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THE LIGHT BEAM SEARCH OVER A NON-DOMINATED SET - AN INTERACTIVE PROCEDURE FOR MULTIPLE-OBJECTIVE ANALYSIS OF LINEAR AND NON-LINEAR PROGRAMS

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Nous présentons une procédure interactive pour l'analyse multicritère de programmes linéaires et nonlinéaires. Dans la phase de dialogue de cette procédure, un échantillon de points non-dominés, comportant le point jugé actuellement le plus intéressant, est présenté au décideur (D). L'échantillon est construit de façon à permettre au D une évaluation facile des points. Envue de celà, nous utilisons une relation de surclassement comme modèle de préférence locale dans le voisinage du point jugé le plus intéressant. La relation de surclassement est utilisée pour définir une sous-région de l'ensemble des points non-dominés à partir de laquelle est construit l'échantillon de points présenté au D. Le D a deux possibilités, ou dégrés de liberté, pour passer d'une sous-région à une autre qui répond mieux à ses préférences. La première possibilité consiste à spécifier un point de référence nouveau qui est ensuite projeté sur l'ensemble non-dominé en vue de trouver un point non-dominé plus intéressant. La deuxième possibilité consiste à choisir parmi des points de l'échantillon proposé un point jugé désormais le plus intéressant. Dans les deux cas, une nouvelle sous-région est définie autour du point qui vient d'être jugé le plus intéressant et elle est ensuite représentée par un échantillon de points. Cette technique pourrait être comparée à une projection d'un faisceau lumineux sur une surface non-dominée à partir d'un spot placé au point de référence. La sous-région illuminée change soit avec le changement du point de référence soit avec le changement du point de la surface jugé le plus intéressant.

The Light Beam Search over a non-dominated set - an interactive procedure for multiple-objective analysis of linear and non-linear programs

Abstract

An interactive procedure for multiple-objective analysis of linear and non-linear programs is presented. At the decision phase of the procedure, a sample of points, composed of the current point and a number of alternative proposals, is presented to the decision maker (DM). The sample is constructed to ensure a relatively easy evaluation of the sample by the DM. To this end we use an outranking relation as a local preference model in a neighbourhood of the current point. The outranking relation is used to define a sub-region of the non-dominated set the sample presented to the DM comes from. The DM has two possibilities, or degrees of freedom, to move from one sub-region to another which better fits his/her preferences. The first possibility consists in specifying a new reference point which is then projected onto the non-dominated set in order to find a better non-dominated point. The second possibility consists in shifting the current point to a selected point from the sub-region. In both cases, a new sub-region is defined around the updated current point. This technique can be compared to projecting a focused beam of light from a spotlight in the reference point onto the non-dominated set; the highlighted sub-region changes when either the reference point or the point of interest in the non-dominated set are changed.

1. INTRODUCTION

In the general case of multiple-objective linear and non-linear mathematical programming, the decision problem consists in selecting the best compromise solution from an infinite multi-dimensional set of non-dominated alternatives. It is commonly acknowledged that interactive procedures are very effective in searching over the non-dominated set for the best compromise. Procedures of this type are characterized by phases of decision alternating with phases of computation. At each computation phase, a solution, or a subset of solutions, is generated for examination in the decision phase. As a result of the examination, the DM inputs some preferential information which intends to improve the proposal(s) generated in the next computation phase.

A number of interactive procedures that present to the DM one point only at each iteration, has been proposed. This class of methods includes such well-known representatives like: STEM (Benayoun et al., 1971), interactive goal programming (see e.g. Lee and Shim, 1986), the reference point method (Wierzbicki, 1980) and Pareto Race (Korhonen and Wallenius, 1988). We claim, however, that presentation of one solution at each iteration does not give the DM a possibility to learn much about the shape of the non-dominated set. In practical situations, the preliminary preferences of the DM are often non-realistic and his/her expectations usually far exceed attainable ranges of objectives. The DM is 'learning' about the problem during the interactive process. Wavering, incoherence and changes of DM's preferences are typical to the process. So, the more the DM learns about the non-dominated set at each iteration, the fewer steps are necessary to find a final solution and the stronger is the conviction of the DM that he/she has found the best compromise. Another drawback of methods from this class is that no information about a neighbourhood of the current point is presented to the DM. So, the DM can miss a possibility of improving the score on one objective at a very small expense of other objectives.

There is also a class of interactive procedures that present to the DM samples of non-dominated points at each iteration. To this class belong such methods like: Zionts-Wallenius method (Zionts and Wallenius, 1976), Jacquet-Lagrèze, Meziani and Słowiński method (Jacquet-Lagrèze et al., 1987), as well as the Reference Direction Approach (Narula et al., 1992) which is an extension of the VIG method (Korhonen, 1987) for the non-linear case, and Computer Graphics-Based method (Korhonen et al., 1992) which is another extension of VIG. At decision phases of such methods, the DM is usually expected to evaluate the presented solutions and specify which one is the best or rank all the solutions in the sample. Authors of these methods make an assumption that evaluation of a finite sample of non-dominated points is relatively easy for the DM.

However, it follows from practical experience and theoretical works made in the field of MCDA that evaluation of an even small finite sets of alternatives can be difficult to the DM. It is rather illusory to expect from the DM an explicit and complete evaluation of the alternatives if, for example, some of them are incomparable.

Instead, he/she gives some preferential information upon which a global preference model can be built.

