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A THEORETICAL FRAMEWORK FOR ANALYSING THE NOTION OF RELATIVE IMPORTANCE OF CRITERIA [1]

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UN CADRE DE TRAVAIL THÉORIQUE POUR L'ANALYSE DE LA NOTION D'IMPORTANCE RELATIVE DES CRITÈRES

Résumé :

L'aide multicritère à la décision requiert le plus souvent l'utilisation de poids, coefficients d'importance, ou bien d'une hiérarchie de critères, de seuils de veto, ... Ceux-ci sont des paramètres d'importance qui visent à différentier le rôle dévolu à chaque critère dans la construction des préférences globales. De nombreux auteurs ont proposé des méthodes d'évaluation de tels paramètres mais peu d'entre eux ont cherché à analyser en détail ce qui est sous-jacent à la notion d'importance relative des critères et à en donner une définition formelle précise.

Dans ce travail, notre objectif est de définir, pour cette notion, un cadre d'analyse valides dans des conditions très générales. Dans ce cadre, il apparait clairement que l'importance des critères est prise en compte de manière très différente dans les diverses procédures d'agrégation. Ce cadre permet d'analyser sous un nouvel angle des questions fondamentales telles que :

- dans quelles conditions est-il possible d'affirmer qu'un critère est au moins aussi important qu'un autre ?
- les paramètres d'importance des diverses procédures d'agrégation sont-ils dépendants ou non du codage des critères ?
- Quels sont les liens existant entre les notions d'importance des critères et de plus ou moins grande compensation des préférences ?

Ce cadre de travail théorique semble suffisamment général pour poursuivre les recherches en vue de définir des méthodes d'évaluation des paramètres d'importance valides d'un point de vue théorique.

Mots clé : Aide Multicritère à la Décision, Importance des critères, Analyse Théorique.

A THEORETICAL FRAMEWORK FOR ANALYSING THE NOTION OF RELATIVE IMPORTANCE OF CRITERIA

Abstract:

Multiple criteria decision aid almost always requires the use of weights, importance coefficients, or even a hierarchy of criteria, veto thresholds, ... These are importance parameters that are used to differentiate the role devoted to each criterion in the construction of comprehensive preferences. Many researchers have studied the problem of how to assign values to such parameters, but few of them have tried to analyse in detail what underlies the notion of importance of criteria and to give a clear formal definition of it.

In this paper, our purpose is to define a theoretical framework so as to analyse the notion of the importance of criteria under very general conditions. Within this framework, it clearly appears that the importance of criteria is taken into account in very different ways in various aggregation procedures. This framework also allows us to shed new light on fundamental questions such as:

- under what conditions is it possible to state that one criterion is more important than another?
- are importance parameters of the various aggregation procedures dependent on or independent of the encoding of criteria?
- what are the links between the two concepts of the importance of criteria and the compensatoriness of preferences?

This theoretical framework seems to us sufficiently general to ground further research in order to define theoretically valid elicitation methods for importance parameters.

Keywords : Multiple Criteria Decision Aid, Importance of Criteria, Theoretical Analysis

INTRODUCTION

This paper deals with Multiple Criteria Decision Aid (MCDA). In the present work, we will suppose defined:

- a set A of alternatives,
- a family F of n semi-criteria $g_1, g_2, ..., g_n$ ($n \geq 3$) pertinent in a given decision context ($F = \{1,2, ..., n\}$).

More precisely, we suppose that the preferences of the Decision Maker (DM) concerning the alternatives of A are formed and argued, in the context considered, by reference to n points of view adequately reflected by the criteria contained in F.

