CAHIER DU LAMSADE

Laboratoire d'Analyse et Modélisation de Systèmes pour l'Aide à la Décision
(Université Paris-Dauphine)
Unité de Recherche Associée au CNRS n° 825

AVERAGE CASE ANALYSIS OF GREEDY ALGORITHMS FOR OPTIMISATION PROBLEMS ON SET SYSTEMS

CAHIER N° 125
octobre 1994

J. BLOT [1]
W. FERNANDEZ de la VEGA [2]
V. Th. PASCHOS [3]
R. SAAD [4]

[1] *CERSEM, Université de Paris I, 90 rue de Tolbiac, 75634 Paris Cedex 13, France.*

[2] *LRI, Université d'Orsay, Bât. 490, 91405 Orsay Cedex, France, e.mail : lalo@lri.lri.fr.*

[3] *LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France, e.mail : paschos@lamsade.dauphine.fr.*

[4] *21, rue Ziane Said la Scale, El-Biar Alger, Algérie.*

Contents

Analyse en moyenne de la performance des algorithmes gloutons pour des problèmes d'optimisation sur des systèmes d'ensembles

Résumé

Nous présentons un cadre général pour l'analyse asymptotique d'algorithmes gloutons pour plusieurs problèmes d'optimisation sur des systèmes aléatoires d'ensembles lorsque le rapport entre la taille de l'ensemble de base et la taille du système d'ensembles reste constant. Les systèmes d'ensembles sont engendrés via des graphes biparties aléatoires et les approximations des chaînes de Markov sont réalisées à l'aide d'équations différentielles ordinaires.

mots-clés: Problème NP-complet, algorithme polynomial approché, approximation en moyenne, couverture d'ensembles, transversal d'un hypergraphe, stable d'un hypergraphe

Average case analysis of greedy algorithms for optimisation problems on set systems

Abstract

A general framework is presented for the asymptotic analysis of greedy algorithms for several optimisation problems such as hitting set, set cover, set packing, etc., applied on random set systems. The probability model used is specified by the size n of the ground set, the size m of the set system and the common distribution of each of its components. The asymptotic behaviour of each algorithm is studied when n and m tend to ∞, with $\frac{m}{n}$ a fixed constant. The main tools used are the generation of random families of sets via random bipartite graphs and the approximation of Markov chains with small steps by solutions of ordinary differential equations.

keywords: NP-complete problem, polynomial time approximation algorithm, average case approximation, set covering, hitting set, hypergraph independent set

1   Introduction

There are many optimisation problems defined on families of sets such as hitting set, set cover, set packing, etc., which are known to be NP-complete, and for the (approximate) solution of which one must thus use either heuristics or greedy algorithms. The purpose of this paper is to present a general framework which permits the asymptotic analysis of greedy algorithms for several of these problems, including in particular those just mentioned.

This framework comprises two main parts:

(i) a representation of random set systems by using degree-constrained random bipartite graphs.

(ii) the introduction of a Markov chain on a certain state space for the analysis of each particular greedy algorithm.

It remains then to analyse the behaviour of this Markov chain. This behaviour can be approximated for problems of large size by the solution of an ordinary differential equation (or a system of such equations) which can be obtained in closed form.

We point out that Markov chains have been used previously, explicitely or implicitly, in the analysis of algorithms by various authors. Let us mention Fernandez de La Vega ([3]) and Frieze Radcliffe and Suen ([5]).

This paper should be considered mainly as a theoretical contribution to the analysis of greedy algorithms on set systems and, accordingly, we have not included numerical results. Such results would be of limited help here since in most cases (a basic obstruction here, comes from the fact that) we do not know the values of the optimum of the objective functions (the approximate values which may be obtained via the use of the first moment method seem to be rather inaccurate).

A preliminary version of this paper has been presented in the LATIN'92 conference ([4]).

The plan of this paper is as follows. In § 2 we introduce the model of random set systems that we use and a useful connection with random bipartite graphs. In § 3, we study in details an algorithm for set covering and hitting set and in § 4 we show how this algorithm can be modified to be used for set packing and hypergraph independent set. Finally, in the appendix, we prove the convergence of the differential system describing the set covering algorithm.

2   Random set systems as random bipartite graphs

2.1   Random sets and random multisets

It will be convenient to work with families of random multisets rather than random sets. Moreover, we will view our families as ordered. In other words, we introduce the uniform measure μ on the set of ordered k-tuples from some finite ground set S and work with μ^m, the m-fold product of μ defined on the set of ordered families of k-tuples of size m. Let us denote by $\nu = \nu_m$ the more conventional mesure which gives the same weight to each m-family of pairwise distinct k-sets. It is easy to see that when sampling on genuine k-sets

with replacement we get m distinct sets with probability near to 1 if only $m = o(n^{k/2})$. Now the probability p of getting a genuine k-set when sampling according to μ^m satisfies clearly

$$p = \frac{(n-1)(n-2)\ldots(n-k+1)}{n^{k-1}} \geq \exp\left\{-\frac{k^2}{2n}\right\}.$$

Thus, setting $\lambda = m/n$, the probability p^m of getting m genuine k-sets when sampling according to μ_m satisfies $p^m \geq \exp\left\{-(1/2)\lambda k^2\right\}$, and this together with the former result, implies that, for a fixed λ, the μ^m measure of the genuine systems of m k-sets drawn from a set of size n exceeds $\exp\{-\lambda k^2\}$. Thus, again for a fixed λ, every event "almost sure" relatively to the sequence of measures μ^m, $m \to \infty$ is also almost sure relatively to the more conventional measure ν_m and this property suffices for our purposes.

2.2 Random set systems and bipartite graphs

Let X and Y denote sets with the same cardinality. Let $\mathcal{P} = \{X_1, X_2, \ldots, X_m\}$ denote a partition of the set X into sets with sizes $m_1, m_2, \ldots, m_m$. Let $P = (p_1, p_2, \ldots, p_m)$ where p_i denotes the number of classes of $\mathcal{P}$ of size i. Similarly, let $\mathcal{Q} = \{Y_1, Y_2, \ldots, Y_n\}$ denote a partition of the set Y into sets with sizes $d_1, d_2, \ldots, d_n$ and let $Q = (q_1, q_2, \ldots, q_n)$ be defined similarly as P.

Now, let Π denote a random pairing between the sets X and Y. If we contract each set of vertices X_i to a single vertex and similarly each set Y_j, we get a bipartite graph, say B, with partitions $\mathcal{M} = (m_1, m_2, \ldots, m_m)$ and $\mathcal{D} = (d_1, d_2, \ldots, d_n)$. If we think of $\mathcal{Q}$ as a fixed ground set, this pairing defines naturally a random set-system $\mathcal{P} = \{X_1, X_2, \ldots, X_m\}$, where we identify X_i with the set of classes Y_j adjacent to it in Π.

Notice that this pairing procedure gives the same measure to all genuine set systems, in which no set contains more than once any element of the ground set and no two sets are equal. Indeed, any such system is given by precisely $\prod_{i=1}^m m_i! \prod_{i=1}^n d_j!$ distinct pairings.

Henceforth, we have at our disposal a procedure for generating random systems of sets with fixed cardinalities, subject moreover to the condition that each element of the ground set belongs to a prescribed number of sets in the system.

2.3 A particular class of set systems distributions

We are in fact mostly interested in the following class of probility distributions on set systems. We postulate that the cardinalities of our random sets $X_1, X_2, \ldots, X_m$ are independent random variables $N_1, N_2, \ldots, N_m$ with a common distribution F defined by its individual probabilities $\{p_j = \Pr[N_1 = j], j = 1, 2, \ldots\}$. Moreover, we postulate that, for each fixed j, the set X_i is, conditionally on $N_i = j$, uniformly distributed between the (multi) subsets of $[n]$ of cardinality j.

Thus, if we choose for F the distribution concentrated on some fixed integer k, we get a random k-uniform set system of size m. If we give a fixed value to the ratio m/n, say $m/n = \lambda$, then, by the law of large numbers, the number n_h of vertices of degree h in the corresponding incidence graph satisfies, for each integer $h \geq 0$, $n_h/n \to e^{-\lambda}\lambda^h/h!$ in

probability as $n \to \infty$.

Now, if we want to generate a random set system according to some distribution in the class just defined, we are faced with two difficulties:

i) the degrees are unbounded,

ii) the partition of the incidence graph is not fixed.

In order to circumvent the first difficulty, we restrict our algorithms to work only on vertices with a bounded (but arbitrarily large) degree Δ and we show that the size of the obtained solutions tends to a limit as $\Delta \to \infty$.

The second difficulty is circumvented by conditioning on the degrees of the graph. Indeed, since the proportion of vertices of each fixed degree in each color class of the incidence graph $B = B_n$ tends to its expectation when $n \to \infty$ and, as it will be seen, the sizes of the obtained solutions are continuous relatively to the ratios n_h/n, we can obtain the limits of these sizes by replacing these ratios by their expectations.

2.4 Removing Vertices from the Incidence Graph

Let us return to the graph B and let us look at what happens when we remove a vertex y randomly chosen from the vertices in Y of degree h in B, i.e, we remove the h corresponding vertices in $V(P)$ (then, all the sets containing y are removed). Since the pairing P between the sets X and Y is random, the subset of $Y \setminus \{y\}$ whose vertices are adjacent to the removed sets in X is also random. It can then easily be checked that the new incidence graph is again random within the graphs which have the same degrees. We note that, since B may eventually be a multigraph, the number of removed sets may (exceptionally) be strictly smaller than h. We will of course take this fact into account in our calculations.

It will be seen that all the algorithms we consider, remove at each step either vertices from Y chosen on the basis of their degrees, or a vertex X_i chosen on the basis of the size $|X_i|$, or both. This implies again that the successive incidence graphs which appear are random within the graphs which have the same partition.

3 Hitting Set and Set Cover

3.1 A Lower Bound for the Random Hitting Set Problem

Let C denote a fixed subset of $[n]$ of cardinality l. The probability that C intersects C_i for some fixed i is $1 - [1 - (l/n)]^k$. Since the C_i's are independent, the probability that C intersects C_i for each i is $p = \left[1 - [1 - (l/n)]^k\right]^m$. Hence, the expectation EY_l of the number Y_l of sets of cardinality l which intersects each C_i is given by

$$\binom{n}{l} \left(1 - \left(1 - \frac{l}{n}\right)^k\right)^m . \tag{1}$$

Setting $m = \lambda n$ and $l = \beta n$, we get

$$\log EY_l \sim n \left(-\lambda \log \frac{1}{1 - (1 - \beta)^k} + \log \frac{1}{\beta^\beta (1 - \beta)^{1-\beta}} \right).$$

This implies $EY_l = o(1)$ whenever

$$\lambda > \lambda_\beta = \frac{\log \frac{1}{\beta^\beta (1-\beta)^{1-\beta}}}{\log \frac{1}{1-(1-\beta)^k}}. \tag{2}$$

The following assertions can be easily deduced from this formula.

Assertion 1. For any fixed k, β tends to 1 when λ tends to ∞.
Assertion 2. For any fixed λ, the ratio $\beta k / \log k$ tends to 1 when k tends to ∞.

Proof of assertion 1.
Putting $\gamma = 1 - \beta$, and using the inequality $\binom{n}{l} \le (ne/l)^l$, we get from (1),

$$E(Y_l)^{1/n} \le \left(1 - \gamma^k \right)^\lambda \left(\frac{e}{\gamma} \right)^\gamma.$$

Suppose for a contradiction that $\gamma \ge \gamma_o > 0$. Then, when $\lambda \to \infty$, the first term on the righthandside tends to 0 whereas one can easily check by taking derivatives that the second term is bounded above by e. Therefore, $EY_l \to 0$ as $\lambda \to \infty$.

Proof of Assertion 2.
Let us prove first that we have $\beta \ge k^{-1} \log k (1 - o(1))$. Inserting in (2) the value $\beta = (\log k - 2 \log \log k)/k$ and using the inequality $\beta \log(\beta^{-1}) \le 1$, we get

$$\lambda_\beta \le \frac{\log \frac{1}{1-\beta}}{\log \frac{1}{1-e^{-\beta k}}} \le \frac{\beta + 2\beta^2}{e^{-\beta k} - e^{-\beta^2 k^2}} \le \frac{2k^{-1} \log k}{k^{-1} \log^2 k (1 - o(1))} = o(1).$$

The proof of the reverse inequality is similar and is omitted.

3.2 The GRECO algorithm

For any family $\mathcal{C}$ of sets, a set H is a hitting set for $\mathcal{C}$ if H has a non-empty intersection with each element of this family and the minimum hitting set problem is that of finding a hitting set of minimum cardinality (see also [6]). We note that this problem is the dual (via the interchange of the two vertex sets of the incidence graph) of the set cover problem.

We note that recently, Lund and Yannakakis in [10] have proved that unless NP $\subseteq$ DTIME$[n^{\text{poly} \log n}]$ (conjecture weaker than P $=$ NP but highly improbable), there is no polynomial time algorithm approximating the optimal solution of these two problems with a ratio smaller than $c \log n$ for a constant $c < 1/4$ (this result is of course a "worst case" result). Consequently, average case studies of approximation algorithms for these

```
begin
        split every set Y_i with degree d_i > η into d_i new vertices of degree 1

        for h ← η to 1 do
            while |X| ≠ 0 do
                    choose randomly one of the Y_i's with maximum d_i;
                    delete Y_i;
                    delete X̄ = X_{i_1}, …, X_{i_h}, the set of the classes in P, incident to Y_i in Π;
                    delete any other vertex in Y adjacent to at least one of the classes of X̄;
                    let X, Y, P, Q denote now the new configuration;
                    let Π denote the survived incidence graph
            od
        od
end;
```

Algorithm 1: **GRECO** algorithm. We suppose that after the splitting of the Y_i performed in the first line of the algorithm, the incidence graph Π is left untouched).

```
begin
        γ ← ⌈n log k/k⌉;
        SC ← {x_1, …, x_γ}
        for j ← 1 to m do
            if C_j ∩ SC = ∅ then add to SC an arbitrarily chosen element of C_j
        od;
end.
```

Algorithm 2. QUICK algorithm

problems are of high theoretical and practical importance. Let us mention that another algorithm due to Karp has been analysed with the same probability model ([7]).