The above mentioned procedures can fail if the DM refuses to accept a substitution between objectives. Such a situation arises when objectives are in strong conflict. In this case, the DM may be simply unable to compare alternatives that are significantly different. Another type of difficulties may appear if the values of objective functions calculated for a feasible solution are uncertain for some reasons. In this case, small differences in the values of the objective functions are meaningless for the DM and alternatives that are not different enough are indifferent.

It is usually assumed that one of the four following situations can appear while comparing two alternatives a and b (Vincke, 1990):

$a\,P\,b$ i.e. a is preferred to b,

$b\,P\,a$ i.e. b is preferred to a,

$a\,I\,b$ i.e. a and b are indifferent,

$a\,?\,b$ i.e. a and b are incomparable.

The *preference* P, *indifference* I and *incomparability* ? relations are the sets of ordered pairs (a, b) such that $a\,P\,b$, $a\,I\,b$, $a\,?\,b$, respectively. The relations are not assumed to be transitive.

However, in order to handle situations where the DM is not able or unwilling to make distinctions between $a\,P\,b$, $a\,I\,b$ and $a\,?\,b$, the use is recommended of a grouped relation S called an *outranking relation* (Roy, 1985): $a\,S\,b$ means that a is at least as good as b; $a \not S\, b$ and $b \not S\, a$ mean that a and b are incomparable.

In order that each particular step of an interactive procedure makes an improvement in the search for the best compromise solution, the sample of points presented to the DM for an examination should meet some requirements. Specifically, the points in the sample should not be indifferent nor incomparable. Otherwise, difficulties in evaluation of the sample can yield additional incoherence in the preferential information supplied by the DM. Moreover, in such a case, the DM can stop the interactive procedure being not able to find a better proposal among the presented points even if the current point is far from the best compromise.

The procedure presented in this paper tries to overcome the drawbacks of the above mentioned interactive procedures. Specifically,

- it uses an outranking relation as a local preference model built in a neighbourhood of a current point,
- the neighbourhood of the current point is composed of non-dominated points that outrank this point, so the neighbourhood includes points that are sufficiently different but comparable; the points from outside the neighbourhood are either incomparable or outranked by the current point,
- the sample of non-dominated points presented to the DM in each decision phase comes from the neighbourhood of the current point,
- the outranking relation used to define the interesting sub-region of the non-dominated set is based on a relatively weak preferential information of an inter- and intra-criteria type,

- the scanning of the non-dominated set is organized such that the sub-region moves in result of either a change of the DM's reference point or a shift of the current point within a neighbourhood of this point.

The last point submits some analogy with projecting of a focused beam of light from a spotlight in the reference point onto the non-dominated set. For this reason, we call our procedure the Light Beam Search or, shortly, LBS (see figure 1).

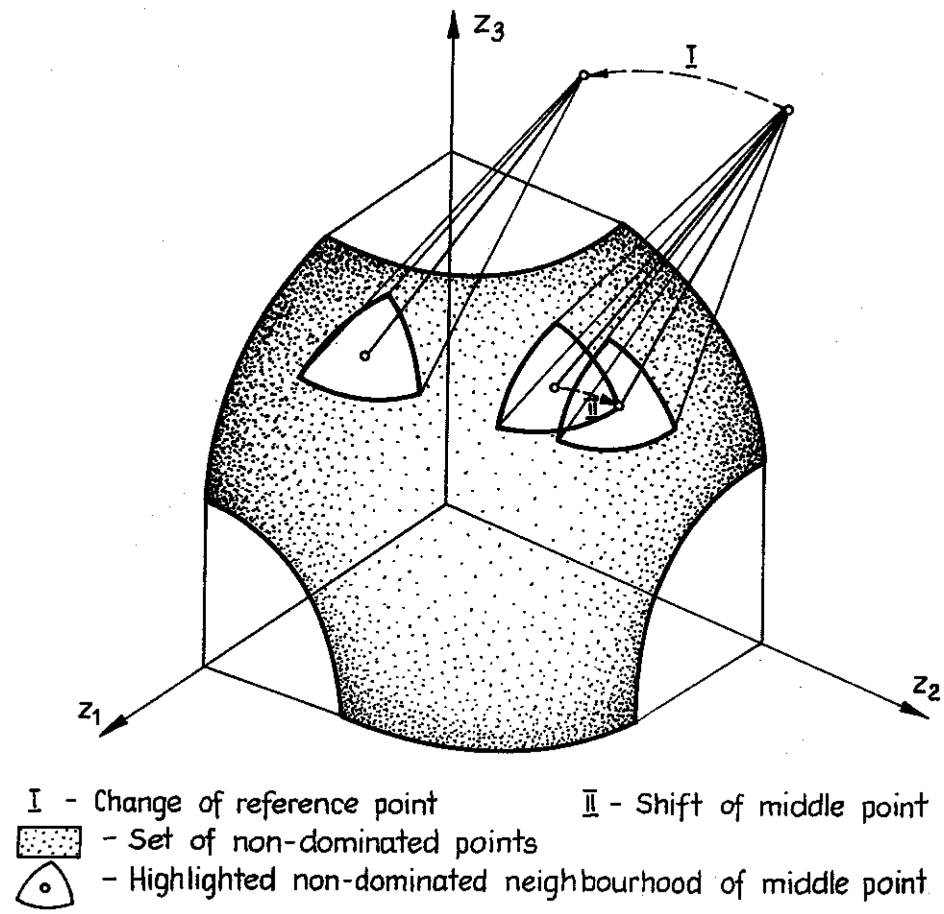

Figure 1. The Light Beam Search over a non-dominated set

The paper is organized in the following way. After a formal statement of the problem and basic definitions, the main idea of the procedure will be presented. Then, the general scheme of the procedure will be outlined. Some characteristic points of the procedure will be described more precisely in the following section. In the next section, an illustrative example will be solved with the LBS procedure. Finally, main features of the procedure will be summarized and some possible directions of further researches will be pointed out.