We denote by $g_j(a)$ the value attributed to the alternative a on the j^{th} criterion; we will call this value the j^{th} performance of a; $g_j(a)$ is a real number. The greater it is, the better the alternative a is for the DM. Specifically, we only suppose that $g_j(a) \geq g_j(b)$ implies that the alternative a is at least as good as the alternative b with respect to the j^{th} point of view. When considering the only point of view formalized by g_j the imprecision, and/or the uncertainty, and/or the inaccurate determination of performances may lead the DM to judge that a is indifferent to b when $g_i(a)=g_i(b)$, $\forall i \neq j$, even if $g_j(a) \neq g_j(b)$. So as to account for such a statement, criteria contained in F are usually considered to be semi-criteria, i.e., such that:

$$\begin{cases} aP_jb \Leftrightarrow g_j(a) > g_j(b) + q_j \\ aI_jb \Leftrightarrow |g_j(a) - g_j(b)| \leq q_j \end{cases}$$

where
- q_j, the indifference threshold, represents the maximum difference of performance compatible with an indifference situation[1];
- aP_jb is to be interpreted as a preference for a over b on criterion g_j;
- aI_jb is to be interpreted as an indifference between a and b on criterion g_j.

P_j is called the preference relation restricted to the j^{th} criterion (the same terminology holds for I_j). (I_j,P_j) defines a semi-order on A.

From a comprehensive point of view, we refer to a position that accounts for the n points of view simultaneouly. From such a comprehensive point of view, when considering two performance vectors $(g_1(a), g_2(a), ..., g_n(a))$ and $(g_1(b), g_2(b), ..., g_n(b))$, the DM can have difficulty for choosing, in a well-thought-out and stable way, one proposition from the three given below:

- b is strictly preferred to a (denoted by bPa),
- a is indifferent to b (denoted by aIb),
- a is strictly preferred to b (denoted by aPb).

[1] In this paper, we will suppose q_j constant; this is not restrictive since any interval-order stemming from a semi-criterion using a variable threshold $q_j(g_j)$ can be represented by another semi-criterion using a constant threshold (see [Roubens & Vincke 85]).

These difficulties stem from zones of imprecision and conflict in the DM's mind. So as not to compel the DM to make arbitrary judgments, it is prudent and usual (see [Roy 85]) to introduce a third relation in order to model the DM's preferences at a comprehensive level. This relation, incomparability, will be denoted by R which reflects the absence of positive reasons for choosing one of the three possibilities given above or the willingness to put off such a choice. Two alternatives a and b are said to be incomparable (aRb) if, of the three assertions aIb, aPb and bPa, none is valid. Thus, for every pair of alternatives (a, b), one and only one of the four following assertions is valid:

$$\left[\begin{array}{l} aPb \\ aIb \\ bPa \\ bRa \end{array}\right. \quad [1]$$

Let us consider the outranking relation $S=P\cup I$, aSb being interpreted as "a is at least as good as b."As $aPb \Rightarrow aSb \Rightarrow$ not [bPa], the four assertions of system [1] correspond to the following assertions:

$$\left[\begin{array}{l} aPb \\ aSb \text{ and } \mathrm{not}[aPb] \\ bPa \\ \mathrm{not}[aSb] \text{ and } \mathrm{not}[bPa] \end{array}\right. \quad [2]$$

In this paper, we consider a comprehensive preference system using three relations (I, P, R)[2]. Let us recall that a standard approach to MCDA consists of grounding the construction of the comprehensive preferences on the restricted preferences (I_j and P_j, $\forall j \in F$) corresponding to the n criteria. This is done through a so-called Multiple Criteria Aggregation Procedure (MCAP). In the MCAP, all criteria are not supposed to play the same role; the criteria are commonly said not to have the same importance. This is why there are, in the MCAP, parameters that aim at specifying the role of each criterion in the aggregation of performances. We will call such parameters **importance parameters**. They aim at accounting for how the DM attaches importance to the points of view modelled by the criteria.

The nature of these parameters varies across MCAP. The way of formalizing the relative importance of each criterion differs from one aggregation model to another. All this is done, for instance, by means of:

- scaling constants in Multiattribute Utility Theory (see [Keeney & Raiffa 76,93]),
- a weak-order on F in lexicographic techniques or a complete pre-order on F in the ORESTE method (see [Roubens 82]),
- intrinsic weights in PROMETHEE methods (see [Brans et al. 84]),
- intrinsic weights combined with veto thresholds in ELECTRE methods (see [Roy & Bertier 73] and [Roy 91]),

[2] The reader will find more details concerning the basic concepts of preference modelling in [Vincke 90].