Let the family of sets $\mathcal{P} = \{X_1, X_2, \ldots, X_m\}$, together with the corresponding partitions P and Q, be defined as above via the random pairing Π. We consider the greedy algorithm 1 for finding a hitting set for $\mathcal{P}$. Informally, our algorithm **GRECO** selects at each step one element chosen at random between the elements hitting the biggest number of sets which have not been hitted before. It depends on an integer parameter η.

Before proceeding to the analysis of this algorithm let us remark that when the product λk is high, the distribution of the degrees tends to uniformity. It turns out that in this case the much simpler algorithm **QUICK** (algorithm 2) provides asymptotically optimal solutions.

Let us proceed to the analysis of **QUICK**.

Plainly the size of the solution is bounded above by $\gamma + \mu$, where μ denotes the number of elements of $\mathcal{C}$ which do not intersect the set $\{x_1, \ldots, x_\gamma\}$. We have

$$E\mu = \lambda n \, \frac{\binom{n-\gamma}{k}}{\binom{n}{k}} \sim \lambda n \left(1 - \frac{\log k}{k} \right)^k .$$

A simple computation yields that the variance of μ is of a smaller order than the square of its expectation. This implies, using Tchebicheff inequality and the familiar inequality $1 + x < e^x$, that $\Pr[\mu \le \lambda n/k] \to_{n \to \infty} 1$.

Thus, again with probability tending to 1, the total size of the solution found by **QUICK** is bounded above by $(n \log k/k) + (\lambda n/k)$.

Assertion 2 then implies that the approximation ratio of **QUICK** tends to 1 when k tends to ∞. Moreover, it can be shown that algorithm **GRECO** works better than **QUICK**; thus, assertion 2 provides also an approximation guarantee for **GRECO** when k tends to ∞.

3.3 Analysis of GRECO on random instances

Let us consider again a random configuration $X, Y, \mathcal{P}, \mathcal{Q}, \Pi$ and the corresponding partitions P and Q. We are interested here in a system of m random sets of size k, drawn from a ground set of size n and we assume that the ratio m/n is fixed, say $m/n = \lambda$, and we let m (and n) go to infinity. We must analyse separately the phases of the algorithm corresponding to the successive removals of the $\mathcal{Q}$-vertices of degree $\eta, \eta - 1, \ldots, 1$. Let h be a positive integer satisfying $1 \le h \le \eta$ and assume that, after having used vertices with degrees greater than h, there remain, for $1 \le j \le h$, precisely m_j $\mathcal{Q}$-vertices of degree j. Then, as it was pointed out before, the conditional distribution of the remaining incidence graph (conditioned by the previous steps of the algorithm) coincides with the "random pairing" distribution corresponding to the n_j's and m_j's (the n_j's being here all equal to k).

Let us remove a (randomly chosen) $\mathcal{Q}$-vertex of degree h. Then:

- $\Pr[h$ new sets in $\mathcal{P}$ are captured$] = 1 - o(1)$;
- the total decrease of the degrees of the remaining vertices has expectation $h(k-1) - o(1)$ and the probability that the degree of any given $\mathcal{Q}$-vertex is decreased by 1 is equal to $[(h(k-1) - o(1))j]/\sum_{1 \le i \le h} im_i$, where j denotes the degree of X;
- the decrease of the total size of the remaining $\mathcal{Q}$-vertices has expectation $h(k-1) + o(1)$ and this implies that, setting $S = h(k-1)/\sum_{1 \le i \le h} im_i$, the expectations $E\Delta m_j$ of the increments Δm_j satisfies

$$\begin{aligned} E\Delta m_j &= (j+1)m_{j+1}S - jm_jS, \ 1 \le j \le h-1, \\ E\Delta m_h &= -1 - hm_hS. \end{aligned}$$

Let $R_{\eta,h} = (1/n)\sum_{1 \le i \le h} im_i$ denote the average degree of the $\mathcal{Q}$ vertices. Notice that the total decrease of the degrees of these vertices in one step is $h + h(k-1) = hk$. Standard results concerning the approximation of "small steps Markov chains" (see for instance

Proposition 4.1 in [9]) imply, setting $y_{\eta,h,j} = m_j/n$ and $\theta = t/n$, that the $y_{\eta,h,j}$'s are well approximated as $n \to \infty$ by the solution $\vec{y}$ of the following system of differential equations (where we use Newton's notation for $dy/d\theta$)

$$(\mathcal{S}_{\eta,h}) \quad \begin{cases} \dot{y}_{\eta,h,j}(\theta) = (j+1)\dfrac{h(k-1)}{R_{\eta,h}-hk\theta}y_{\eta,h,j+1}(\theta) - j\dfrac{h(k-1)}{R_{\eta,h}-hk\theta}y_{\eta,h,j}(\theta), \quad 1 \le j \le h-1 \\ \dot{y}_{\eta,h,h}(\theta) = -1 - h\dfrac{h(k-1)}{R_{\eta,h}-hk\theta}y_{\eta,h,h}(\theta) \end{cases}$$

in the sense that, $\sup_{\theta} |y_{\eta,h,j}(\theta) - (1/n)m_j(\theta)| \to 0$ as $n \to \infty$, in probability for each $1 \le j \le h$ if only the initial conditions $y_{\eta,h,j}(0)$, $1 \le j \le \eta$ tend to fixed limits.
This concludes the description of the systems $(\mathcal{S}_{\eta,h})$, $1 \le h \le \eta$.

Let us mention that the use of three indices η, h and j which may seem luxurious to the reader, is in fact needed for the mathematical study of the above systems (see the appendix).
Let us emphasize now how we use these systems to analyse **GRECO** with parameter η. We integrate first $(\mathcal{S}_{\eta,\eta})$ under the following initial conditions:

$$y^I_{\eta,\eta,1}(0) = e^{-\lambda} + \sum_{j=\eta+1}^{\infty} \frac{e^{-\lambda}\lambda^j}{(j-1)!}$$

$$y^I_{\eta,\eta,j}(0) = \frac{e^{-\lambda}\lambda^j}{j!}, \quad 1 \le j \le \eta \tag{3}$$

where $\lambda = m/n$ is the average degree of the $\mathcal{Q}$-vertices of Π.
Observe that for each j, the corresponding RHS of the above equalities is equal to the almost sure limit of $m_j(0)/n$ as $n \to \infty$. Also, $R_{\eta,\eta} = \sum_{j=1}^{\eta} jy^I_{\eta,\eta,j}$. We denote by $\theta_{\eta,\eta}$ the first time at which the last coordinate becomes 0:

$$\theta_{\eta,\eta} = \min\{\theta > 0 : y_{\eta,\eta,\eta}(\theta) = 0\}.$$

We set $y^I_{\eta,\eta-1,j} = y_{\eta,\eta,j}(\theta_{\eta,\eta})$, $1 \le j \le \eta - 1$, and $R_{\eta,\eta-1} = \sum_{j=1}^{\eta-1} jy^I_{\eta,\eta-1,j}$. Then, we turn to the system $(\mathcal{S}_{\eta,\eta-1})$ of dimension $(\eta-1) \times (\eta-1)$ which we integrate with the initial conditions: $\theta = 0$ and $y_{\eta,\eta-1,j}(0) = y^I_{\eta,\eta-1,j}$, $1 \le j \le \eta - 1$. We define $\theta_{\eta,\eta-1}$ by

$$\theta_{\eta,\eta-1} = \min\{\theta > 0 : y_{\eta,\eta-1,\eta-1}(\theta) = 0\}$$

and we repeat inductively this procedure for each $h \ge 1$, taking as initial conditions for the system $(\mathcal{S}_{\eta,h})$ the final conditions for $(\mathcal{S}_{\eta,h+1})$.

We define

$$\tau_\eta = \sum_{h=1}^{\eta} \theta_{\eta,h}.$$

Our aim is to study the asymptotic behaviour of the sequence $(\tau_\eta)_{\eta \in I\!N}$. We observe first that, using inductively the approximation theorem of [9], we get immediately the following theorem.

Theorem 1. *There exists a function $f(.,.,.)$ such that, for any fixed values of the parameters k, η and λ, the size $S = S(k, \eta, \lambda, n)$ of the solution found by* **GRECO** *when*

7

applied with the parameter η to a random instance of m k-sets of size k drawn randomly from a set of size $m = n\lambda^{-1}$, satisfies

$$\frac{S}{n.f(k,\eta,\lambda)} \to 1$$

in probability as $n \to \infty$.

We will prove in the appendix that the sequence $(\tau_\eta)_{\eta \in \mathbb{N}^*}$ tends to a limit as $\eta \to \infty$. This implies immediately the following theorem.

Theorem 2. *There exists a function $g(.,.)$ such that, for any fixed values of the parameters k and λ and any positive ϵ, the size $S = S(k,\eta,\lambda,n)$ of the solution found by* **GRECO** *when applied with the parameter η to a random instance of m k-sets of size k drawn randomly from a set of size $m = n\lambda^{-1}$, satisfies*

$$1 - \epsilon \le \frac{S}{n.g(k,\lambda)} \le 1 + \epsilon$$

in probability as $n \to \infty$, if only η is sufficiently large.

3.4 Proof of the convergence of of the sequence $(\tau_\eta)_{\eta \in \mathbb{N}^*}$

3.4.1 Explicit integration of the differential systems

Let us firs reformulate the system $(\mathcal{S}_{\eta,h})$. By introducing $\beta = (k-1)/k$ and $\alpha_{\eta,h} = (1/hk)R_{\eta,h}$, $1 \le h \le \eta$, the vectors of $\mathbb{R}^h$: $y_{\eta,h}(\theta) = (y_{\eta,h,j}(\theta))_{1 \le j \le h}^T$, $e_h = (0,0,\ldots,-1)^T$ and the matrix M_h of dimension $h \times h$ with

$$M_h = \begin{cases} M_{h,j,j} &= -j \\ M_{h,j,j+1} &= j+1 \\ M_{h,i,j} &= 0 \qquad \text{elsewhere} \end{cases}$$

we can reformulate $(\mathcal{S}_{\eta,h})$ in a vectorial form as follows:

$$(\mathcal{S}_{\eta,h}) \qquad \dot{y}_{\eta,h}(\theta) = \frac{\beta}{\alpha_{\eta,h} - \theta} M_h y_{\eta,h}(\theta) + e_h.$$

We define also

$$\tau_\eta = \sum_{h=1}^{\eta} \theta_{\eta,h}.$$

Our aim is to study the asymptotic behaviour of the infinite sequence $(\tau_\eta)_{\eta \in \mathbb{N}}$.

We consider $(\mathcal{S}_{\eta,h})$ when $\theta \in [0, \alpha_{\eta,h}[$. It is a vectorial first-order linear non-homogeneous non-autonomous o.d.e. (ordinary differential equation). The system's coefficients are continuous functions of the variable θ on $[0, \alpha_{\eta,h}[$. So, Cauchy's theorem about the existence and uniqueness of the solutions of linear Cauchy's problems is applicable here,

and moreover the non-extendable solution of $(\mathcal{S}_{\eta,h})$ under the initial conditions $y_{\eta,h}(0) = y_{\eta,h}^I$, is defined everywhere on $[0, \alpha_{\eta,h}[$, ([1], p. IV, 17).