2. PROBLEM STATEMENT AND BASIC DEFINITIONS

The general multiple-objective programming problem is formulated as

$$\max\{f_1(\mathbf{x}) = z_1\}$$
$$\cdots\cdots\cdots \qquad \text{(P1)}$$
$$\max\{f_k(\mathbf{x}) = z_k\}$$

s.t.

$$\mathbf{x} \in D$$

where $\mathbf{x} = [x_1, \ldots, x_n]$ is a vector of decision variables, functions f_j are continuous and differentiable and condition $\mathbf{x} \in D$ can be stated using continuous and differentiable constraints.

Problem (P1) can also be stated more succinctly as

$$\max\{\mathbf{z}\} \qquad \text{(P2)}$$

s.t.

$$\mathbf{z} \in Z$$

where $\mathbf{z} = [z_1, \ldots, z_k]$ is a vector of objective functions $z_j = f_j(\mathbf{x})$ and Z is an image of set D in the objective space.

Point $\mathbf{z}' \in Z$ is *non-dominated* if there is no $\mathbf{z} \in Z$ such that $z_j \geq z_j' \ \forall j$, and $z_i > z_i'$ for at least one i. Point $\mathbf{z}' \in Z$ is *weakly non-dominated* if there is no $\mathbf{z} \in Z$ *such that* $z_j > z_j' \ \forall j$. The set of all non-dominated points is the *non-dominated set*. For other definitions concerning non-dominance and efficiency, see e.g. Wierzbicki (1986).

The point $\mathbf{z}^*$ composed of the best attainable objective function values is called the *ideal point*

$$z_j^* = \max\ \{f(\mathbf{x}) \mid \mathbf{x} \in D\} \qquad j = 1, \ldots, k.$$

Another useful definition is that of the *achievement scalarizing function* in the objective space:

$$s(\mathbf{z}, \Lambda, \rho) = \max_j\left\{\lambda_j\left(z_j^{\text{o}} - z_j\right)\right\} + \rho\sum_{j=1}^{k}\left(z_j^{\text{o}} - z_j\right) \qquad (1)$$

where $\mathbf{z}^{\text{o}}$ is a reference point, $\varepsilon_j > 0$ is moderately small, $\Lambda = [\lambda_1, \ldots, \lambda_k]$ is a weighting vector, $\lambda_i \geq 0$, $\sum_{j=1}^{k}\lambda_j = 1$ and ρ is a sufficiently small positive number.

3. MAIN IDEA OF THE LIGHT BEAM SEARCH PROCEDURE

The LBS procedure falls into the category of interactive procedures with generation of finite samples of non-dominated points at each computation phase. A sample is composed of a current point, called the *middle point*, obtained at a previous iteration, and a number of non-dominated points from its neighbourhood. In order to define the neighbourhood the sample represents, we use an outranking relation S as a local preference model. Precisely, for a current middle point, the sub-region is defined as a set of non-dominated points that are not worse than the middle point, i.e. outrank the middle point. The sub-region is called the *outranking neighbourhood* of the middle point. The sample is composed of points that are obtained by independent optimization of particular objectives in the outranking neighbourhood, called the *characteristic neighbours* of the middle point. Moreover, the DM is able to scan more precisely the inner of the neighbourhood through the objective function trajectories between any two characteristic neighbours or between a characteristic neighbour and the middle point. Other methods for exploration of the neighbourhood can also be used.

The formal expression of the conditions that must be satisfied to validate the assertion $a\,S\,b$ can be influenced by many factors. In the presented procedure, following the approaches proposed in various versions of ELECTRE methods (Roy, 1990), the following factors will be taken into account:

- the discrimination power of the DM's preferences with respect to particular objectives that will be modelled with *indifference* and *preference thresholds* (i.e. the intra-criteria information),
- the inter-criteria information specified in the form of the *veto thresholds.*

Similarly to ELECTRE IV, we will not use the inter-criteria information in the form of importance coefficients which might be too difficult to define; we assume, however, that one objective is not more important that all the others together. Let us notice that the ratio of veto and preference thresholds of a criterion is related with its importance; the lower is the ratio the greater is the importance (Roy, 1980).

In the traditional preference modelling it is assumed that every difference on a single objective z_j is significant to the DM. However, in practice, there exists an interval in which the DM does not feel any difference between two elements or refuses to accept a preference for one of the alternatives. This fact was already pointed out by Poincaré (1935 p.69), but it was Luce (1956) who introduced this fundamental remark to preference modelling. This situation can be modelled with the indifference threshold q_j given by the DM.