- tradeoffs or substitution rates in the Zionts and Wallenius procedures (see [Zionts & Wallenius 76, 83]),
- aspiration levels in interactive methods (see [Vanderpooten & Vincke 89]),
- eigen vectors of a pairwise comparison matrix in the AHP method (see [Saaty 80]).

What exactly is covered by this notion of importance? What does a decision-maker mean by assertions such as "criterion g_j is more important than criterion g_i," "criterion g_j has a much greater importance than criterion g_i?", etc. Let us recall, by way of comparison, that the assertion "b is preferred to a" reflects the fact that, if the decision-maker must choose between b and a, he is supposed to decide in favor of b. So, it is possible to test whether this assertion is valid or not. There is no similar possibility for comparing the importance of criteria. These remarks lead more generally to question us on what gives meaning in practice to this notion of importance. We will come back to this question at the end of the paper by showing that the framework presented in the paper helps to progress in the understanding of how to elicit this notion. Moreover, the way this notion is taken into account, within the framework of different models mentioned above, by means of importance parameters reveals deep differences, as will be shown below.

This paper has a double purpose:

i) to propose a formal framework allowing us to give sense to the notion of relative importance of criteria under very general conditions,

ii) to use this formal framework to give some partial answers to questions such as:
   - under what conditions is it possible to state that one criterion is more important than another?
   - are importance parameters of the various MCAPs dependent on or independent of the encoding of criteria?
   - what are the links between the two concepts of importance of criteria and the compensatoriness of preferences?

It seems to us that the proposed theoretical framework increases our capability to interpret the information obtained through interacting with the DM in order to express formally his positions concerning the respective role of the various criteria through numerical values assigned to importance parameters. This question is not directly addressed in this paper (certain aspects of this problem are studied in [Mousseau 93]). In any case, the theoretical framework proposed here highlights some basic traps to be avoided when assigning values to importance parameters.

We will introduce, in section 1, a formal definition of the notion of relative importance of criteria as well as the theoretical framework we propose to analyse it. In section 2, we will present some results on the theoretical framework introduced. This will lead us to specify, in section 3, how the relative importance of criteria is taken into account in some basic aggregation procedures. In the fourth section, we will show how the proposed theoretical framework sheds new light on certain fundamental problems. In the last section, we will consider the two ways in which it is possible to give meaning to the notion of importance of criteria and show the interest of the proposed framework in both approaches.

1. THE THEORETICAL FRAMEWORK PROPOSED

Many authors have studied the problem of the elicitation of the relative importance of criteria[3] (RIC), but few have tried to give a precise definition to this notion[4].

1.1. What information characterizes the notion of relative importance of criteria?

The first statement we should make when trying to give a formal definition to the notion of RIC is the following: the information underlying this notion is much richer than that contained in the importance parameters of the various multicriteria models. In fact, these parameters are mainly scalars and constitute a simplistic way of taking RIC into account, as this notion is, by its nature, of a functional type.

In the comparison of two alternatives a and b, when one or several criteria are in favor of a and one or several others in favor of b, the way each MCAP solves this conflict and determines the comprehensive preference refers to the importance attached to each criterion (and to the logic of the aggregation used). Thus, the result of such conflicts (i.e., the comprehensive preference situation between a and b) constitutes the elementary data providing information on the relative importance of the criteria in conflict.

In order to delimit exhaustively the importance of a criterion, we should analyse the contribution of any preference at the restricted level of a criterion to the comprehensive level (for each pair of alternatives). Nevertheless, when two alternatives are indifferent on criterion g_j, the comprehensive preference situation will generally give no significant information concerning the importance of this criterion.