To solve $(\mathcal{S}_{\eta,h})$, we consider the associated homogeneous o.d.e.:

$$(\mathcal{H}_{\eta,h}) \qquad \dot{y}(\theta) = \frac{\beta}{\alpha_{\eta,h} - \theta} M_h y(\theta).$$

The resolvant of this homogeneous system is $\Phi(\theta) = \exp\{\beta \ln(\alpha_{\eta,h}/\alpha_{\eta,h} - \theta) M_h\}$. The calculation of the characteristic polynomial of M_h shows that M_h possesses h distinct eigenvalues, namely $-1, -2, \ldots, -h$, hence M_h is diagonalizable. If b_j is an eigenvector associated to eigenvalue $-j$, for each j in $\{1, 2, \ldots, h\}$, then $\mathcal{B} = (b_1, \ldots, b_h)$ is a basis of $\mathbb{R}^h$ and, in terms of $\mathcal{B}$, M_h is expressed by means of the diagonal matrix $D_h = \left(-i\delta_i^j\right)_{i,j}$, where δ_i^j is the Kronecker symbol. Denoting by P the matrix of the change of coordinates from $\mathcal{B}$ to the canonical basis, we have $M_h = P D_h P^{-1}$. For each j, we take $b_j = (b_{j,1}, \ldots, b_{j,h})^T$ where

$$b_{j,i} = \begin{cases} (-1)^{i+j} \binom{j}{i} & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i > j \end{cases}$$

The matrix P is then defined as

$$P_{i,j} = \begin{cases} (-1)^{i+j} \binom{j}{i} & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i > j \end{cases}$$

and its inverse matrix $P^{-1} = Q$ is defined as

$$Q_{i,j} = \begin{cases} \binom{j}{i} & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i > j \end{cases}$$

Then $\Phi(\theta) = \exp\{\beta \ln(\alpha_{\eta,h}/\alpha_{\eta,h} - \theta) P D_h P^{-1}\} = P \exp\{\beta \ln(\alpha_{\eta,h}/\alpha_{\eta,h} - \theta) D_h\} P^{-1}$, and we can explicitly compute its components, obtaining so:

$$\begin{cases} \Phi_{i,j}(\theta) = 0 & \text{if} \quad i > j \\ \Phi_{i,j}(\theta) = \binom{j}{i} \left(\frac{\alpha_{\eta,h}}{\alpha_{\eta,h} - \theta}\right)^{-\beta i} \left(1 - \left(\frac{\alpha_{\eta,h}}{\alpha_{\eta,h} - \theta}\right)^{-\beta}\right)^{j-i} & \text{if} \quad i \le j \end{cases} \tag{4}$$

So, we have expicitely integrated $(\mathcal{H}_{\eta,h})$. The solution of $(\mathcal{S}_{\eta,h})$ under the initial conditions $y_{\eta,h}(0) = y_{\eta,h}^I$ is , for $\theta \in [0, \alpha_{\eta,h}[$, ([1], p. IV, 21):

$$y_{\eta,h}(\theta) = \Phi(\theta) y_{\eta,h}^I + \Phi(\theta) \left(\int_0^\theta \Phi^{-1}(\sigma) d\sigma\right) e_h \tag{5}$$

Let

$$u_{\eta,h}(\theta) = \Phi(\theta) \left(\int_0^\theta \Phi^{-1}(\sigma) d\sigma\right) e_h. \tag{6}$$

Since $u_{\eta,h}$ is the solution of $(\mathcal{S}_{\eta,h})$ under the initial conditions $u_{\eta,h}(0) = 0$, one can verify by a straightforward calculation that, for every $j = 1, \ldots, h$ and for every $\theta \in [0, \alpha_{\eta,h}[$,

$$u_{\eta,h,j}(\theta) = -\binom{h}{j} \int_0^\theta \left(\frac{\alpha_{\eta,h} - \theta}{\alpha_{\eta,h} - t}\right)^{-\beta j} \left(1 - \left(\frac{\alpha_{\eta,h} - \theta}{\alpha_{\eta,h} - t}\right)^\beta\right)^{h-j} dt. \tag{7}$$

So, with expressions (4,5,6,7), we have explicitely resolved the system $(\mathcal{S}_{\eta,h})$ under the initial conditions $y_{\eta,h}(0) = y_{\eta,h}^I$.

3.4.2 The computation of $\theta_{\eta,h}$

Using the results of § 3.4.1, we have, $\forall \theta \in [0, \alpha_{\eta,h}[$,

$$y_{\eta,h,h}(\theta) = \left(\frac{\alpha_{\eta,h}}{\alpha_{\eta,h} - \theta}\right)^{-\beta h} y_{\eta,h,h}^I + \frac{1}{\beta h - 1} \left(\frac{\alpha_{\eta,h}^{-\beta h+1}}{(\alpha_{\eta,h} - \theta)^{-\beta h}} - (\alpha_{\eta,h} - \theta)\right).$$

Assuming that $y_{\eta,h,h}^I > 0$, we search for these $\theta \in [0, \alpha_{\eta,h}[$ verifying $y_{\eta,h,h}(\theta) = 0$. By a straightforward computation, we obtain: if $y_{\eta,h,h}^I > 0$, $k \geq 2$ and $h \geq 2$, then there exists a unique $\theta_{\eta,h} \in [0, \alpha_{\eta,h}[$ such that $y_{\eta,h,h}(\theta_{\eta,h}) = 0$ given by

$$\theta_{\eta,h} = \alpha_{\eta,h} \left[1 - \left(1 + \frac{\beta h - 1}{\alpha_{\eta,h}} y_{\eta,h,h}^I\right)^{\frac{1}{-\beta h+1}}\right]. \tag{8}$$

3.4.3 The signs of $y_{\eta,h,j}^I$

Let us start by establishing the following assertions.

(A1) Let $1 \leq h \leq \eta$ and suppose that $\forall j$, $1 \leq j \leq h$, $y_{\eta,h,j}^I > 0$. Then:
(i) $\forall j \in \{1, 2, \ldots, h - 1\}$, $\forall \theta \in [0, \theta_{\eta,h}[$, $y_{\eta,h,j}(\theta) > 0$;
(ii) $\forall j \in \{1, 2, \ldots, h - 1\}$, $y_{\eta,h-1,j}^I = y_{\eta,h,j}(\theta_{\eta,h}) \geq 0$. ∎
To show (i), we reason by contradiction in assuming that there exist $\theta \in [0, \theta_{\eta,h}[$ and j, $1 \leq j \leq h - 1$, such that $y_{\eta,h,j}(\theta) \leq 0$. Since $y_{\eta,h,j}(0) = y_{\eta,h,j}^I > 0$, we have $\theta > 0$. By an argument of connectedness, there exists $\theta_1 \in]0, \theta]$ such that $y_{\eta,h,j}(\theta_1) = 0$. Let us denote by $J = \{j : 1 \leq j \leq h - 1, y_{\eta,h,j}([0, \theta_{\eta,h}[) \ni 0\}$, $J \neq \emptyset$. For each $j \in J$, let us introduce

$$\begin{aligned}
\theta(j) &= \inf\{\theta \in [0, \theta_{\eta,h}[: y_{\eta,h,j}(\theta) = 0\} = \inf\{[0, \theta_{\eta,h}] : y_{\eta,h,j}(\theta) = 0\} \\
&= \inf([0, \theta_{\eta,h}] \cap y_{\eta,h,j}^{-1}(\{0\})).
\end{aligned}$$

Since this last set is closed and bounded below, it contains its greatest lower bound, thus $y_{\eta,h,j}(\theta(j)) = 0$. As, on the other hand, $y_{\eta,h,j}(0) > 0$ and $y_{\eta,h,j}$ is continuous, then $y_{\eta,h,j}$ is positive on a neighbourhood of 0, hence $\theta(j) = 0$. By definition of $\theta(j)$, for all $\theta \in [0, \theta(j)[$, we have $y_{\eta,h,j}(\theta) > 0$. Let us introduce $\hat{\theta} = \min\{\theta(j) : j \in J\}$; we have $0 < \hat{\theta} < \theta_{\eta,h}$ and let $\hat{j} = \max\{j \in J : \theta(j) = \hat{\theta}\}$. Note that j cannot be equal to $h - 1$. In fact, if

the contrary holds, we have $\forall \theta \in [0, \hat{\theta}[$, $y_{\eta,h,h-1}(\theta) > 0$ and $y_{\eta,h,h-1}(\hat{\theta}) = 0$, which implies $\dot{y}_{\eta,h,h-1}(\hat{\theta}) \leq 0$. By using $(\mathcal{S}_{\eta,h})$, we get

$$0 \geq \dot{y}_{\eta,h,h-1}(\hat{\theta}) = h\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}}y_{\eta,h,h}(\hat{\theta}) - (h-1)\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}}y_{\eta,h,h-1}(\hat{\theta}) = h\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}}y_{\eta,h,h}(\hat{\theta}) > 0$$

which is impossible.

Consequently, we have proved that $1 \leq \hat{j} \leq h - 2$. Let us note that if $\hat{j} \leq j \leq h$ and if $\theta \in [0, \hat{\theta}]$, then $y_{\eta,h,j}(\theta) > 0$ because $j > \hat{j} \implies (j \notin J \vee \theta(j) > \hat{\theta})$.

Since $y_{\eta,h,\hat{j}}(\theta) > 0$ for $\theta \in [0, \hat{\theta}]$, and $y_{\eta,h,\hat{j}}(\hat{\theta}) > 0$, we have $\dot{y}_{\eta,h,\hat{j}}(\hat{\theta}) \leq 0$. This, by using $(\mathcal{S}_{\eta,h})$, implies that

$$
\begin{aligned}
0 \geq \dot{y}_{\eta,h,\hat{j}}(\hat{\theta}) &= (\hat{j}+1)\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}}y_{\eta,h,\hat{j}+1}(\hat{\theta}) - \hat{j}\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}}y_{\eta,h,\hat{j}}(\hat{\theta}) \\
&= (\hat{j}+1)\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}}y_{\eta,h,\hat{j}+1}(\hat{\theta}) > 0
\end{aligned}
$$

a contradiction.

This justifies (i). On the other hand, (ii) is a consequence of (i) because of the continuity of $y_{\eta,h,j}$.

(A2) Under the hypotheses of **(A1)** let $1 \leq j \leq h-1$. If $y_{\eta,h,j}(\theta_{\eta,h}) = 0$, then $\forall i$, $j \leq i \leq h$ we have $y_{\eta,h,i}(\theta_{\eta,h}) = 0$. ∎

In fact, with **(A1)** we know that for all $\theta \in [0, \theta_{\eta,h}[$, $y_{\eta,h,j}(\theta) > 0$ and $y_{\eta,h,j}(\theta_{\eta,h}) = 0$, which implies

$$
\begin{aligned}
0 \geq \dot{y}_{\eta,h,j}(\theta_{\eta,h}) &= (j+1)\frac{\beta}{\alpha_{\eta,h} - \theta_{\eta,h}}y_{\eta,h,j+1}(\theta_{\eta,h}) - j\frac{\beta}{\alpha_{\eta,h} - \hat{\theta}_{\eta,h}}y_{\eta,h,j}(\theta_{\eta,h}) \\
&= (j+1)\frac{\beta}{\alpha_{\eta,h} - \theta_{\eta,h}}y_{\eta,h,j+1}(\theta_{\eta,h}) \geq 0.
\end{aligned}
$$

Thus, $y_{\eta,h,j+1}(\theta_{\eta,h}) = 0$.

We can repeat the same reasoning for $y_{\eta,h,j+2}$ and so on.

(A3) Under the hypotheses of **(A1)** we have:
$(\forall j = 1, \ldots, h-1, \ y_{\eta,h,j}(\theta_{\eta,h}) > 0) \iff (y_{\eta,h,h-1}(\theta_{\eta,h}) > 0)$. ∎
In fact, the first implication is obvious and the converse results from **(A2)**.

We also remark that $y_{\eta,h,h-1}(\theta_{\eta,h}) = 0 \iff \dot{y}_{\eta,h,h-1}(\theta_{\eta,h}) = 0$. In fact, it suffices to remark that by the definition of $\theta_{\eta,h}$, $\dot{y}_{\eta,h,h-1}(\theta_{\eta,h}) = -(h-1)(\beta/\alpha_{\eta,h} - \theta_{\eta,h})y_{\eta,h,h-1}(\theta_{\eta,h})$.

3.4.4 Relations between $\theta_{\eta,\mathrm{h}}$ and $\alpha_{\eta,\mathrm{h}}$

We multiply the j-th equation of $(\mathcal{S}_{\eta,h})$ by j and next, we add the h resulting equations; we thus obtain $\sum_{j=1}^{h} j\dot{y}_{\eta,h,j}(\theta) = -h - [\beta/(\alpha_{\eta,h} - \theta)]\sum_{j=1}^{h} jy_{\eta,h,j}(\theta)$.
Denoting by $z_{\eta,h}(\theta) = \sum_{j=1}^{h} jy_{\eta,h,j}(\theta)$, we see that $z_{\eta,h}$ is a solution of the o.d.e.:

$$\dot{z}_{\eta,h}(\theta) = -h - \frac{\beta}{\alpha_{\eta,h} - \theta}z_{\eta,h}(\theta) \tag{9}$$

11

under the initial condition $z_{\eta,h}(0) = R_{\eta,h}$. We note that equation (9) is an one-dimensional linear non-homogeneous o.d.e. with variable coefficient (continuous function of θ) and consequently, the theorem about the existence and uniqueness of the non-extendable solutions of Cauchy's problems is usable in (9). Integrating (9), for instance using the method of constants variations, we obtain, $\forall \theta \in [0, \alpha_{\eta,h}[$,

$$z_{\eta,h}(\theta) = R_{\eta,h}\left(1 - \frac{\theta}{\alpha_{\eta,h}}\right)^\beta + \frac{\alpha\eta,h}{-\beta+1}h\left[\left(1 - \frac{\theta}{\alpha_{\eta,h}}\right) - \left(1 - \frac{\theta}{\alpha_{\eta,h}}\right)^\beta\right] = R_{\eta,h} - hk\theta.$$

On the other hand, $z_{\eta,h}(\theta_{\eta,h}) = \sum_{j=1}^{h} j y_{\eta,h,j}(\theta_{\eta,h}) = \sum_{j=1}^{h-1} j y_{\eta,h-1,j}^I = R_{\eta,h-1}$, that implies $R_{\eta,h-1} = R_{\eta,h} - kh\theta_{\eta,h}$, $2 \leq h \leq n-1$, or equivalently

$$\alpha_{\eta,h-1} = \left(\frac{h}{h-1}\right)(\alpha_{\eta,h} - \theta_{\eta,h}). \tag{10}$$

By introducing the 2-variable function $T_h(\alpha,y) = \alpha\left[1 - [1 + (\beta h - 1)/\alpha y]^{1/(-\beta h+1)}\right]$ for $\alpha > 0$, $y > 0$, we see, using expression (8), that $\theta_{\eta,h} = T_h(\alpha_{\eta,h}, y_{\eta,h,h}^I)$. One can also verify that if $k \geq 3$, $h \geq 2$, $\alpha > 0$, $y > 0$ and $k\alpha \geq y$ then

$$\frac{1}{-\beta h + 1}\ln\left(1 - \frac{-\beta h + 1}{\alpha}y\right) \geq \frac{1}{1-(k-1)(h-1)}\ln((k-1)(h-1)).$$