Moreover, experience shows that, usually, there is no precise value giving the limit between the indifference and preference but there exists an intermediary region where the DM hesitates between the indifference and preference or gives different answers, depending on the way he/she is questioned. This remark has led to the introduction of the preference threshold p_j. In general, the indifference and preference thresholds are functions of z_j; moreover:

$$p_j(z_j) \geq q_j(z_j) \geq 0$$

The indifference and preference thresholds allow to distinguish between the three following preference relations with respect to z_j for any ordered pair $(a,\ b)$ of alternatives:

$a\,I_j\,b$ i.e. a and b are *equivalent* $\Leftrightarrow$ $-q_j\left(z_j^a\right) \leq z_j^a - z_j^b \leq q_j\left(z_j^b\right),$

$a\,Q_j\,b$ i.e. a is *weakly preferred* to b $\Leftrightarrow$ $q_j\left(z_j^a\right) < z_j^a - z_j^b < p_j\left(z_j^b\right),$

$a\,P_j\,b$ i.e. a is *significantly preferred* to b $\Leftrightarrow$ $p_j\left(z_j^a\right) \le z_j^a - z_j^b$,

The veto threshold v_j allows to take into account the possible difficulties of comparing the relative value of two alternatives when one is significantly better than the other on a subset of objectives, but much worse on at least one other objective. In general, the veto threshold is also a function of z_j.

The outranking relation has already been used as a preference model in the Cone Contraction Method with Visual Interaction for Multiple-Objective Non-Linear Programmes (Jaszkiewicz and Słowiński, 1992). In that method, however, it is used as global preference model. The construction of an outranking relation follows the methodology proposed for the ELECTRE III method (Roy, 1978) and the relation is built on a representative sample of non-dominated points. As the indifference, preference and veto thresholds, in general, depend on z_j, the DM should specify these thresholds in the form of mathematical functions, $q_j(z_j)$, $p_j(z_j)$ and $v_j(z_j)$. If the functions are complicated, it is practically impossible for the DM to specify them explicitly. In the Light Beam Search procedure the outranking relation is used as a local preference model in the neighbourhood of a middle point, so a single value of each threshold is sufficient for a given middle point. Of course, the DM can update the values of the thresholds for every new middle point.

The outranking relation has also been used as a local preference model in the method proposed by Lotfi et al. (1992). However, their method is purposed for multiple objective analysis of problems with finite set of alternatives only. In this case, the whole neighbourhood can be generated and presented to the DM. Moreover, as the authors do not use any additional preferential information, the definition of the neighbourhood seems somewhat arbitrary.

4. GENERAL SCHEME OF THE INTERACTIVE PROCEDURE

The following is a general scheme of the procedure presented in a Pascal-like form:

> Fix the points of the best and the worst values of objectives; make the former one the first reference point;
> Ask the DM to specify the preferential information of inter- and intra-criteria type;
> Find a starting middle point;
> **repeat**
>> Present the middle point to the DM;
>> Calculate the characteristic neighbours of the middle point and present them to the DM;
>> Allow the DM to scan the inner of the current neighbourhood;
>> **if** *the DM wants to store the middle point* **then**
>>> Add it to the set of stored points;
>>
>> **case**
>>> *The DM wants to define a new reference point:*
>>>> Ask the DM to specify the aspiration levels on particular objectives;

Project the reference point onto the non-dominated set;

The DM wants a point from the neighbourhood to be the new middle point:

Ask the DM to select the new middle point;

The DM wants to return to one of the stored points:

Use the stored point as a new middle point;

The DM wants to update the preferential information

Ask the DM to specify the new preferential information;

end

until *the DM feels satisfied with a point found during the interactive process;*

5. DETAILED DESCRIPTION OF THE PARTICULAR STEPS

The procedure starts by asking the DM to specify (subjective) best and worst values of objectives, z_j^*, z_{j*} $(j = 1,..,k)$, respectively. If he/she is unable to do so, the best values are fixed at individual maxima of particular objectives (ideal point) and the worst values are set equal to minimal values of objectives at the points corresponding to the individual maxima. The point of the best values $\mathbf{z}^*$ becomes the first reference point, $\mathbf{z}^0$.

Then, the DM is asked to give the preferential information for each objective, i.e. the indifference and, optionally, the preference and veto threshold. At this stage the DM should decide if he/she wants to specify the preference and/or veto thresholds, however, he/she is able to change these settings at every step of the procedure.

In the next step, the starting middle point $\mathbf{z}^c$ is computed. The point is obtained by projecting point $\mathbf{z}^*$ of the best values of objectives onto the non-dominated set in the direction defined by point $\mathbf{z}^*$ and point $\mathbf{z}_*$ of the worst values of objectives. The achievement scalarizing function (1) is used to this end.

Then, the *characteristic neighbours* of the middle point are computed. The characteristic neighbour, with respect to objective z_j is a point $\mathbf{z}^j$ from the outranking neighbourhood of point $\mathbf{z}^c$ that maximizes the distance from $\mathbf{z}^c$ in direction of the greatest locally feasible improvement of objective z_j $(j = 1,..,k)$. An *attainable characteristic neighbour* $\mathbf{z}'^j$ is a point obtained in result of a projection of point $\mathbf{z}^j$ onto the non-dominated set $(j = 1,..,k)$.