When the preference model is of a (I, P, R) type, it does not seem restrictive to delimit the information concerning the relative importance of a criterion using only preference and outranking relations (see system [2] in the introduction). *On such a basis, we will assume that the empirical content of the RIC notion refers to the nature and variety of cases in which a preference restricted to a given criterion g_j leads us to accept the same preference on the comprehensive level, or only an outranking in the same direction, or to refuse the inverse preference*. This conception may be synthesized as follows:

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[3] A critical overview of the literature in the field can be found in [Mousseau 92].

[4] For an interesting exception, see [Podinovskii 88, 94].

Basic empirical hypothesis: The relative importance of criterion g_j is characterized by[5]:

- the set of "*situations*" in which aP_jb and aPb hold simultaneously,
- the set of "*situations*" in which aP_jb and aSb hold simultaneously,
- the set of "*situations*" in which aP_jb and not[bPa] hold simultaneously.

1.2. Notations

Let us denote by:

X_i the ordered set of possible performances on criterion g_i ($X_i \subset \mathbb{R}$).

$X = \prod_{i=1}^{n} X_i$ the set of performance vectors.

$\underline{x} = (x_1, x_2, ..., x_n) \in X$ a performance vector corresponding to an alternative a such that $g_i(a)=x_i$, $\forall i \in F$.

$X_{-j} = \prod_{i \neq j} X_i$ the set of performance vectors from which the j^{th} component is removed.

$\underline{x}_{-j} = (x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_n) \in X_{-j}$

In order to simplify the notations, we will denote by:

$$\underline{x} = (\underline{x}_{-j}, x_j)$$
$$(\underline{x}_{-j}, x_j)\ P\ (\underline{y}_{-j}, y_j) \Leftrightarrow \underline{x}\ P\ \underline{y}$$

The dominance relation Δ relatively to F is defined by:

$$\forall \underline{x},\underline{y} \in X^2,\quad \underline{x}\ \Delta\ \underline{y} \Leftrightarrow x_i \geq y_i,\quad \forall\ i \in F$$

The data of the preferences (I_j, P_j) restricted to the n viewpoints, together with the comprehensive preference system[6] (I, P, R), define what we call a preference structure Ψ on X.

Considering a preference structure Ψ on X, the following sets are defined for all $x_j, y_j \in X_j^2$ such that $x_j\ P_j\ y_j$.

$$\Delta_j^P(x_j, y_j) = \{\ (\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } (\underline{x}_{-j},x_j)\ P\ (\underline{y}_{-j},y_j)\ \}$$
$$\Delta_j^S(x_j, y_j) = \{\ (\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } (\underline{x}_{-j},x_j)\ S\ (\underline{y}_{-j},y_j)\ \}$$
$$\Delta_j^V(x_j, y_j) = \{\ (\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \text{not}[(\underline{y}_{-j},y_j)\ P\ (\underline{x}_{-j},x_j)]\ \}$$

The set $\Delta_j^P(x_j, y_j)$ ($\Delta_j^S(x_j, y_j)$ and $\Delta_j^V(x_j, y_j)$, respectively) contains all combinations of performances ($\underline{x}_{-j}$ and $\underline{y}_{-j}$) that it is possible to combine with x_j and y_j, when $x_jP_jy_j$, so as to obtain $\underline{x}P\underline{y}$ ($\underline{x}S\underline{y}$ and *not*[$\underline{y}P\underline{x}$], respectively) on the comprehensive preference level.

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[5] In this characterization the contribution of criterion g_j to the preferences at the comprehensive level is considered only through the situations of strict preference at the restricted level of criterion g_j. This does not seem restrictive when all criteria are semi-criteria, but it can become restrictive when F contains pseudo-criteria (see [Roy & Vincke 84]).

[6] The question of where does the comprehensive preferences come from will be addressed in section 5.

Since those sets are defined only $\forall x_j, y_j \in X_j^2$ such that $x_j P_j y_j$, we will not mention this condition of definition explicitly each time we refer to those sets.