By taking for a short while $X = (k-1)(h-1) \geq 2$, the previous inequality allows us to obtain $[1/(-\beta h + 1)]\ln[1 - (-\beta h + 1)/\alpha y] \geq \ln(X)/(1-X) \geq \ln(1/2)$; therefore, taking the exponential mapping, we get $[1 + (\beta h - 1)/\alpha y]^{[1/(-\beta h+1)]} \geq 1/2$; consequently, if $k \geq 3$, $h \geq 2$, $\alpha > 0$, $y > 0$ and $k\alpha \geq y$, then

$$T_h(\alpha,y) \leq \frac{\alpha}{2}. \tag{11}$$

From the above expression, since $k\alpha_{\eta,h} = (1/h)R_{\eta,h} \geq y_{\eta,h,h}^I$, we deduce, $\forall k \geq 3$, $\forall h \geq 2$

$$\theta_{\eta,h} \leq \frac{1}{2}\alpha_{\eta,h}.$$

3.4.5 An upper bound of $y_{\eta,h,j}$

By adding the h equations of the system $(\mathcal{S}_{\eta,h})$, we obtain after some calculation the following expression: $\sum_{j=1}^{h} \dot{y}_{\eta,h,j}(\theta) = -1 - (\beta/\alpha_{\eta,h})y_{\eta,h,1}(\theta)$. Since $y_{\eta,h,1}$ is non-negative on $[0, \theta_{\eta,h}]$, the derivative of the function $\sum_{j=1}^{h} y_{\eta,h,j}$ is non-positive on the same interval, hence this function is monotonically decreasing on $[0, \theta_{\eta,h}]$, and consequently, $\forall \theta \in [0, \theta_{\eta,h}]$

$$\sum_{j=1}^{h} y_{\eta,h,j}(\theta) \leq \sum_{j=1}^{h} y_{\eta,h,j}^I. \tag{12}$$

By taking $\theta = \theta_{\eta,h}$, we get $\sum_{j=1}^{h-1} y_{\eta,h-1,j}^I \leq \sum_{j=1}^{h} y_{\eta,h,j}^I$ which implies that $\forall h \leq \eta$

$$\sum_{j=1}^{h} y_{\eta,h,j}^I \leq \sum_{j=1}^{\eta} y_{\eta,\eta,j}^I \leq 1. \tag{13}$$

Also, from the above expression and expression (12), we deduce, $\forall n \in I\!N^*$, $\forall h \in I\!N^*$, $h \leq \eta$, $\forall j \in I\!N^*$, $j \leq h$, $\forall \theta \in [0, \theta_{\eta,h}]$

$$0 \leq y_{\eta,h,j}(\theta) \leq 1. \tag{14}$$

On the other hand, using expressions (5) and (6), we get:

$$y_{\eta,h,j}(\theta) = \sum_{p=1}^{h} \Phi_{j,p}(\theta) y_{\eta,h,p}^I + u_{\eta,h,p}(\theta).$$

We easily verify, using expression (7), that $u_{\eta,h,p}(\theta) \leq 0$; thus using expression (4), we obtain

$$y_{\eta,h,j}(\theta) \leq \sum_{p=j}^{h} \binom{p}{j} \left(\frac{\alpha_{\eta,h} - \theta}{\alpha_{\eta,h}} \right)^{j\beta} \left[1 - \left(\frac{\alpha_{\eta,h} - \theta}{\alpha_{\eta,h}} \right)^{\beta} \right]^{p-j} y_{\eta,h,p}^I. \tag{15}$$

In what follows, we will need inequality (16), whose demonstration is elementary:

$$x^i e^{(1-x)\lambda} \leq 1, \quad x \in]0,1], \lambda > 0, i \in I\!N, i \geq \lambda. \tag{16}$$

We are going to establish, by downward induction, inequalities (17) and (18), for all integers $\eta, h, j \in I\!N$, $\lambda \leq j \leq h \leq \eta$ and $\forall \theta \in [0, \theta_{\eta,h}]$.

$$y_{\eta,h,j}^I \leq \frac{e^{-\lambda} \lambda^j}{j!}. \tag{17}$$

In fact, we initialize the induction at $h = \eta$ with expression (3). We suppose true the inequalities for a fixed h. Then, in the step $h - 1$, by putting $x = [(\alpha_{\eta,h} - \theta_{\eta,h})/\alpha_{\eta,h}]^{\beta}$, we use expression (15) to obtain, for $\theta = \theta_{\eta,h}$,

$$y_{\eta,h-1,j}^I = y_{\eta,h,j}(\theta_{\eta,h}) \leq \sum_{p=j}^{h} \binom{p}{j} x^j (1-x)^{p-j} y_{\eta,h,p}^I;$$

next, using the induction hypothesis, we get for $y_{\eta,h-1,j}^I$:

$$
\begin{aligned}
y_{\eta,h-1,j}^I &\leq \sum_{p=j}^{h} \binom{p}{j} x^j (1-x)^{p-j} \frac{e^{-\lambda} \lambda^p}{p!} \\
&\leq e^{-\lambda} x^j \frac{\lambda^j}{j!} \sum_{p=j}^{h} \frac{1}{(p-j)!} (1-x)^{p-j} \lambda^{p-j} = \frac{e^{-\lambda} \lambda^j}{j!} x^j e^{(1-x)\lambda}
\end{aligned}
$$

and, finally, with expression (16) we obtain (17).

$$y_{\eta,h,j}(\theta) \leq \frac{e^{-\lambda} \lambda^j}{j!}. \tag{18}$$

In fact, by taking now $x = [(\alpha_{\eta,h} - \theta)/\alpha_{\eta,h}]^{\beta}$ and by using (15), we get:

$$y_{\eta,h,j}(\theta) \leq \sum_{p=j}^{h} \binom{p}{j} x^j (1-x)^{p-j} y_{\eta,h,p}^I.$$

13

Next, using expressions (16) and (17), we obtain

$$y_{\eta,h,j}(\theta) \le \sum_{p=j}^{h} \binom{p}{j} x^j (1-x)^{p-j} \frac{e^{-\lambda}\lambda^p}{p!} \le \frac{e^{-\lambda}\lambda^j}{j!} x^j e^{(1-x)\lambda} \le \frac{e^{-\lambda}\lambda^j}{j!}$$

q.e.d.

Let us now introduce the constant

$$\Xi = \frac{1}{\displaystyle\min_{j \in I\!\!N, 1 \le j \le \lambda} \left\{ \frac{e^{-\lambda}\lambda^j}{j!} \right\}}. \tag{19}$$

We note that Ξ does not depend on λ and also that $\Xi > 1$, since $e^{-\lambda}\lambda^j/j! < e^{-\lambda}e^{\lambda} = 1$. Using Ξ, we can prove the following expression.

$$0 \le y_{\eta,h,j}(\theta) \le \Xi \frac{e^{-\lambda}\lambda^j}{j!}, \quad \forall \eta, h, j \in I\!\!N \ \lambda \le j \le h \le \eta, \forall \theta \in [0, \theta_{\eta,h}] \tag{20}$$

In fact, if $j \ge \lambda$, then by using (18), ($\Xi > 1$) we get $y_{\eta,h,j}(\theta) \le e^{-\lambda}\lambda^j/j! \le \Xi e^{-\lambda}\lambda^j/j!$. On the other hand, if $j < \lambda$, then by expression (14) we get

$$y_{\eta,h,j}(\theta) \le 1 \le \Xi \frac{e^{-\lambda}\lambda^j}{j!}$$

q.e.d.

3.4.6 An upper bound of the sequence $(\tau_\eta)_{\eta \in I\!\!N^*}$

Using expression (10) and for $h \ge 2$, we get:

$$\theta_{\eta,h} = \alpha_{\eta,h} - \frac{h-1}{h}\alpha_{\eta,h-1} = \alpha_{\eta,h} - \alpha_{\eta,h-1} + \frac{1}{h}\alpha_{\eta,h-1}.$$

Consequently,

$$\tau_\eta = \sum_{h=1}^{\eta} \theta_{\eta,h} = \theta_{\eta,1} + \sum_{h=2}^{\eta}(\alpha_{\eta,h} - \alpha_{\eta,h-1}) + \sum_{h=2}^{\eta} \frac{1}{h}\alpha_{\eta,h-1} = \theta_{\eta,1} + \alpha_{\eta,\eta} - \alpha_{\eta,1} + \sum_{h=1}^{\eta-1} \frac{1}{h+1}\alpha_{\eta,h}.$$

Thus,

$$\tau_\eta = (\theta_{\eta,1} - \alpha_{\eta,1}) + \alpha_{\eta,\eta} + \sum_{h=1}^{\eta-1} \frac{1}{h+1}\alpha_{\eta,h}.$$

We know that $\theta_{\eta,1} \le \alpha_{\eta,1}$ and, by using expression (3) we obtain

$$\tau_\eta \le \frac{1}{\eta k} \sum_{j=1}^{\eta} j \frac{e^{-\lambda}\lambda^j}{j!} + \sum_{h=1}^{\eta-1} \frac{1}{h+1} \frac{1}{hk} \sum_{j=1}^{h} j y_{\eta,h,j}^I \le \frac{e^{-\lambda}}{\eta} \frac{\lambda}{k} \sum_{j=0}^{\eta-1} \frac{\lambda^j}{j!} + \sum_{h=1}^{\eta-1} \frac{1}{(h+1)hk} \sum_{j=1}^{h} j \frac{\Xi e^{-\lambda}\lambda^j}{j!}$$

the last inequality obtained using (20); hence

$$\tau_\eta \le \frac{e^{-\lambda}}{\eta} \frac{\lambda}{k} \sum_{j=0}^{\eta-1} \frac{\lambda^j}{j!} + \frac{\Xi e^{-\lambda}\lambda}{k} \sum_{h=1}^{\eta-1} \frac{1}{h(h+1)} \sum_{j=0}^{h-1} \frac{\lambda^j}{j!}.$$

14

Note that

$$\sum_{h=1}^{\eta-1} \frac{1}{h(h+1)} \sum_{j=0}^{h-1} \frac{\lambda^j}{j!} \le e^\lambda \sum_{h=1}^{\eta-1} \frac{1}{h(h+1)} = e^\lambda \sum_{h=1}^{\eta-1} \left(\frac{1}{h} - \frac{1}{h+1}\right) = e^\lambda \left(1 - \frac{1}{\eta}\right)$$

which leads to

$$\tau_\eta \le \frac{e^{-\lambda}\lambda}{k} \left(\frac{1}{\eta} \sum_{j=0}^{\eta-1} \frac{\lambda^j}{j!}\right) + \frac{\Xi\lambda}{k}\left(1 - \frac{1}{\eta}\right). \tag{21}$$

Note also that $(1/\eta)\sum_{j=0}^{\eta-1}(\lambda^j/j!) \le \max_{0 \le j \le n-1}\{\lambda^j/j!\}$. By denoting by $\lfloor\lambda\rfloor$ the floor of λ, we easily verify that: $\max_{0 \le j \le \lambda-1}\{\lambda^j/j!\} = \lambda^{\lfloor\lambda\rfloor-1}/(\lfloor\lambda\rfloor - 1)!$ and $\max_{\lambda-1 \le j}\{\lambda^j/j!\} = \lambda^{\lfloor\lambda\rfloor}/\lfloor\lambda\rfloor!$ and, since $\lambda \ge \lfloor\lambda\rfloor$, we have $\lambda^{\lfloor\lambda\rfloor}/\lfloor\lambda\rfloor! \ge \lambda^{\lfloor\lambda\rfloor-1}/(\lfloor\lambda\rfloor - 1)!$; consequently, we get $\max_{j\in I\!\!N}\{\lambda^j/j!\} = \lambda^{\lfloor\lambda\rfloor}/\lfloor\lambda\rfloor!$. Whenever $0 < \lambda < 1$, we have $\lfloor\lambda\rfloor = 0$, $\lfloor\lambda\rfloor! = 1$ and $\lambda^j/j! \le 1 = \lambda^0/0! = \lambda^{\lfloor\lambda\rfloor}/\lfloor\lambda\rfloor!$, therefore, $\forall\lambda > 0$,

$$\max_{j\in I\!\!N}\left\{\frac{\lambda^j}{j!}\right\} = \frac{\lambda^{\lfloor\lambda\rfloor}}{\lfloor\lambda\rfloor!}. \tag{22}$$

On the other hand, $1 - (1/\eta) \le 1$ and by (21) and (22) we deduce that

$$\tau_\eta \le \frac{e^{-\lambda}\lambda}{k} \frac{\lambda^{\lfloor\lambda\rfloor}}{\lfloor\lambda\rfloor!} + \frac{\Xi\lambda}{k} = \frac{\lambda}{k}\left[\frac{e^{-\lambda}\lambda^{\lfloor\lambda\rfloor}}{\lfloor\lambda\rfloor!} + \Xi\right]$$

and consequently,

$$\sup_{\eta\ge 2} \tau_\eta \le \frac{\lambda}{k}\left[\frac{e^{-\lambda}\lambda^{\lfloor\lambda\rfloor}}{\lfloor\lambda\rfloor!} + \Xi\right] < +\infty. \tag{23}$$

Since Ξ depends only on λ (see also (19)), the proposed upper bound only depends on λ and k.

3.4.7 Variations with respect to α

We consider for each $h \in I\!\!N^*$ and for each $\alpha > 0$, the following differential system in $I\!\!R^h$:

$$(\mathcal{C}_{h,\alpha}) \qquad \dot{y}(\theta) = \frac{\beta}{\alpha - \theta} M_h y(\theta) + e_h.$$

when $\theta \in [0, \alpha[$.