In order to test if a point $\mathbf{z}$ outranks the middle point, first, the following numbers are calculated:

$m_s(\mathbf{z}, \mathbf{z}^c)$ - the number of the objectives for which point $\mathbf{z}$ is indifferent, or weakly or strictly preferred, to $\mathbf{z}^c$,

$m_q(\mathbf{z}^c, \mathbf{z})$ - the number of the objectives for which point $\mathbf{z}^c$ is weakly preferred to $\mathbf{z}$,

$m_p(\mathbf{z}^c, \mathbf{z})$ - the number of the objectives for which point $\mathbf{z}^c$ is strictly preferred to $\mathbf{z}$,

$m_v(\mathbf{z}^c, \mathbf{z})$ - the number of the objectives being in a strong opposition to the assertion $\mathbf{z}\, S\, \mathbf{z}^c$, i.e. card $\{j: z_j^c - v_j \geq z_j, j = 1,..,k\}$.

The construction of the outranking relation depends on the type of preferential information supplied by the DM. If the DM has specified all of the thresholds, we propose to use the following definition of the outranking relation:

$$\mathbf{z}\, S^a\, \mathbf{z}^c \Leftrightarrow \begin{cases} m_v(\mathbf{z}^c,\mathbf{z}) = 0 \text{ and} \\ m_p(\mathbf{z}^c,\mathbf{z}) \leq 1 \text{ and} \\ m_q(\mathbf{z}^c,\mathbf{z}) + m_p(\mathbf{z}^c,\mathbf{z}) \leq m_s(\mathbf{z},\mathbf{z}^c) \end{cases}$$

If the DM has decided not to specify the veto thresholds, we should not assume that for every objective he/she is ready to accept any worsening of the objective even if a subset of other objectives is significantly improved, i.e. we should not assume that the veto threshold does not exist. Such a situation indicates that at the particular stage of the interactive process, the DM is not able or unwilling to specify the value of this threshold explicitly. In this case, we propose to use the following definition of the outranking relation:

$$\mathbf{z}\, S^b\, \mathbf{z}^c \Leftrightarrow \begin{cases} m_p(\mathbf{z}^c,\mathbf{z}) = 0 \text{ and} \\ m_q(\mathbf{z}^c,\mathbf{z}) \leq m_s(\mathbf{z},\mathbf{z}^c) \end{cases}$$

Let us observe that $S^b \subseteq S^a$ and that $S^b = S^a$ if $v_j = p_j\ \forall j$. If, for an objective z_j, the DM is ready to accept the worsening of its value greater that p_j, some points that outrank the middle point can be left outside the outranking neighbourhood. However, the neighbourhood will be still composed of points that are comparable with the middle point.

In a similar way we can analyze the situation when the DM has decided not to specify the preference threshold. In this case we should not assume that the DM feels no difference between the weak and strict preferences. Such a situation indicates that the DM is not able or unwilling to specify explicitly the value allowing to distinguish between the two relations. In this case, we propose to use the following definition of the outranking relation:

$$\mathbf{z}\, S^c\, \mathbf{z}^c \Leftrightarrow \begin{cases} m_v(\mathbf{z}^c,\mathbf{z}) = 0 \text{ and} \\ m_q(\mathbf{z}^c,\mathbf{z}) \leq 1 \end{cases}$$

Let us observe that $S^c \subseteq S^a$ and that $S^c = S^a$ if $p_j = q_j\ \forall j$.

Finally, if the DM has decided to specify the preference thresholds only, we propose to use the following definition of the outranking relation:

$$\mathbf{z}\, S^d\, \mathbf{z}^c \Leftrightarrow m_q(\mathbf{z}^c,\mathbf{z}) = 0$$

Let us observe that $S^d \subseteq S^b \subseteq S^a$ and $S^d \subseteq S^c \subseteq S^a$, moreover, $S^d = S^a$ if $v_j = p_j = q_j\ \forall j$.

In order to find the k characteristic neighbours, gradients of particular objectives are projected onto linear approximation of constraints which are active in point $\mathbf{z}^c$ (cf. gradient projection methods for non-linear optimization, Rosen, 1960). Let h be the number of active constraints in point $\mathbf{z}^c$. The linear constraints can be presented in a matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{z}_j^c = f_j(\mathbf{x}^c), j = 1,\ldots,k,\ a_{ti} = \partial c_t / \partial x_i$ are elements of matrix $\mathbf{A}$, t is an index of an active constraint, $t = 1,..,h;\ i = 1,\ldots,n$. Next, the projection matrix $\mathbf{P}$ is calculated

$$\mathbf{P} = \mathbf{I} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}$$

Matrix $\mathbf{P}$ and gradients of particular objectives $\nabla_{\mathbf{x}} f_j$ are used to obtain directions $\Delta\mathbf{x}^j$ in the space of variables