1.3. Formal definition of the relative importance of criteria

The following definition consists of a simplification of the one proposed in [Roy 91a] analyzing the more general case of pseudo-criteria. Using the notations given above, this definition formalizes the basic empirical hypothesis of section 1.1.

Definition 1.1: the importance of a semi-criterion g_j is characterized by the sets $\Delta_j^P(x_j, y_j)$, $\Delta_j^S(x_j, y_j)$, $\Delta_j^V(x_j, y_j)$ (defined $\forall x_j, y_j \in X_j^2$ such that $x_j P_j y_j$). The values attributed to the importance parameters must be grounded on the range and the shape of the preceeding sets.

This highlights the restricted nature of the importance parameters relative to the complexity of the notion of RIC.

Let us illustrate this definition with some important specific cases:

- $\Delta_j^P(x_j, y_j) = X_{-j}^2$ means that $(\underline{x}_{-j}, x_j) P (\underline{y}_{-j}, y_j)$ holds $\forall\ x_j, y_j \in X_j^2$ such that $x_j P_j y_j$ and $\forall\ (\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2$. Criterion g_j imposes its viewpoint and is thus a dictator (see 3.2).
- $\Delta_j^P(x_j, y_j) = X_{-j}^2(S) = \left\{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \forall i \in F \setminus \{j\}\ x_i \geq y_i - q_i \text{ (i.e., such that } x_i S_i y_i)\right\}$ means that the only cases in which $x_j P_j y_j$ and $(\underline{x}_{-j}, x_j) P (\underline{y}_{-j}, y_j)$ hold simultaneously are the cases in which no other criterion is opposed to g_j ($x_i S_i y_i \Leftrightarrow \text{not}[y_i P_i x_i]$). In other words, criterion g_j imposes its decision on the comprehensive level only when no other criterion is opposed to it.
- $\Delta_j^S(x_j, y_j) = X_{-j}^2(S)$ and $\Delta_j^P(x_j, y_j) \subset \Delta_j^S(x_j, y_j)$ means that the importance of criterion g_j is reduced (relative to the preceding case), since even when no criterion is opposed to it, g_j does not always impose a strict preference on the comprehensive level.
- $\Delta_j^S(x_j, y_j) = \Delta_j^V(x_j, y_j)\ \forall\ x_j, y_j \in X_j^2$ such that $x_j P_j y_j$ means that the only cases in which $x_j P_j y_j$ appears compatible with $\underline{y} P \underline{x}$ are those where $\underline{x} S \underline{y}$ holds. Let us remark that other cases can exist, particularly when a criterion g_j opposes a veto to $\underline{y} S \underline{x}$ (see 2.2.4).

2. INTERPRETATION AND PROPERTIES

2.1. Interpretation

As aPb $\Rightarrow$ aSb $\Rightarrow$ not [bPa], the following sequence of inclusions holds (see 2.2.1):

$$\forall\ x_j, y_j \in X_j^2,\ \Delta_j^P(x_j, y_j) \subseteq \Delta_j^S(x_j, y_j) \subseteq \Delta_j^V(x_j, y_j)$$

If F satisfies the cohesion axiom (see appendix A), $\Delta_j^S(x_j, y_j) \supseteq X_{-j}^2(S)$

Moreover, $\Delta_j^P(x_j, y_j) \supseteq X_{-j}^2(S)$ holds if $\forall j \in F$, $[x_j P_j y_j \text{ and } x_i = y_i\ \forall i \neq j] \Rightarrow xPy$.

From this fact, it is possible to give a concrete interpretation of the differences (see figure 1):

$C_j(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in \Delta_j^P(x_j,y_j)$ such that $(\underline{x}_{-j},\underline{y}_{-j}) \notin X_{-j}^2(S)\}$

Reflects the *ability of criterion g_j to* ***convince***, i.e., the cases in which the comprehensive preference between two alternatives is in favor of the partial preference on criterion g_j in spite of the opposition of other criteria.

