction of the parallelotopes $[f_{iMin}, f_{iMax}]$ and $[\alpha_i, \beta_i]$, with interval analysis. The following remark is important for implementing the system : how many reference points to use in the method in order to find the global solution of (1) ? In other words, what is the stopping rule for the procedure ? The problem posed is the global convergence to a final solution with a "satisfactory cost" (number of evaluations function). The number R of reference points is unknown. We can try to compute it by the following reasoning :

- i) to obtain the global solution of (1), we must have at least one reference point (in the vicinity of the ideal point) which allows us to find this solution;
- ii) assuming that p is the probability to find such a point, then, if we take R points randomly selected and mutually independent, the probability q to obtain at least one point which will determine the global maximum is $q = 1 - (1 - p)^R$. Then, we have one equation and three parameters p , q and R . If we suppose p and q fixed, then we can compute (a priori) the required number R of reference points before starting the GMC method : $R = \text{Log}(1-q)/\text{Log}(1-p)$.

Note that p and q depend on the kind of problem treated, type of decomposition, and type of classification selected for the variables². The necessary information for implementing the GMC method is the following:

- priority of criterion;
- interval of variation of the quantity f_i ; $f_i \in [f_{i\text{Min}}, f_{i\text{Max}}]$. The greater the priority given to criterion f_i , the more the length of the corresponding interval is reduced.
- inter-criteria information; the latter is obtained from the value of dual variables associate to the (PM) problem. The Lagrange multipliers are obtained as a by-product of GRG code, and there are used as "indicators" of constraints which can be eliminated from the process (according to their priority).

We can now present the GMC procedure in a step-by-step way. It consists of sequential solution to a nonlinear programming problem.

² We will study later the specific problem of separable functions, by using a similar method to the GMC.

Step 0 : Define the interval of variation for all objective functions f and f_i , by using techniques based on the interval analysis. For instance, if we have a function $f(x) = x_1x_2^2 - x_3^3$ with the hyperrectangle H defined by $1 \leq x_1 \leq 4$, $1 \leq x_2 \leq 3$ and $2 \leq x_3 \leq 6$; we obtain $x_2^2 \in [1, 9]$, $x_3^3 \in [8, 216]$ and $x_1x_2^2 \in [1, 36]$ and thus that $f(x) \in [-215, 32]$. Set $f(x) \in [f_{gMin}, f_{gMax}]$ where $K = f_{gMax}$ and $\epsilon' = f_{gMin}$. Here, we have $K = 32$ and $\epsilon' = -215$.

The point FD_G^* of coordinates $(f_{1gMax}, f_{2gMax}, \dots, f_{pgMax})$ dominates the ideal point $F_G^* = (f_{1Max}, f_{2Max}, \dots, f_{pMax})$ in the criterion space. f_{igMax} is the maximum value for the objective i determined by interval analysis, whereas f_{iMax} is the real (but unknown) maximum value of objective i . Define the first reference point FD_G^* as : $FD_G^* = (K_1, K_2, \dots, K_p)^t$ where $K_i = f_{igMax}$. Define the Nadir point N as : $N = (N_1, N_2, \dots, N_p)^t$, where $N_i = f_{igMin}$, i.e the lowest value of objective determined by the interval analysis. Note that this stage does not require any optimization. Its complexity is the same as a specific evaluation procedure for polynomial functions, i.e $O(n)$.

Step 1 : Find the (first) globally non dominated solution : $u^0 = (u_1^0, u_2^0, \dots, u_p^0)$ in the criterion space and the corresponding globally efficient solution in the decision variables space, $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$. The point u^0 is the closest to FD_G^* in the criterion space, according to the Augmented Tchebycheff norm ; it is obtained by solving problem (TP). The weights $\theta_i > 0$ are chosen a priori, to keep u^0 in the Nadir point direction; θ_i are direction coefficients of the straight line segment proceeding from FD_G^* to N ; $\theta_i = 1/(K_i - N_i)$; $i=1,2,\dots, p$. The solution u^0 becomes an initial point for generating another globally non dominated solution u^1 and its corresponding globally efficient solution x^1 , by using a direction parallel to $(FD_G^* - N)^3$.

³ We have tested another strategies (cf. M'SILTI and TOLLA, 1992)

Step 2 : Make a global optimality analysis : A specific investigation procedure based on the solution of problem (GB) (or another as for instance, the tunnelling method) is used in order to decide whether u^0 is a global solution with respect to definition 1.