$$\Delta\mathbf{x}^j = \mathbf{P}\nabla_{\mathbf{x}} f_j, \quad j = 1,\ldots,k$$

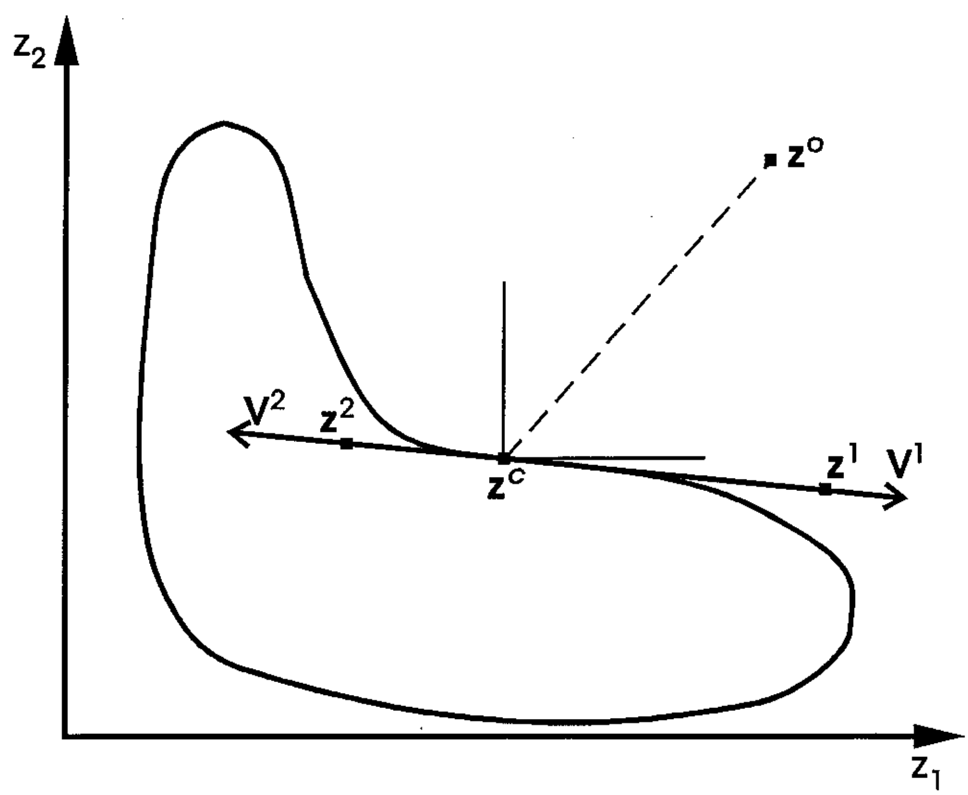

Figure 2. Characteristic neighbours found using a gradient projection onto a linear approximation of active constraints in $\mathbf{z}^c$

$\Delta\mathbf{x}^j$ is the feasible direction of the greatest improvement of objective $z_j = f_j(\mathbf{x})$. Directions $\Delta\mathbf{x}^j$ are used in turn to define corresponding directions $\mathbf{V}^j$ in the objective space

$$\mathbf{V}^j = [\sum_{i=1}^{n}(\Delta x_i \partial f_1 / \partial x_i),\ldots,\sum_{i=1}^{n}(\Delta x_i \partial f_k / \partial x_i)]$$

Then, the following mathematical programming problem is solved in order to maximize objective z_j in direction $\mathbf{V}^j$ ($j = 1,\ldots,k$)

$$\begin{array}{lll} & \max \alpha & \qquad\qquad (\mathrm{P3}) \\ \text{s.t.} & \mathbf{z}^j \, S \, \mathbf{z}^c, & \mathbf{z}^j = \mathbf{z}^c + \alpha \mathbf{V}^j \\ & \alpha \geq 0 & \end{array}$$

Problem (P3) is a small mathematical programming problem with one variable only. Points, $\mathbf{z}^j$ ($j = 1,\ldots,k$) following from solving of k problems (P3) give characteristic neighbours (see figure 2).

Attainable characteristic neighbours are obtained in result of projection of points $\mathbf{z}^j$ ($j = 1,..,k$) onto the non-dominated set in direction connecting $\mathbf{z}^j$ with point $\mathbf{z}^*$ (see figure 3).

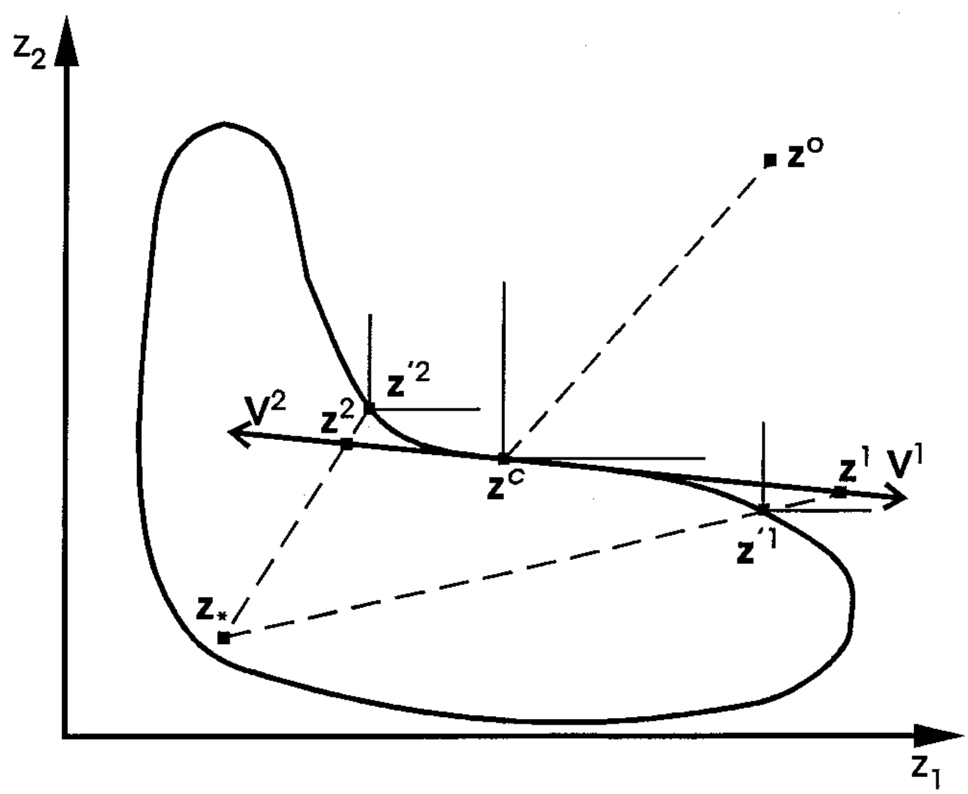

Figure 3. Attainable characteristic neighbours found by projecting points $\mathbf{z}^1$ and $\mathbf{z}^2$ onto the non-dominated set