$E_j(x_j,y_j) = \Delta_j^S(x_j,y_j) \backslash \Delta_j^P(x_j,y_j)$

Reflects the *ability of criterion g_j to* ***equilibrate***, i.e., the cases in which the partial preference on criterion g_j allows us to obtain an indifference between the two alternatives in spite of the opposition of criteria.

$O_j(x_j,y_j) = \Delta_j^V(x_j,y_j) \backslash \Delta_j^S(x_j,y_j)$

Reflects the *ability of criterion g_j to* ***oppose***, i.e., the cases in which the partial preference on criterion g_j is opposed to other criteria in such a way that the two alternatives are incomparable on the comprehensive level (see result 2.2.4).

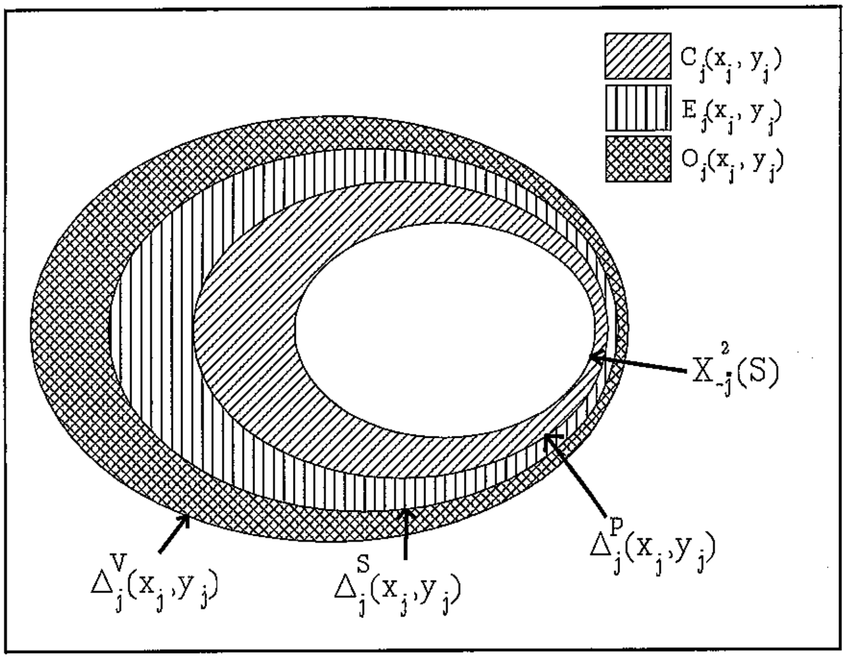

Figure 1 : Interpretation of the sets $C_j(x_j,y_j)$, $E_j(x_j,y_j)$ and $O_j(x_j,y_j)$ in the case where $\Delta_j^P(x_j, y_j) \supseteq X_{-j}^2(S)$.

The illustrations presented in section 3 will allow the reader to verify that it is interesting to compare the relative size of the sets $C_j(x_j,y_j)$, $E_j(x_j,y_j)$ and $O_j(x_j,y_j)$ in order to analyse how these three different aspects of the notion of importance are taken into account in several MCAPs.

2.2. Results

In what follows, the family of criteria F will be supposed consistent, i.e., will verify three axioms given in appendix A. The proofs of the results presented in this section can be found in appendix B.

2.2.1 Embedded nature of the sets $\Delta_j^H(x_j, y_j)$, $H \in \{P,S,V\}$.

Result 2.1: $\forall j \in F$, $\forall x_j, y_j \in X_j^2$, it holds $\left[\begin{array}{l} \Delta_j^P(x_j,y_j) \subseteq \Delta_j^S(x_j,y_j) \subseteq \Delta_j^V(x_j,y_j) \\ \Delta_j^S(x_j,y_j) \supseteq X_{-j}^2(S) \end{array}\right.$

Moreover, if criterion g_j is inductive, i.e., such that $[g_i(a)=g_i(b)\ \forall i \neq j \text{ and } aP_jb] \Rightarrow aPb$ then: $\Delta_j^P(x_j,y_j) \supseteq X_{-j}^2(S)$