- i) Compare the following values : u_i^0 and K_i ; u_i^0 is the current value of objective i ; we have $(u_i^0 - K_i) < 0$; $i = 1, 2, \dots, p$ and $\forall x \in X$.

Solve problem (GB) by using a parametric variation of weights λ_i so as to filter the best solution.

We try to compute the "extreme" globally non dominated solutions, i.e the coordinates of the ideal point . In other words, we try to localize the area of ideal point by grid technique in the criterion space (Fig.8). If we obtain a good estimation of this point, then the global solution will be obtained directly as the closest to the ideal point.

- ii) If we obtain another globally efficient solution u^1 by solving (GB) better than u^0 , then we set $u^0 \leftarrow u^1$ and we go to step 1. The procedure will reduce the length of interval $[u_i^0 - K_i]$ by reducing the value of K_i at each iteration : $K_i \leftarrow K_i - \Gamma \cdot \epsilon_i$, where $\Gamma = 2^m$;
 $m = 1, 2, \dots$, and ϵ_i is suitably chosen to avoid oscillations.

The procedure continues until two successive solutions are "identical". In practice, in certain cases, we have used the following strategy : solve program (GB) every M iterations of the Tchebycheff routine, where M is a fixed number of efficient solutions.

Convergence of the procedure (computing speed) : Generally, the procedure is finished after R iterations at most, with $R \leq n$; R is the number of reference points fixed a priori ($R = 10$, for instance for certain problems; this value is changed for another test). Note that this does not mean that the final solution is necessarily obtained after 10 iterations of the procedure. At each iteration, we adjust the interval of variation of all objective functions. Then, we reduce progressively the set S by introducing new constraints of special type (M'SILTI and TOLLA, (1992)). This aspect is very important in practice because, if we do not take any precautions, we can introduce bounds which do not correspond to the feasible solutions.

The procedure is based on the construction of "grid" above the set S in the criterion space. If we have p objectives, then q^p calculations are necessary (q is the arbitrary number of bounds for each objective at each iteration, Fig.8). In fact, this number is an upper bound for the computation; indeed, if certain of these bounds are fixed at a high level, a feasible solution does not probably exist. The problem is then to reduce the computation speed (by minimizing the number of evaluations function), thanks to the information obtained by the intervall analysis and improving of a such solution by the (GB) program.

The sequence of U_j must quickly converge to the ideal point, i.e for a certain number R of reference points (R is arbitrary fixed in our test problems). Consequently, we control the quantity $|| U_i - U_{i+1} || \leq \epsilon$.

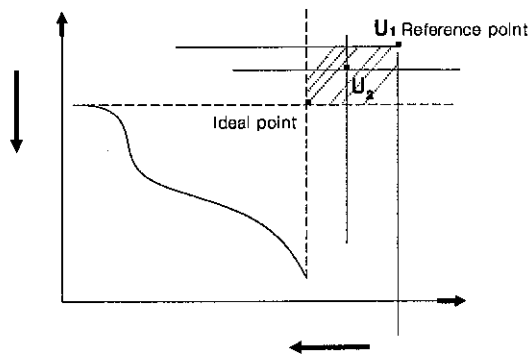


Fig.8 : definition of grid

We are presently experimenting with a new strategy of filtering adapted from the "tunnelling method" itself derived from the "deflation technique" of GOLDSTEIN and PRICE (1970) , which consists in solving the following program :

$$Max \ g(x) = \frac{f(a) - f(x)}{(x - a)^2}$$

We remark that if a is the local solution of (1), then $[f(a) - f(x)] > 0$, $\forall x \in X$. In other words, if we find a point $z \in X$ such that $g(z) < 0$, then $f(a) < f(z)$ and a is not a global solution of problem (1). In our test problems, we take a as a globally efficient solution obtained by the Tchebycheff routine. In the general case where f is a polynome of degree m , we consider the following "filtering function" :

$$T(x) = \frac{f(a) - f(x)}{||x - a_1||^2 \dots ||x - a_t||^2}$$

where $a_1, a_2, \dots, a_t$ are the previous efficient solutions with $f(a_1) < f(a_2) < \dots < f(a_t)$. The term of the denominator avoids obtaining a local solution already found. This solution strategy allows us to combine advantages of both methods interval analysis and tunnelling technique in order to optimize the stopping rule of the procedure. We will present in another paper the results obtained. This method is operational on a micro-computer of type 80386 with co-processor integrated. The computational and implementation aspects are detailed in other papers (M'SILTI and TOLLA , 1992). We have tested many (medium) problems, i.e having up to 200 constraints and 300 variables. However, taking into account the great difficulty of solving nonlinear problems in the general case, we have preferred to give priority working security, i.e global convergence of the method and the CPU time to the detriment of the size of the problem. The results concerning this study are summed in the following table:

Problem	Number of variables	Number of constraints	average CPU time
1	2	3	0.0001 s
2	4	2	2.5 s
3	7	6	11 s
4	16	13	27 s
5	21	16	58 s
6	27	13	1 mn 14 s
7	23	19	1 mn 55 s
8	21	11	1 mn 10 s

The average CPU time (with a sufficiently weak error tolerance) on problems test 1 to 8 depends on the number of reference points R arbitrarily fixed. The same problem is tested for different values of R . But we impose in all cases the size of subset of efficient solutions about 10. Consider the following example :

$$\begin{aligned}
 &\text{Max } f(x) = x_1^2 + (x_2 - 2)^2 - 7 \\
 &\text{s.t.} \\
 &\quad -x_1^3 + x_1^2 + x_1 - x_2 \leq 0 \\
 &\quad (x_1 + 3)^2 + (x_2 - 2)^2 - 36 \leq 0 \\
 &\quad -(x_1 + 3)^2 - (x_2 - 6)^2 + 16 \leq 0 \\
 &\quad -15 \leq x_1 \leq 15 \\
 &\quad -15 \leq x_2 \leq 15
 \end{aligned}$$

The decomposition chosen is $f(x) = f_1(x) + f_2(x) = (x_1^2 - 7) + (x_2 - 2)^2$

The objective function f has four local maxima :

$$a = (-1.107, 2.476) ; f(a) = -5.547$$

$$b = (-0.416, 0.829) ; f(b) = -5.456$$

$$c = (2.036, -1.261) ; f(c) = 7.782$$

$$d = (0.969, 6.500) ; f(d) = 14.100$$

The GMC method has found the global maximum, d , in one iteration of the procedure ($R = 1$ reference point).

4. Conclusion :

The methodology based on the notion of "global efficiency" enables us to build a "good" solution to problem (1). The weighted sum approach (GB) has an important advantage, that is the computation of weights λ_i does not require any precision. We consider only the parametric aspect of their variation. The central process can be adjusted, depending on our particular choices and allows to imagine a great number of variants of the GMC method. The system stores the movements of reference points in order to restart from any point. Moreover, it specifies the "status" of an intermediate solution due to λ_i/λ_j , which is a privileged information.

Many improvements can be made in the MMP strategy, specially in the way of handling the "feasibility" of solutions in the new problem. Our research essentially consists in the two following points : inter-criteria information (i.e. λ_i/λ_j) and characterization of efficient solutions for specific problems. The first point expresses in fact the concept of compensation in the MMP terminology ; it is defined in our case by the weights λ_i and corresponds to the trade-off between the objectives concerned. The second point is more general in the MMP analysis ; the following questions resume the investigation in this domain :

- What is to be done for defining the dominance relation in the case where certain objective function are correlated (i.e there exists a dependence relation of f_i and f_j for example) ? In a such a case, have we to eliminate this redundancy ?

- What is to be done about the dependence between two criteria and their degree of correlation ? Have we to introduce thresholds in the definition of objective functions, and if the response is positive, what is to be done for defining the dominance relation with thresholds ?

In other words, in the MMP framework, if we assume independence of criteria, then the strategy of resolution of problem (1) is the weighted sum approach. The only problem is to define the λ_i ; we are not concerned by their interpretation. In the other case, the solution to initial problem must be defined by the specification of the relative importance of criteria (lexicographic approach for instance). At this stage, we remark that a method based on the combination of a "local optimizer" (GRG) and certain multiobjective routines has solved some highly nonlinear problems. We try to adapt a special technique derived from dynamic programming and strategies of random investigation, for improving the search of reference points and to avoid clustering of the successive solutions, in order to guarantee the reliability of the solution selected (sensitivity analysis) for specific functions. Another interesting point is the development of a graphic interface so as to represent the set of non dominated solutions and to implement the sensitivity analysis based on the specific techniques.

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