One can see that $(\mathcal{S}_{\eta,h}) = (\mathcal{C}_{h,\alpha_{\eta,h}})$.

We denote by $\varphi_h(\theta, \alpha, z^I)$ the value at the time θ of the solution of $(\mathcal{C}_{h,\alpha})$ that takes the value z^I at the time 0. Then, function $T_h(\alpha, z_h^I)$ introduced above denotes the first time the h-th coordinate $(\varphi_{h,h})$ of φ_h becomes 0. We can see that $y_{\eta,h}(\theta) = \varphi(\theta, \alpha_{\eta,h}, y_{\eta,h}^I)$, $\forall\theta \in [0, \alpha_{\eta,h}[$. We have already noted that $\theta_{\eta,h} = T_h(\alpha_{\eta,h}, y_{\eta,h,h}^I)$.

The homogeneous system associated with $(\mathcal{C}_{h,\alpha})$ is denoted $(\mathcal{L}_{h,\alpha})$ and the resolvant of $(\mathcal{L}_{h,\alpha})$ is denoted by $\Phi_h(\theta, \alpha)$.

With computations similar to the ones performed for the systems $(\mathcal{S}_{\eta,h})$, we establish the following: $\Phi_h(\theta, \alpha) = \exp\{\beta \ln[\alpha/(\theta - \alpha)] M_h\}$, with

$$\Phi_{h,i,j}(\theta, \alpha) = \begin{cases} 0 & \text{if } i > j \\ \binom{j}{i}\left(\frac{\alpha}{\alpha-\theta}\right)^{-\beta i}\left[1 - \left(\frac{\alpha}{\alpha-\theta}\right)^{-\beta}\right]^{j-i} & \text{if } i \le j \end{cases} \tag{24}$$

15

We have also $\forall j = 1, \ldots, h$, $\varphi_{h,j}(\theta) \geq 0$ in $[0, T_h(\alpha, z_h^I)]$ when $z_{h,i}^I > 0$.

The solution of $(\mathcal{C}_{h,\alpha})$ that is equal to z^I at the time 0 is $\varphi(\theta, \alpha, z^I) = \Phi_h(\theta, \alpha)z^I + \psi_h(\theta, \alpha)$, where

$$\psi_{h,j}(\theta, \alpha) = -\binom{h}{j} \int_0^\theta \left(\frac{\alpha - \theta}{\alpha - t}\right)^{\beta j} \left[1 - \left(\frac{\alpha - \theta}{\alpha - t}\right)^\beta\right]^{h-j} dt. \tag{25}$$

Starting from expression (25), we have, $\forall j \leq h$ and $\forall \theta \in [0, T_h(\alpha, z_h^I)]$

$$\psi_{h,j}(\theta) \leq 0. \tag{26}$$

Moreover, with a similar reasoning as in the case of expression (18), if $\forall j = 1, \ldots, h$, $z_j^I \leq \Xi e^{-\lambda} \lambda^j / j!$, then $\forall j = 1, \ldots, h$, $\forall \theta \in [0, T_h(\alpha, z_h^I)]$ we have

$$\varphi_{h,j}(\theta, \alpha, z^I) \leq \Xi \frac{e^{-\lambda} \lambda^j}{j!}. \tag{27}$$

Also by arguments similar to the ones of § 3.4.4 we establish that function

$$\theta \mapsto \sum_{j=1}^h j\varphi_{h,j}(\theta, \alpha, z^I)$$

is solution of the scalar o.d.e. $\dot{x}(\theta) = -h - [\beta/(\alpha - \theta)]x(\theta)$. We then deduce that

$$\sum_{j=1}^h j\varphi_{h,j}(\theta, \alpha, z^I) = \left(\sum_{j=1}^h j z_j^I\right)\left(1 - \frac{\theta}{\alpha}\right)^\beta + \frac{h\alpha}{-\beta + 1}\left[\left(1 - \frac{\theta}{\alpha}\right) - \left(1 - \frac{\theta}{\alpha}\right)^\beta\right].$$

Supposing also that $\sum_{j=1}^h j z_j^I \leq hk\alpha$, after some easy algebra we obtain: $\forall h \in I\!N^*$, $\forall \alpha > 0$, $\forall z^I \in I\!R^{+h}$ such that $\sum_{j=1}^h j z_j^I \leq hk\alpha$

$$\sum_{j=1}^h j\varphi_{h,j}(\theta, \alpha, z^I) \leq hk(\alpha - \theta). \tag{28}$$

We now compute a lower bound for $R_{\eta,\eta}$ and $\alpha_{\eta,\eta}$. In fact, we prove that $\forall \lambda > 0$, $\exists \eta(\lambda) \in I\!N^*$ such that $\forall \eta \in I\!N^*$, $\eta \geq \eta(\lambda)$

$$R_{\eta,\eta} \geq \frac{\lambda}{2}$$
$$\alpha_{\eta,\eta} \geq \frac{\lambda}{2k\eta}. \tag{29}$$

Since $\lim_{\eta \to \infty} \sum_{i=0}^{\eta-1} (\lambda^i/i!) = e^\lambda > e^\lambda/2$ and $\left(\sum_{i=0}^{\eta-1} (\lambda^i/i!)\right)_\eta$ is a monotonic increasing sequence, then $\exists n(\lambda) \in I\!N^*$ such that $\forall \eta \geq \eta(\lambda)$, $\sum_{i=0}^{\eta-1} (\lambda^i/i!) \geq e^\lambda/2$. Consequently,

$$R_{\eta,\eta} = e^{-\lambda} \lambda \sum_{i=0}^{\eta-1} \frac{\lambda^i}{i!} \geq e^{-\lambda} \lambda \frac{e^\lambda}{2} = \frac{\lambda}{2}$$

and

$$\alpha_{\eta,\eta} = \frac{R_{\eta,\eta}}{\eta k} \geq \frac{\lambda}{2k\eta}$$

16

q.e.d.

Let us now prove the following: $\forall \eta \in \mathbb{N}^*, \forall h \in \mathbb{N}^*, h \leq \eta$

$$y^I_{\eta,h,h} \geq \theta_{\eta,h}. \tag{30}$$

Recall that

$$0 \leq \frac{\partial T_h}{\partial y}(\alpha, y) \leq 1 \tag{31}$$

and also that $\forall \alpha > 0$, $T_h(\alpha, 0) = 0$. Using the mean value theorem, we have

$$
\begin{aligned}
0 \leq \theta_{\eta,h} &= T_h(\alpha_{\eta,h}, y^I_{\eta,h,h}) \\
&= |T_h(\alpha_{\eta,h}, y^I_{\eta,h,h}) - T_h(\alpha_{\eta,h}, 0)| \left(\sup_{y \in [0, y^I_{\eta,h,h}]} \left| \frac{\partial T_h}{\partial y}(\alpha_{\eta,h}, y) \right| \right) |y^I_{\eta,h,h} - 0| \\
&\leq 1|y^I_{\eta,h,h}| = y^I_{\eta,h,h}.
\end{aligned}
$$

This concludes the proof of expression (30).

Moreover, by some easy calculations, we can prove the following $\forall p \in \mathbb{N}^*$, $p \geq 2\lambda$, $\forall h, \eta \in \mathbb{N}^*$ such that $\eta - 1 \geq h \geq 2\lambda$

$$
\frac{\lambda^p}{p!} \leq \frac{\lambda^{p-1}}{(p-1)!} - \frac{\lambda^p}{p!}
$$
$$
\sum_{p=h}^{\eta-1} \frac{\lambda^p}{p!} \leq \frac{\lambda^h}{(h-1)!} - \frac{\lambda^{\eta-1}}{(\eta-1)!}. \tag{32}
$$

We now prove that $\forall \eta \in \mathbb{N}^*, \forall h \in \mathbb{N}^*$ such that $h < \eta$ we have

$$h\alpha_{\eta,h} \geq (h+1)\alpha_{\eta,h+1} - (h+1)y^I_{\eta,h+1,h+1}. \tag{33}$$

In fact, using expression (10), we have $R_{\eta,h} = R_{\eta,h+1} - (h+1)k\theta_{\eta,h+1}$ or

$$\frac{R_{\eta,h}}{k} = \frac{R_{\eta,h+1}}{k} - (h+1)\theta_{\eta,h+1}.$$

Recall also that $\alpha_{\eta,h} = R_{\eta,h}/(hk)$. So, we get

$$h\alpha_{\eta,h} = (h+1)\alpha_{\eta,h+1} - (h+1)\theta_{\eta,h+1}.$$

Moreover, using expression (30): $-\theta_{\eta,h+1} \geq -y^I_{\eta,h+1,h+1}$. Combining the last two expressions, we obtain expression (33).

Let us now prove that $\forall \eta \in \mathbb{N}^*, \eta \geq 2\lambda$ and $\forall h \in \mathbb{N}^*, 2\lambda \leq h \leq \eta$ we have

$$\alpha_{\eta,h} \geq \frac{\eta}{h}\alpha_{\eta,\eta} - \frac{e^{-\lambda}\lambda^h}{h!}. \tag{34}$$

Using expression (33), $\forall l \in \mathbb{N}^*$ such that $\eta > l \geq h \geq 2\lambda$, we have

$$
\begin{aligned}
l\alpha_{\eta,l} - (l+1)\alpha_{\eta,l+1} &\geq -(l+1)y^I_{\eta,l+1,l+1} \geq -(l+1)\frac{e^{-\lambda}\lambda^{l+1}}{(l+1)!} \\
&\overset{(17)}{\geq} \frac{e^{-\lambda}\lambda^{l+1}}{l!} = e^{-\lambda}\lambda\frac{\lambda^l}{l!}.
\end{aligned}
$$

17

Summing these inequalities, we obtain (using expression (32))

$$\sum_{l=h}^{\eta-1} l\alpha_{\eta,l} - \sum_{l=h}^{\eta-1}(l+1)\alpha_{\eta,l+1} \geq -e^{-\lambda}\lambda\sum_{l=h}^{\eta-1}\frac{\lambda^l}{l!}$$

Using also expression (32), we get $h\alpha_{\eta,h} - \eta\alpha_{\eta,\eta} \geq -e^{-\lambda}\lambda^h/(h-1)!$, and expression (34) follows.

Also, $\forall \eta \in I\!N^*$, $\eta > 2\lambda$, $\eta > \eta(\lambda)$ (expression (29)) and $\forall h \in I\!N^*$ such that $2\lambda \leq h \leq \eta$, we have

$$\alpha_{\eta,h} \geq \frac{\lambda}{2hk} - \frac{e^{-\lambda}\lambda^h}{h!}. \tag{35}$$

In fact, using expressions (34) and (29), $\alpha_{\eta,h} \geq (\eta/h)\alpha_{\eta,\eta} - e^{-\lambda}\lambda^h/h! \geq (\eta/h)(\lambda/2\eta k) - e^{-\lambda}\lambda^h/h!$ and expression (35) follows.

We finally prove that $\exists h_* \in I\!N^*$ such that $\forall \eta \in I\!N^*$, $\eta > 2\lambda$, $\eta \geq \eta(\lambda)$ (expression (29)), $\forall \eta \in I\!N^*$, $\eta > h \geq h_*$, $h \geq 2\lambda$, we have

$$\alpha_{\eta,h} \geq \frac{\lambda}{4hk}. \tag{36}$$

Since $\left(\lambda^{h-1}/(h-1)!\right)_h$ is the general term of a convergent series, $\lambda^{h-1}/(h-1)! \to_{h\to\infty} 0$. Thus, $\lambda^h/(h-1)! = \lambda\lambda^{h-1}/(h-1)! \to_{h\to\infty} \lambda.0 = 0$ or $h\lambda^h/h! = \lambda^h/(h-1)! \to_{h\to\infty} 0$; consequently, $\exists h_* \in I\!N^*$ such that $\forall h \geq h_*$, $h\lambda^h/h! \leq \lambda/(4ke^{-\lambda})$ or $e^{-\lambda}\lambda^h/h! \leq \lambda/(4hk)$. Thus, using expression (35), we deduce expression (36).

Let us now introduce, $\forall h \in I\!N^*$, the norm $\|\cdot\|_h$ in $I\!R^h$, defined as $\|x\|_h = \sum_{i=1}^h (i/h)|x_i|$.