In the decision phase, the middle point and its characteristic neighbours are presented to the DM. Both numerical and graphical forms of presentation should be used to help the DM in evaluating large amounts of information. Moreover, the DM is able to scan more precisely the region between any two characteristic neighbours or between a characteristic neighbour and the middle point. For this purpose, the line segment connecting the points in the objective space is projected onto the non-dominated set. The obtained subset of the non-dominated points is called the *profile* of the neighbourhood. As in non-linear case getting a continuous profile is practically impossible, we choose a finite numbers of points lying on the line segment and project them onto the non-dominated set (see figure 4). The points resulting from the projection are then presented to the DM. A similar technique of scanning a sub-region of the non-dominated set has been used in Jaszkiewicz and Słowiński (1992). Some other techniques of local characterisation of the non-dominated set can also be used at this step.

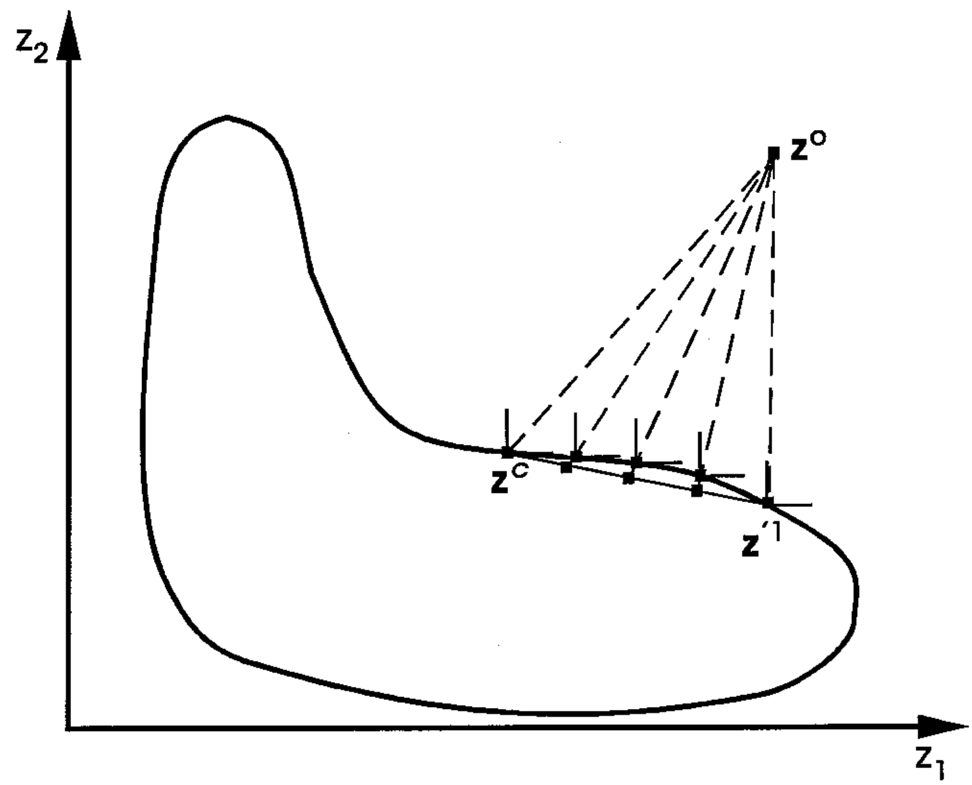

Figure 4. Finding an approximation of a profile of the non-dominated set

The procedure stops if one of the presented points is satisfactory to the DM on all objectives. Otherwise, he/she can continue the scanning using two degrees of freedom. The first degree consists in modifying the aspiration levels, i.e. the reference point. The new reference point is then projected onto the non-dominated set in order to find the new middle point. The second degree of freedom consists in selecting one of the points from the neighbourhood to be the new middle point for the same reference point. Then a new outranking neighbourhood is generated (see figure 1).

Before continuing the scanning, the DM can store the current middle point. He/she is allowed to restore any of the stored points at any time.

Finally, the DM is able to modify the preferential information given for each objective, i.e. the indifference, preference an veto thresholds. He/she can also change the type of the outranking relation. It influences the construction of the outranking relation and the size of the new outranking neighbourhood.