Then, $\exists c_1 > 0$ such that $\forall h \in I\!N^*$, $h > 2\lambda$, $\forall \eta \in I\!N^*$, $\eta \geq h$, $\forall \alpha \geq \min\{\alpha_{\eta,h}, \alpha_{\eta+1,h}\}$, $\forall z^I \in I\!R^{+h}$ such that $\forall j = 1, \ldots, h$, $z_j^I \leq \Xi e^{-\lambda}\lambda^j/j!$ and $z_h^I \leq k\alpha$ we have

$$\left\|\frac{\partial}{\partial\theta}\varphi_h(T_h(\alpha, z_h^I), \alpha, z^I)\right\|_h \leq c_1. \tag{37}$$

Let us fix η and h such that $\eta \geq h > 2\lambda$. Let also $\overline{\theta} = T_h(\alpha, z_h^I)$.
Then, $\forall j = 1, \ldots, h-1$, we have

$$\frac{j}{h}\left|\frac{\partial}{\partial\theta}\varphi_{h,j}(\overline{\theta}, \alpha, z^I)\right| \leq \frac{j(j+1)}{h}\frac{\beta}{\alpha - \overline{\theta}}(\varphi_{h,j+1}(\overline{\theta}, \alpha, z^I) + \varphi_{h,j}(\overline{\theta}, \alpha, z^I)).$$

Since $z_h^I \leq k\alpha$ (expression (11), we have $\overline{\theta} \leq (1/2)\alpha$, hence $\beta/(\alpha - \overline{\theta}) \leq 2\beta/\alpha$.
Let us suppose that $\alpha \geq \alpha_{\eta,h}$, the reasoning being similar in the case $\alpha \geq \alpha_{\eta+1,h}$. So, $\beta/(\alpha - \overline{\theta}) \leq 2\beta/\alpha_{\eta,h}$ and since $h > 2\lambda$, using expression (36), we get $\beta/(\alpha - \overline{\theta}) \leq 8\beta hk/\lambda = (8(k-1)/\lambda)h$. Thus, using the hypotheses of expression (37) and expression (27), we have

$$\frac{j}{h}\left|\frac{\partial}{\partial\theta}\varphi_{h,j}(\overline{\theta}, \alpha, z^I)\right| \leq \frac{j(j+1)}{h}\frac{8(k-1)}{\lambda}h(\varphi_{h,j+1}(\overline{\theta}, \alpha, z^I) + \varphi_{h,j}(\overline{\theta}, \alpha, z^I))$$

$$\leq \frac{8(k-1)}{\lambda}j(j+1)\left(\Xi\frac{e^{-\lambda}\lambda^{j+1}}{(j+1)!} + \Xi\frac{e^{-\lambda}\lambda^j}{j!}\right).$$

18

Hence,

$$\sum_{j=1}^{h} \frac{j}{h} \left| \frac{\partial}{\partial \theta} \varphi_{h,j}(\overline{\theta}, \alpha, z^I) \right| \leq \frac{8(k-1)}{\lambda} \Xi e^{-\lambda} \sum_{j=1}^{h-1} j(j+1) \left(\frac{\lambda^j}{j!} + \frac{\lambda^{j+1}}{(j+1)!} \right)$$

$$\leq \frac{8(k-1)}{\lambda} \Xi e^{-\lambda} 2 e^{\lambda} (\lambda^2 + \lambda) \leq 16(k-1)\Xi(\lambda+1).$$

On the other hand,

$$\frac{h}{h} | \frac{\partial}{\partial \theta} \varphi_{h,h}(\overline{\theta}, \alpha, z^I)| \leq 1 + \frac{\beta}{\alpha - \overline{\theta}} \varphi_{h,h}(\overline{\theta}, \alpha, z^I) \leq 1 + \frac{8(k-1)}{\lambda} h k \Xi \frac{e^{-\lambda} \lambda^h}{h!} \leq 1 + 8(k-1)\Xi$$

and, finally,

$$\sum_{j=1}^{h} \frac{j}{h} \left| \frac{\partial}{\partial \theta} \varphi_{h,j}(\overline{\theta}, \alpha, z^I) \right| \leq 8(k-1)\Xi[2(\lambda+1)+1] + 1 = c_1$$

q.e.d.

We now prove that $\exists c_2 > 0$ such that $\forall h \in I\!N^*$, $h \leq 2\lambda$, $\forall \eta \in I\!N^*$, $\eta \geq h$, $\forall \alpha > 0$, $\forall z^I \in I\!R^{+h}$ such that $\sum_{j=1}^{h} j z_j^I \leq h k \alpha$, we have

$$\left\| \frac{\partial}{\partial \theta} \varphi_h(T_h(\alpha, z_h^I), \alpha, z^I) \right\|_h \leq c_2. \tag{38}$$

Let us fix η, h, α and put $\overline{\theta} = T_h(\alpha, z_h^I)$. If $j \leq h-1$, then

$$\left| \frac{\partial}{\partial \theta} \varphi_{h,j}(\overline{\theta}, \alpha, z^I) \right| = \left| \frac{\beta}{\alpha - \overline{\theta}} \left[(j+1)\varphi_{h,j+1}(\overline{\theta}, \alpha, z^I) - j \varphi_{h,j}(\overline{\theta}, \alpha, z^I) \right] \right|$$

$$\leq \frac{\beta}{\alpha - \overline{\theta}} [(j+1)\varphi_{h,j+1}(\overline{\theta}, \alpha, z^I) + j \varphi_{h,j}(\overline{\theta}, \alpha, z^I)]$$

$$\leq \frac{\beta}{\alpha - \overline{\theta}} \sum_{i=1}^{h} i \varphi_{h,i}(\overline{\theta}, \alpha, z^I)$$

and using (28),

$$\frac{\beta}{\alpha - \overline{\theta}} \sum_{i=1}^{h} i \varphi_{h,i}(\overline{\theta}, \alpha, z^I) \leq \frac{\beta}{\alpha - \overline{\theta}} h k (\alpha - \overline{\theta}) = \beta h k = h(k-1)$$

$$\leq 2\lambda(k-1) \left| \frac{\partial}{\partial \theta} \varphi_{h,h}(\overline{\theta}, \alpha, z^I) \right| \leq 1 + \frac{\beta}{\alpha - \overline{\theta}} \varphi_{h,h}(\overline{\theta}, \alpha, z^I)$$

$$\leq 1 + \frac{\beta}{\alpha - \overline{\theta}} h k (\alpha - \overline{\theta}) = 1 + h(k-1) \leq 2\lambda(k-1) + 1.$$

So,

$$\left\| \frac{\partial}{\partial \theta} \varphi_h(\overline{\theta}, \alpha, z^I) \right\|_h = \sum_{i=1}^{h} \frac{i}{h} \left| \frac{\partial}{\partial \theta} \varphi_{h,i}(\overline{\theta}, \alpha, z^I) \right| \leq \sum_{i=1}^{h} \frac{h}{h} \left| \frac{\partial}{\partial \theta} \varphi_{h,i}(\overline{\theta}, \alpha, z^I) \right|$$

$$= \sum_{i=1}^{h-1} \left| \frac{\partial}{\partial \theta} \varphi_{h,i}(\overline{\theta}, \alpha, z^I) \right| + \left| \frac{\partial}{\partial \theta} \varphi_{h,h}(\overline{\theta}, \alpha, z^I) \right|$$

$$\leq \sum_{i=1}^{h-1} 2\lambda(k-1) + 2\lambda(k-1) + 1$$

$$\leq (2\lambda)^2(k-1) + 2\lambda(k-1) + 1 = c_2$$

q.e.d.

Using (37) and (38), putting $c_3 = \max\{c_1, c_2\}$, noting that $hz_h^I \leq \sum_{i=1}^h jz_j^I \leq hk\alpha$, or $z_h^I \leq k\alpha$, we deduce that $\exists c_3$, $\forall h \in I\!N^*$, $\forall \eta \in I\!N^*$, $\eta \geq h$, $\forall \alpha \geq \min\{\alpha_{\eta,h}, \alpha_{\eta+1,h}\}$, $\forall z^I \in I\!R^{+h}$ such that $\forall j = 1, \ldots, h$, $z_j^I \Xi e^{-\lambda} \lambda^j / j!$ and $\sum_{i=1}^h jz_j^I \leq hk\alpha$ we have

$$\left\| \frac{\partial}{\partial \theta} \varphi_h(T_h(\alpha, z_h^I), z^I) \right\|_h \leq c_3. \tag{39}$$

We now prove that $\forall v \in I\!R^h$, $\forall \alpha > 0$, $\forall \theta \in [0, T_h(\alpha, v)]$

$$\|\Phi_h(\theta, \alpha)v\|_h \leq \|v\|_h. \tag{40}$$

We have

$$\|\Phi_h(\theta, \alpha)v\|_h = \sum_{i=1}^h \frac{i}{h} \left| \sum_{j=1}^h \Phi_{h,i,j}(\theta, \alpha)v_j \right| \leq \sum_{i=1}^h \frac{i}{h} \sum_{j=1}^h |\Phi_{h,i,j}(\theta, \alpha)| \, |v_j| \,.$$

By expression (24), putting $x = [(\alpha - \theta)/\alpha]^\beta$ ($0 \leq x \leq 1$ or $1 - x \geq 0$), we have $\Phi_{h,i,j}(\theta, \alpha) \geq 0$. So,

$$
\begin{aligned}
\|\Phi_h(\theta, \alpha)v\|_h &\leq \sum_{i=1}^h \frac{i}{h} \sum_{j=1}^h \Phi_{h,i,j}(\theta, \alpha) |v_j| = \frac{1}{h} \sum_{j=1}^h \sum_{i=1}^h i \frac{j!}{i!(j-i)!} x^i (1-x)^{j-i} |v_j| \\
&= \frac{i}{h} \sum_{j=1}^h jx \left(\sum_{l=0}^{j-1} \binom{j-1}{l} x^l (1-x)^{j-1-l} \right) |v_j| \\
&= \frac{1}{h} \sum_{j=1}^h jx(x+1-x)^{j-1} |v_j| = x\|v\|_h
\end{aligned}
$$

q.e.d.

Next, we prove that $\forall h \in I\!N^*$, $\forall \alpha > 0$, $\forall \theta < \alpha$

$$\left\| \frac{1}{\alpha} \psi_h(\theta, \alpha) \right\|_h \leq 1. \tag{41}$$

Let us put $x(t) = [(\alpha - \theta)/(\alpha - t)]^\beta$ ($x(t) \leq 1$). Then

$$
\begin{aligned}
\left\| \frac{1}{\alpha} \psi_h(\theta, \alpha) \right\|_h &= \frac{1}{\alpha} \sum_{i=1}^h \frac{i}{h} |\psi_{h,i}(\theta, \alpha)| = \frac{1}{\alpha} \sum_{i=1}^h \frac{i}{h} \int_0^\theta x(t)^i (1-x(t))^{h-i} dt \\
&= \frac{1}{\alpha} \int_0^\theta \left[\sum_{i=1}^h \binom{h-1}{i-1} x(t)^{i-1} (1-x(t))^{h-1-(i-1)} x(t) \right] dt \\
&= \frac{1}{\alpha} \int_0^\theta (x(t) + 1 - x(t))^{h-1} x(t) dt = \frac{1}{\alpha} \int_0^\theta x(t) dt \\
&= \frac{1}{\alpha} \int_0^\theta \left(\frac{\alpha - \theta}{\alpha - t} \right)^\beta dt \leq \frac{\theta}{\alpha} \leq 1
\end{aligned}
$$

20

q.e.d.

Also, $\forall h \in \mathbb{N}^*$, $\forall \alpha > 0$, $\forall z^I \in \mathbb{R}^h$, $\forall \theta \in [0, T_h(\alpha, z_h^I)]$, we have

$$\frac{\partial}{\partial \alpha} \varphi_h(\theta, \alpha, z^I) = \frac{-\theta}{\alpha} \frac{\partial}{\partial \theta} \varphi_h(\theta, \alpha, z^I) + \frac{1}{\alpha} \psi_h(\theta, \alpha). \tag{42}$$

We fix h, α, z^I. We have already seen that $\theta \in [0, T_h(\alpha, z_h^I)]$ and also that

$$\varphi_h(\theta, \alpha, z^I) = \Phi_h(\theta, \alpha) z^I + \psi_h(\theta, \alpha).$$

Let us put $\chi_h(\theta, \alpha, z^I) = \Phi_h(\theta, \alpha) z^I$, so,

$$\varphi_h(\theta, \alpha, z^I) = \chi_h(\theta, \alpha, z^I) + \psi_h(\theta, \alpha).$$

Let us introduce the function $w : [0, 1[\mapsto \mathbb{R}^h$, $w(s) = \exp\{\beta \ln[1/(1-s)] M_h\} z^I$; then, $\chi_h(\theta, \alpha, z^I) = w(\theta/\alpha)$.
We also introduce the function $v : [0, 1[\mapsto \mathbb{R}^h$, $v(s) = (v_1(s), \ldots, v_h(s))^T$, where

$$v_j(s) = -\binom{h}{j} \int_0^s \left(\frac{1-s}{1-r}\right)^{\beta j} \left[1 - \left(\frac{1-s}{1-r}\right)^\beta\right]^{h-j} dr.$$

Performing a variable changing $t = \alpha r$, we obtain $\psi_h(\theta, \alpha) = \alpha v(\theta/\alpha)$.
From the above, we deduce that $\varphi_h(\theta, \alpha, z^I) = w(\theta/\alpha) + \alpha v(\theta/\alpha)$.
Therefore,

$$\frac{\partial}{\partial \theta} \varphi_h(\theta, \alpha, z^I) = \dot{w}\left(\frac{\theta}{\alpha}\right) \frac{1}{\alpha} + \alpha \dot{v}\left(\frac{\theta}{\alpha}\right) \frac{1}{\alpha} = \dot{w}\left(\frac{\theta}{\alpha}\right) \frac{1}{\alpha} + \dot{v}\left(\frac{\theta}{\alpha}\right).$$

On the other hand,

$$\frac{\partial}{\partial \alpha} \varphi_h(\theta, \alpha, z^I) = \dot{w}\left(\frac{\theta}{\alpha}\right)\left(\frac{-\theta}{\alpha^2}\right) + v\left(\frac{\theta}{\alpha}\right) + \dot{v}\left(\frac{\theta}{\alpha}\right)\left(\frac{-\theta}{\alpha^2}\right)$$

and after some easy algebra, expression (42) is proved.

We now prove the following: $\exists c_4 > 0$ such that $\forall h \in \mathbb{N}^*$, $\forall \eta \in \mathbb{N}^*$, $\eta \geq h$, we have

$$\|\varphi_{h+1}(\theta_{\eta+1,h+1}, \alpha_{\eta+1,h+1}, y_{\eta+1,h+1}^I) - \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y_{\eta+1,h+1}^I)\|_{h+1}$$
$$\leq c_4 |\alpha_{\eta+1,h+1} - \alpha_{\eta,h+1}|. \tag{43}$$

To shorten the writing, we put $z^I = y_{\eta+1,h+1}^I$. So, $\theta_{\eta+1,h+1} = T_{h+1}(\alpha_{\eta+1,h+1}, z_{h+1}^I)$ and $\theta_{\eta,h+1} = T_{h+1}(\alpha_{\eta,h+1}, z_{h+1}^I)$. Let us also denote by $[x_1; x_2]$ a convex segment (notation which does not signify that $x_1 \leq x_2$).
Let us introduce, for $\alpha \in [\alpha_{\eta,h+1}; \alpha_{\eta+1,h+1}]$, the function $g(\alpha) = \varphi_{h+1}(T_{h+1}(\alpha, z_{h+1}^I), \alpha, z^I)$.
Thus

$$\|\varphi_{h+1}(\theta_{\eta+1,h+1}, \alpha_{\eta+1,h+1}, y_{\eta+1,h+1}^I) - \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y_{\eta+1,h+1}^I)\|_{h+1}$$
$$= \|g(\alpha_{\eta+1,h+1}) - g(\alpha_{\eta,h+1})\|_{h+1}.$$

Using the mean value theorem, we get

$$\|g(\alpha_{\eta+1,h+1}) - g(\alpha_{\eta,h+1})\|_{h+1} \leq \left(\sup_{\alpha \in [\alpha_{\eta,h+1};\alpha_{\eta+1,h+1}]} \|\dot{g}(\alpha)\|_{h+1}\right) |\alpha_{\eta+1,h+1} - \alpha_{\eta,h+1}|.$$

On the other hand

$$\dot{g}(\alpha) = \frac{\partial\varphi_{h+1}}{\partial\theta}(T_{h+1}(\alpha, z_{h+1}^I), \alpha, z^I)\frac{\partial T_{h+1}}{\partial\alpha}(\alpha, z_{h+1}^I) + \frac{\partial\varphi_{h+1}}{\partial\alpha}(T_{h+1}(\alpha, z_{h+1}^I), \alpha, z^I)$$

$$\stackrel{(42)}{=} \left(\frac{\partial T_{h+1}}{\partial\alpha}(\alpha, z_{h+1}^I) - \frac{\theta}{\alpha}\right)\frac{\partial\varphi_{h+1}}{\partial\theta}(T_{h+1}(\alpha, z_{h+1}^I), \alpha, z^I) + \frac{1}{\alpha}\psi_{h+1}(T_{h+1}(\alpha, z_{h+1}^I), \alpha).$$

Consequently

$$\|\dot{g}(\alpha)\|_{h+1} \leq \left(\left|\frac{\partial T_{h+1}}{\partial\alpha}(\alpha, z_{h+1}^I)\right| + \frac{\theta}{\alpha}\right)\left\|\frac{\partial\varphi_{h+1}}{\partial\theta}(T_{h+1}(\alpha, z_{h+1}^I), \alpha, z^I)\right\|_{h+1}$$

$$+ \left\|\frac{1}{\alpha}\psi_{h+1}(T_{h+1}(\alpha, z_{h+1}^I), \alpha)\right\|_{h+1}.$$

Let us remark here that one can prove by elementary techniques that

$$\left|\frac{\partial T_h(\alpha, y)}{\partial\alpha}\right| \leq 4. \tag{44}$$

So, using this inequality as well as $\theta < \alpha$ and (41), we get

$$\|\dot{g}(\alpha)\|_{h+1} \leq 5\left\|\frac{\partial\varphi_{h+1}}{\partial\theta}(T_{h+1}(\alpha, z_{h+1}^I), \alpha, z^I)\right\|_{h+1} + 1.$$

We can also remark that the hypotheses made on α and z^I for the proof of (39) remain valid here also; therefore, $\forall \alpha \in [\alpha_{\eta,h+1}; \alpha_{\eta+1,h+1}]$, we have $\|\dot{g}(\alpha)\|_{h+1} \leq 5c_3 + 1$, q.e.d.

The following inequality also holds $\forall \eta, h \in \mathbb{N}^*, h \leq \eta$:

$$|\alpha_{\eta+1,h} - \alpha_{\eta,h}| \leq \frac{1}{k}\|y_{\eta+1,h}^I - y_{\eta,h}^I\|_h. \tag{45}$$

In fact,

$$|\alpha_{\eta+1,h} - \alpha_{\eta,h}| \leq \left|\frac{1}{hk}\sum_{i=1}^h iy_{\eta+1,h,i}^I - \frac{1}{hk}\sum_{i=1}^h iy_{\eta+1,h,i}^I\right| = \frac{1}{k}\left|\sum_{i=1}^h \frac{i}{h}y_{\eta+1,h,i}^I - \sum_{i=1}^h \frac{i}{h}y_{\eta,h,i}^I\right|$$

$$= \frac{1}{k}\left|\|y_{\eta+1,h}^I\|_h - \|y_{\eta,h}^I\|_h\right| \leq \frac{1}{k}\|y_{\eta+1,h}^I - y_{\eta,h}^I\|_h$$

the last two inequalities holding because $y_{\eta+1,h}^I, y_{\eta,h}^I \in \mathbb{R}^h$ and because $\|\cdot\|_h$ is a norm of $\mathbb{R}^h$ respectively, q.e.d.

We now prove that $\forall \eta, h \in \mathbb{N}^*, h < \eta$, we have

$$\|\varphi_{h+1}(\theta_{\eta+1,h+1}, \alpha_{\eta+1,h+1}, y_{\eta+1,h+1}^I) - \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y_{\eta+1,h+1}^I)\|_{h+1}$$

$$\leq \|y_{\eta+1,h+1}^I - y_{\eta,h+1}^I\|_{h+1}. \tag{46}$$

Plainly

$$\|\varphi_{h+1}(\theta_{\eta+1,h+1}, \alpha_{\eta+1,h+1}, y^I_{\eta+1,h+1}) \quad - \quad \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y^I_{\eta+1,h+1})\|_{h+1}$$
$$\leq \quad \|\Phi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1})(y^I_{\eta+1,h+1} - y^I_{\eta,h+1})\|_{h+1}.$$

Using this inequality and expression (40), we obtain (46).

Let us continue by showing the following: $\exists c_5 > 0$ such that $\forall \eta, h \in I\!N^*$, $h < \eta$, we have

$$\|y^I_{\eta+1,h} - y^I_{\eta,h}\|_h \leq c_5 \|y^I_{\eta+1,h+1} - y^I_{\eta,h+1}\|_{h+1}. \tag{47}$$

Let us start the proof of expression (47) by noting that, if $x = (x_1, \ldots, x_h, 0)^T$ is a vector of $I\!R^{h+1}$ and if we denote by x' the sub-vector $(x_1, \ldots, x_h)^T$ of x, then

$$\|x'\|_h = \left(1 + \frac{1}{h}\right) \|x\|_{h+1} \leq 2\|x\|_{h+1}.$$

$$\|y^I_{\eta+1,h} - y^I_{\eta,h}\|_h =$$
$$\|y_{\eta+1,h+1}(\theta_{\eta+1,h+1}) - y_{\eta,h+1}(\theta_{\eta,h+1})\|_h \leq$$
$$2\|\varphi_{h+1}(\theta_{\eta+1,h+1}, \alpha_{\eta+1,h+1}, y^I_{\eta+1,h+1}) - \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y^I_{\eta,h+1})\|_{h+1} \leq$$
$$2\|\varphi_{h+1}(\theta_{\eta+1,h+1}, \alpha_{\eta+1,h+1}, y^I_{\eta+1,h+1}) - \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y^I_{\eta+1,h+1})\|_{h+1}$$
$$+2\|\varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y^I_{\eta+1,h+1}) - \varphi_{h+1}(\theta_{\eta,h+1}, \alpha_{\eta,h+1}, y^I_{\eta,h+1})\|_{h+1} \overset{(26,45,46)}{\leq}$$
$$\left(\frac{2c_4}{k} + 1\right) \|y^I_{\eta+1,h+1} - y^I_{\eta,h+1}\|_{h+1} \quad = \quad c_5 \|y^I_{\eta+1,h+1} - y^I_{\eta,h+1}\|_{h+1}$$

q.e.d.

Also, $\exists c_6 > 0$ such that $\forall \eta \in I\!N^*$, $\eta > 2\lambda$, we have

$$\sup_{\theta \in [0,\theta_{\eta+1,\eta+1}]} \|\dot{y}_{\eta+1,\eta+1}(\theta)\|_{\eta+1} \leq c_6. \tag{48}$$

Let $j = 1, \ldots, \eta$; then

$$\dot{y}_{\eta+1,\eta+1,j}(\theta) \quad = \quad (j+1)\frac{\beta}{\alpha_{\eta+1,\eta+1} - \theta} y_{\eta+1,\eta+1,j+1}(\theta)$$
$$-j\frac{\beta}{\alpha_{\eta+1,\eta+1} - \theta} y_{\eta+1,\eta+1,j}(\theta) \quad \text{or}$$

$$\frac{j}{\eta+1}|\dot{y}_{\eta+1,\eta+1,j}(\theta)| \quad \leq \quad \frac{j(j+1)}{\eta+1}\frac{\beta}{\alpha_{\eta+1,\eta+1} - \theta}(y_{\eta+1,\eta+1,j}(\theta) + (y_{\eta+1,\eta+1,j+1}(\theta)).$$

Also,

$$\frac{\beta}{\alpha_{\eta+1,\eta+1} - \theta} \leq \frac{\beta}{\alpha_{\eta+1,\eta+1} - \theta_{\eta+1,\eta+1}} \leq \frac{2\beta}{\alpha_{\eta+1,\eta+1}} \leq \frac{8\beta(n+1)k}{\lambda} = \frac{8(n+1)(k-1)}{\lambda}.$$

Therefore, using (20), we get

$$\frac{j}{\eta+1}|\dot{y}_{\eta+1,\eta+1,j}(\theta)| \leq \frac{8(k-1)}{\lambda}\Xi e^{-\lambda}j(j+1)\left[\frac{\lambda^j}{j!} + \frac{\lambda^{j+1}}{(j+1)!}\right]$$

23

or,

$$\sum_{j=1}^{\eta} \frac{j}{\eta+1} |\ddot{y}_{\eta+1,\eta+1,j}(\theta)| \leq \frac{8(k-1)}{\lambda} \Xi e^{-\lambda} 2e^{\lambda}(\lambda^2+\lambda) = 16(k-1)\Xi(\lambda+1).$$

On the other hand,

$$\ddot{y}_{\eta+1,\eta+1,\eta+1}(\theta) \leq 1 + \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta} y_{\eta+1,\eta+1,\eta+1}(\theta) \leq 1 + 8(k-1)\Xi\frac{e^{-\lambda}\lambda^\eta}{\eta!} \leq 1 + 8(k-1)\Xi.$$

Thus finally, $\|\ddot{y}_{\eta+1,\eta+1}(\theta)\|_{\eta+1} \leq 8(k-1)\Xi(4(\lambda+1)+1)+1 = c_6$, q.e.d.

Let us now prove that $\exists c_7 > 0$ such that $\forall \eta \in I\!N^*$, $\eta \leq 2\lambda - 1$, we have

$$\sup_{\theta \in [0,\theta_{\eta+1,\eta+1}]} \|\ddot{y}_{\eta+1,\eta+1}(\theta)\|_{\eta+1} \leq c_7. \tag{49}$$

Let $j = 1, \ldots, \eta$; then

$$
\begin{aligned}
\ddot{y}_{\eta+1,\eta+1,i}(\theta) &\leq \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta}\left((i+1)y_{\eta+1,\eta+1,i+1}(\theta) + i\frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta}y_{\eta+1,\eta+1,i}(\theta)\right) \\
&\leq \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta}\sum_{j=1}^{h} j y_{\eta+1,\eta+1,j} \\
&\overset{(13)}{\leq} \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta}(h+1)k(\alpha_{\eta+1,\eta+1}-\theta) \\
&\leq \beta(n+1)k < 2\lambda(k-1).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\ddot{y}_{\eta+1,\eta+1,\eta+1}(\theta) &\leq 1 + \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta} y_{\eta+1,\eta+1,\eta+1}(\theta) \\
&\leq 1 + \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta}\sum_{j=1}^{\eta+1} j y_{\eta+1,\eta+1,j}(\theta) \\
&\overset{(13)}{=} 1 + \frac{\beta}{\alpha_{\eta+1,\eta+1}-\theta}(n+1)k(\alpha_{\eta+1,\eta+1}-\theta) \\
&= 1 + \beta(n+1)k \leq 1 + (k-1)2\lambda.
\end{aligned}
$$

Hence, $\|\ddot{y}_{\eta+1,\eta+1}(\theta)\|_{\eta+1} \leq [\sum_{i=1}^{\eta}(i/\eta+1)][(k-1)2\lambda] + [1+(k-1)2\lambda] \leq 4\lambda(k-1)+1 = c_7$, q.e.d.