6. ILLUSTRATIVE EXAMPLE

To illustrate the work of the Light Beam Search procedure the following example will be solved

$$\max\left\{f_1 = x_1^2 - 0.1x_2 - 0.1x_3\right\}$$

$$\max\left\{f_1 = x_2^2 - 0.1x_1 - 0.1x_3\right\}$$

$$\max\left\{f_1 = x_3^2 - 0.1x_1 - 0.1x_2\right\}$$

s.t.

$$x_1^3 + x_2^3 + x_3^3 \leq 24$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

The procedure starts by fixing the points of the best and the worst values of objectives, $\mathbf{z}^*$ and $\mathbf{z}_*$, respectively. Assume that the DM is not willing to specify the points and they are fixed automatically as ideal and nadir points

$$\mathbf{z}^* = [8.32, 8.32, 8.32]$$

$$\mathbf{z}_* = [-0.288, -0.288, -0.288]$$

Then the DM is asked to decide what kind of the preferential information he wants to specify. The DM decides to specify all the thresholds and gives the following values

Objective	q_j	p_j	v_j
f_1	0.1	0.5	1.5
f_2	0.2	0.8	3
f_3	0.4	0.9	6

The procedure finds the starting middle point

$$\mathbf{z}^c = [3.6, 3.6, 3.6]$$

An outranking neighbourhood is constructed around the middle point. Attainable characteristic neighbours coming from the neighbourhood are calculated

$$\mathbf{z}^1 = [4.380, 3.185, 3.185]$$

$$\mathbf{z}^2 = [3.185, 4.380, 3.185]$$

$$\mathbf{z}^3 = [3.396, 3.396, 3.995]$$

The points are presented to the DM. The DM has an idea about desired values of objectives and decides to specify a reference point. He/she gives the following point

$$\mathbf{z}^o = [7.2, 5.8, 2.8]$$

The point given by the DM is non-attainable. The procedure projects it onto the non-dominated set. The new middle point

$$\mathbf{z}^c = [5.338, 3.938, 0.938]$$

is a result of the projection. The attainable characteristic neighbours found for the new middle points are the following

$$\mathbf{z}^1 = [7.106, 1.797, 0.508]$$

$$\mathbf{z}^2 = [3.742, 5.636, 0.697]$$

$$\mathbf{z}^3 = [5.047, 3.720, 1.722]$$

The DM feels that at this point the values of some thresholds are not appropriate and decides to change the preferential information. This time he/she gives the following values of the thresholds

Objective	q_j	p_j	v_j
f_1	0.05	0.3	0.8
f_2	0.2	0.5	3
f_3	0.3	0.7	6

With the new preferential information the procedure constructs a new outranking neighbourhood and calculates new attainable characteristic neighbours

$$\mathbf{z}^1 = [6.724, 2.336, 0.619]$$

$$\mathbf{z}^2 = [4.505, 4.884, 0.815]$$

$$\mathbf{z}^3 = [5.047, 3.720, 1.722]$$

The DM thinks that point $\mathbf{z}^3$ is better than the middle points and selects it to be the new middle point. New attainable characteristic neighbours are found and presented to the DM

$$\mathbf{z}^1 = [5.884, 2.937, 1.411]$$

$$\mathbf{z}^2 = [4.210, 4.755, 1.485]$$

$$\mathbf{z}^3 = [4.749, 3.508, 2.393]$$

The DM decides to scan more precisely the profile between the middle point $\mathbf{z}^c$ and its characteristic neighbour $\mathbf{z}^3$. The profile is constructed by projection of the line segment that connects point $\mathbf{z}^c$ with $\mathbf{z}^3$. The sample of the non-dominated points from the profile is presented to the DM

$$\mathbf{w}^1 = [5.006, 3.691, 1.819]$$

$$\mathbf{w}^2 = [4.964, 3.662, 1.916]$$

$$\mathbf{w}^3 = [4.922, 3.632, 2.013]$$

$$\mathbf{w}^4 = [4.879, 3.602, 2.109]$$

$$\mathbf{w}^5 = [4.836, 3.571, 2.204]$$

$$\mathbf{w}^6 = [4.793, 3.540, 2.299]$$

The DM feels that point $\mathbf{w}^2$ is satisfactory on all objectives. Thus it yields the best compromise. In the space of variables the point correspond to the following solution

$$x_1 = 2.306$$

$$x_2 = 2.011$$

$$x_3 = 1.532$$

At this point the first constraint is active.

7. SUMMARY AND CONCLUSIONS

A new interactive procedure for multiple-objective analysis of linear and non-linear programs has been presented. At the decision phase of the procedure a sample of non-dominated points is presented to the DM. The sample comes from an outranking neighbourhood of a given middle point. The DM has two degrees of freedom to move from one neighbourhood to another: by changing his/her aspiration levels or by selecting one of the points from the neighbourhood to be the new middle point. The procedure is called the Light Beam Search. Its microcomputer implementation working under MS-Windows environment is available from the authors upon request. The class of non-linear problems that can be solved with the implementation depends on the type of a non-linear solver used; non-convex problems require specialized solvers giving a relatively good chance of obtaining the global optimum.

Although, the presented procedure has been developed for analysis of continuous mathematical programming problems, it can be extended for discrete programming problems and for problems with finite sets of alternatives. In these cases, it could be possible to present all points from the outranking neighbourhood to the DM. Such techniques like filtering and clustering can be used to reduce the number of points presented at the decision phase if the number of points belonging to the outranking neighbourhood would be too large.

In general, a considerable computational effort is required to find the characteristic neighbours of the middle point in an exact way. A parallel computing can be used to accelerate calculations at the computation phase. Moreover, even on a sequential computer, it is possible to minimize the durations of particular steps by overlapping the computation and decision phases. To this end, the middle point should be presented to the DM as soon as it is found. The DM should also be able to use various forms of presentation of this point and to change it before all the characteristic neighbours are found. Furthermore, every characteristic neighbour should be presented to the DM as soon as it is calculated. As a dialogue with the user does not cause a great burden of the CPU it is possible to perform the calculations in the background.

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