Using (48) and (49) and putting $c_8 = \max\{c_6, c_7\}$, we obtain the following, $\forall \eta \in I\!N^*$

$$\sup_{\theta \in [0,\theta_{\eta+1,\eta+1}]} \|\ddot{y}_{\eta+1,\eta+1}(\theta)\|_{\eta+1} \leq c_8. \tag{50}$$

We now prove that $\exists c_9 > 0$ such that $\forall \eta \in I\!N^*$, we get

$$\|y_{\eta+1,\eta}^I - y_{\eta,\eta}^I\|_\eta \leq c_9 \frac{e^{-\lambda}\lambda^{\eta+1}}{(\eta+1)!}. \tag{51}$$

24

Using the remark in the beginning of the proof of (47), we have

$$
\begin{aligned}
\|y_{\eta+1,\eta}^{I} - y_{\eta,\eta}^{I}\|_{\eta} &= \|y_{\eta+1,\eta+1}(\theta_{\eta+1,\eta+1}) - (y_{\eta,\eta}^{I}, 0)\|_{\eta} \\
&\leq 2\|y_{\eta+1,\eta+1}(\theta_{\eta+1,\eta+1}) - (y_{\eta,\eta}^{I}, 0)\|_{\eta+1} \\
&= 2\sum_{i=1}^{\eta} \frac{i}{\eta+1} \left| y_{\eta+1,\eta+1,i}(\theta_{\eta+1,\eta+1}) - \frac{e^{-\lambda}\lambda^{i}}{i!} \right| \\
&\leq 2\sum_{i=1}^{\eta+1} \frac{i}{\eta+1} \left| y_{\eta+1,\eta+1,i}(\theta_{\eta+1,\eta+1}) - \frac{e^{-\lambda}\lambda^{i}}{i!} \right| \\
&= 2\sum_{i=1}^{\eta+1} \frac{i}{\eta+1} |y_{\eta+1,\eta+1,i}(\theta_{\eta+1,\eta+1}) - y_{\eta+1,\eta+1,i}(0)| \\
&= 2\sum_{i=1}^{\eta+1} \|y_{\eta+1,\eta+1}(\theta_{\eta+1,\eta+1}) - y_{\eta+1,\eta+1}(0)\|_{\eta+1} \\
&= 2\left\| \int_{0}^{\theta_{\eta+1,\eta+1}} \dot{y}_{\eta+1,\eta+1}(\theta)d\theta \right\|_{\eta+1} \\
&\leq 2\int_{0}^{\theta_{\eta+1,\eta+1}} \|\dot{y}_{\eta+1,\eta+1}(\theta)\|_{\eta+1}d\theta \overset{(50)}{\leq} 2c_{8}\int_{0}^{\theta_{\eta+1,\eta+1}} d\theta \\
&= 2c_{8}\theta_{\eta+1,\eta+1} \leq 2c_{8}\frac{e^{-\lambda}\lambda^{\eta+1}}{(\eta+1)!} \overset{(30)}{=} c_{9}\frac{e^{-\lambda}\lambda^{\eta+1}}{(\eta+1)!}
\end{aligned}
$$

q.e.d.

Finally, let us prove the following: $\forall \eta, h \in I\!N^{*}$, $h \leq \eta$, we have

$$
|\theta_{\eta+1,h} - \theta_{\eta,h}| \leq 4\left(1 + \frac{1}{k}\right)\|y_{\eta+1,h}^{I} - y_{\eta,h}^{I}\|_{h}. \tag{52}
$$

By (44) and (31), we have

$$
\begin{aligned}
|\theta_{\eta+1,h} - \theta_{\eta,h}| &= |T_{h}(\alpha_{\eta+1,h}, y_{\eta+1,h,h}^{I}) - T_{h}(\alpha_{\eta,h}, y_{\eta,h,h}^{I})| \\
&\leq 4(|\alpha_{\eta+1,h} - \alpha_{\eta,h}| + |y_{\eta+1,h,h}^{I} - y_{\eta,h,h}^{I}|).
\end{aligned}
$$

So, using (45), we obtain expression (52).

3.4.8   The convergence of the sequence $(\tau_{\eta})_{\eta \in I\!N^{*}}$

We prove first that the series of general term $(\sum_{h=1}^{\eta} |\theta_{\eta+1,h} - \theta_{\eta,h}|)_{\eta \geq 1}$ is convergent in $I\!R^{+}$. In fact, this series is a positive terms one, hence in order to prove its convergence we shall use the comparison principle.

By combining expressions (52), (51) and (47), we can obtain

$$
\sum_{h=1}^{\eta} |\theta_{\eta+1,h} - \theta_{\eta,h}| \leq \sum_{h=1}^{\eta} c_{5}^{\eta-h} c_{9}\frac{e^{-\lambda}\lambda^{\eta+1}}{(\eta+1)!} = 4\left(1 + \frac{1}{k}\right)c_{9}c_{5}^{\eta}\frac{e^{-\lambda}\lambda^{\eta+1}}{(\eta+1)!}\sum_{h=1}^{\eta}\left(\frac{1}{c_{5}}\right)^{h}.
$$

Since $c_5 \geq 2 > 1$, we get

$$\sum_{h=1}^{\eta} |\theta_{\eta+1,h} - \theta_{\eta,h}| \leq 4 \left(1 + \frac{1}{k}\right) \frac{c_9}{c_5} \frac{e^{-\lambda}(\lambda c_5)^{\eta+1}}{(\eta + 1)!} \frac{1}{c_5 - 1}.$$

Putting $c_{10} = 4\left[1 + (1/k)\right](c_9/c_5)[1/(c_5 - 1)] > 0$, the quantity $\sum_{h=1}^{\eta} |\theta_{\eta+1,h} - \theta_{\eta,h}|$ is bounded above by $c_{10} e^{-\lambda}(\lambda c_5)^{\eta+1}/(\eta + 1)!$; hence,

$$\sum_{\eta=1}^{\infty} \left(\sum_{h=1}^{\eta} |\theta_{\eta+1,h} - \theta_{\eta,h}|\right) \leq c_{10} \frac{e^{-\lambda}(\lambda c_5)^{\eta+1}}{(\eta + 1)!} \leq c_{10} e^{-\lambda} e^{\lambda c_5} < +\infty$$

q.e.d.

Let us now prove that the series of general term $(\theta_{\eta+1,\eta+1})_{\eta \geq 1}$ converges in $I\!\!R^+$.
Plainly, using (30), we get $\theta_{\eta+1,\eta+1} \leq y_{\eta+1,\eta+1,\eta+1}^I = e^{-\lambda}(\lambda)^{\eta+1}/(\eta + 1)!$.
Since it is a positive term series, it suffices to apply the comparison principle, by noting that the series of general term $(e^{-\lambda}\lambda^{\eta+1}/(\eta + 1)!)_{\eta \geq 1}$ converges in $I\!\!R^+$, q.e.d.

The series of general term $(|\tau_{\eta+1} - \tau_{\eta}|)_{\eta \geq 1}$ converges in $I\!\!R^+$.
In fact,

$$\begin{aligned}
|\tau_{\eta+1} - \tau_{\eta}| &= \left|\sum_{h=1}^{\eta+1} \theta_{\eta+1,h} - \sum_{h=1}^{\eta} \theta_{\eta,h}\right| = \left|\theta_{\eta+1,\eta+1} + \sum_{h=1}^{\eta}(\theta_{\eta+1,h} - \theta_{\eta,h})\right| \\
&\leq \theta_{\eta+1,\eta+1} + \sum_{h=1}^{\eta} |\theta_{\eta+1,h} - \theta_{\eta,h}|.
\end{aligned}$$

The convergence follows from the comparison principle and the propositions proved above.

We conclude the section by proving that the sequence $(\tau_{\eta})_{\eta \in I\!\!N^*}$ converges in $I\!\!R^+$.
Using the proposition proved above, the series of general term $(\tau_{\eta+1} - \tau_{\eta})_{\eta \geq 1}$ is absolutely convergent in $I\!\!R$, thus convergent in $I\!\!R$.
Let us remark that $\tau_{\eta} = \sum_{q=1}^{\eta-1}(\tau_{q+1} - \tau_q) + \tau_1$. Moreover,

$$\lim_{\eta \to \infty} \tau_{\eta} = \sum_{q=1}^{\infty}(\tau_{q+1} - \tau_q) + \tau_1.$$

These two remarks conclude the convergence of the sequence $(\tau_{\eta})_{\eta \in I\!\!N^*}$ in $I\!\!R^+$.

4   Set packing and hypergraph independent set

4.1   The PACK algorithm

We consider the greedy algorithm 3 for finding a set packing for $\mathcal{P}$. Informally, **PACK** algorithm selects at each step one set chosen at random between the sets of smallest cardinality which do not intersect any previously selected set. By exchanging the two color classes of the incidence graph, this algorithm gives an independent set in the hypergraph $\mathcal{P}, \mathcal{Q}$, i.e. a set of vertices no two of which belong to the same hyperedge.

4.2   Analysis of PACK on random instances

4.2.1   Analysis of PACK on standard distribution

The analysis of **PACK** on random instances can be carried over along the same lines than for **GRECO**. However, this leads prima facie to a more complicated system of differential equations because, in the case of **PACK**, we have three succesive removals at each step (see the description of **PACK**).

The following observation ([2]) permits a simpler analysis: at any stage of the execution of **PACK**, the conditional distribution of the remaining sets $X'_1, \ldots, X'_h$ (relatively to the previous unraveling of the algorithm) coincides with the uniform distribution; we can then compute the expectation of the number of these sets which will be tried before finding one which does not intersect the partial packing as a function $t(j)$ of the size j of the partial packing which satisfies $t(j) \sim [1 - (jk/n)]^{-k}$.

Thus, the expectation of the number $T(s)$ of trials before a packing of size s is obtained satisfies

$$ET(s) = \sum_{j=1}^{s} t(j) \sim \frac{n}{k(k-1)} \cdot \left[\frac{1}{(1 - \frac{sk}{n})^{k-1}} - 1 \right]. \tag{53}$$

Using the fact that the trials are independent, Tchebichev's inequality implies $T(s) \sim ET(s)$ and solving (53) for s, one gets the following estimate for the size of the obtained packing:

$$s = \frac{n}{k} \left[1 - \frac{1}{[1 + \alpha k(k-1)]^{\frac{1}{k-1}}} \right].$$

4.2.2   The differential system simulating PACK for solving the hypergraph independent set problem

When analysing **PACK** on the hypergraph independent set problem, we interchange the color classes of the incidence graph and, since it is not true that for any given value h the joint distribution of the sets Y_i with $|Y_i| = h$ is uniform, we must abandon the preceding stopping times method and come back to Markov chains. The expectations of

27

the increments during the stage when vertices of degree h enter the independent set are

$$E\Delta m_j = ((j+1)m_{j+1} - jm_j)S, \quad j \neq h,$$
$$E\Delta m_h = -1 + ((h+1)m_{h+1} - hm_h))Sd,$$

where $S = h(k-1)/(\sum im_i)$.

With the same notation as in § 3.3, we get the differential system

$$\dot{y}_{n,h,j} = (j+1)y_{n,h,j+1}\frac{h(k-1)}{R_{n,h} - hk\theta} - jy_{n,h,j}\frac{h(k-1)}{R_{n,h} - hk\theta}, \quad 0 \leq j \leq h-1,$$

$$\dot{y}_{n,h,h} = -1 + (h+1)y_{n,h,h+1}\frac{h(k-1)}{R_{n,h} - hk\theta} - hy_{n,h,h}\frac{h(k-1)}{R_{n,h} - hk\theta}.$$

A similar method, as for system $(\mathcal{S}_{n_h})$ in the case of hitting set, could be developed to solve the above system.

5   Conclusions

We have proposed a general method to analyse the behaviour of approximation algorithms for hard optimization problems defined on random set systems. We note that this method is quite general and its application could be extended even to graph problems. In fact, a graph $G = (V, E)$ can be seen as a hypergraph whose vertex set is E, each vertex of V being a hyperedge containing its incident edges. Hence, problems like minimum dominating set, minimum vertex covering, maximum independent set, etc., could be solved by algorithms, the behaviour of which could be analyzed using the framework described.

We remark here that, when working on average case approximation, a well-known difficult mathematical problem is the precise estimation of the expectation of the size of the optimal solution. Such an estimation would enrich our method, by permitting to obtain close estimations for the expected values of the approximation ratios for the algorithms under study.

References

[1] N. Bourbaki, *Fonctions d'une Variable Réelle* (Hermann, Paris, 1976).

[2] W. Fernandez de la Vega, Sur la cardinalité maximum des couplages d'hypergraphes aléatoires uniformes, *Discrete Mathematics* **40**, (1982) 315-318.

[3] W. Fernandez de la Vega, Induced trees in sparse random graphs, *Graphs and Combinatorics* **2**, (1986) 227-231.

[4] W. Fernandez de la Vega, V. Th. Paschos, R. Saad, Average case analysis of a greedy algorithm for the minimum hitting set problem, in: *Proc. LATIN'92*, Lecture Notes in Computer Science, Vol. 583 (Springer, Berlin, 1992) 130-138.

[5] A. Frieze, A. J. Radcliffe, S. Suen, Analysis of a simple greedy matching algorithm on random cubic graphs, in: *Proc. 4th Ann. Symposium on Discrete Algorithms* (ACM Press, 1993) 341-351.

[6] M. R. Garey and D. S. Johnson, *Computers and Intractability - A Guide to the Theory of NP-Completeness* (Freeman, New York, 1979).

[7] R. M. Karp, The probabilistic analysis of some combinatorial search algorithms, in: J. F. Traub, ed., *Algorithms and Complexity: New Directions and Recent Results* (Academic Press, New York, 1976) 1-19.

[8] K. Knopp, *Infinite Sequences and Series* (Dover Publ., New York, 1956).

[9] T. G. Kurtz, Solutions of ordinary differential equations as limits of pure jump Markov processes, *J. Appl. Prob.* **7** (1970) 49-58.

[10] C. Lund and M. Yannakakis, On the hardness of approximating minimization problems, in: *Proc. STOC'93* (ACM Press, 1993) 286-293.