2.2.2 Expanding nature of the sets $\Delta_j^H(x_j, y_j)$ $H \in \{P,S,V\}$.

Result 2.2: $\forall\ x_j, y_j, u_j, v_j \in X_j^4$ such that $u_j \geq x_j$, $y_j \geq v_j$, $x_j P_j y_j$,
$\Delta_j^H(x_j, y_j) \subseteq \Delta_j^H(u_j, v_j)$, $\forall\ H \in \{P,S,V\}$

2.2.3 Borderline of the sets $\Delta_j^H(x_j, y_j)$, $H \in \{P,S,V\}$.

As Δ denotes the dominance relation relatively to F (see 1.2), we denote by Δ_{-j} the dominance relation defined on X_{-j}^2.

Result 2.3: If $(\underline{x}_{-j}, \underline{y}_{-j}) \in \Delta_j^H(x_j, y_j)$,
then, $\forall \underline{u}_{-j}, \underline{v}_{-j} \in X_{-j}^2$ verifying $\left[\begin{array}{l} \underline{u}_{-j}\ \Delta_{-j}\ \underline{x}_{-j} \\ \underline{y}_{-j}\ \Delta_{-j}\ \underline{v}_{-j} \end{array}\right.$, it holds $(\underline{u}_{-j}, \underline{v}_{-j}) \in \Delta_j^H(x_j, y_j)$ $(H \in \{P,S,V\})$

As a consequence, there exists a minimal subset from which it is possible to derive all the other elements of $\Delta_j^H(x_j, y_j)$. This subset will be called the borderline of $\Delta_j^H(x_j, y_j)$.
Definition 2.1[7]: $(\underline{x}_{-j}, \underline{y}_{-j})$ will be a borderline element of $\Delta_j^H(x_j, y_j)$ iff

$$\forall (\underline{u}_{-j}, \underline{v}_{-j}) \in \Delta_j^H(x_j, y_j),\ \left.\begin{array}{l} \underline{x}_{-j}\,\Delta_{-j}\,\underline{u}_{-j} \\ \underline{v}_{-j}\,\Delta_{-j}\,\underline{y}_{-j} \end{array}\right] \Rightarrow\ (\underline{u}_{-j}, \underline{v}_{-j}) = (\underline{x}_{-j}, \underline{y}_{-j})$$

The knowledge of the set of all the borderline elements (i.e., the borderline of $\Delta_j^H(x_j,y_j)$) is sufficient to reconstruct the whole set $\Delta_j^H(x_j,y_j)$, $H \in \{P,S,V\}$. An illustrative example of borderline is given in the appendix C.

[7] In the case of a familly of criteria containing some criteria for which X_j is continuous, if $\Delta_j^H(x_j,y_j)$ is an open set, it is necessary to replace this set by its topological closure.

Let us denote by $C'_j(x_j,y_j)$, $E'_j(x_j,y_j)$ and $O'_j(x_j,y_j)$ the borderlines of $\Delta_j^P(x_j, y_j)$, $\Delta_j^S(x_j, y_j)$ and $\Delta_j^V(x_j, y_j)$, respectively. If $\forall j\in F$, X_j is discrete[8], we have the following inclusions:

$$\forall j\in F,\ \forall x_j,y_j\in X_j \quad \left[\begin{array}{l} C'_j(x_j,y_j) \subset C_j(x_j,y_j) \\ E'_j(x_j,y_j) \subset E_j(x_j,y_j) \\ O'_j(x_j,y_j) \subset O_j(x_j,y_j) \end{array}\right.$$

Thus, we will call these three borderlines, the conviction, the equilibration and the opposition borderlines, respectively. Since the knowledge of these three borderlines allows us to characterize the form and the extent of the sets $\Delta_j^H(x_j, y_j)$ (H=P,S,V), we can claim, according to definition 1.1, that all the information concerning the importance of the criterion g_j is contained in $C'_j(x_j,y_j)$, $E'_j(x_j,y_j)$ and $O'_j(x_j,y_j)$, $\forall x_j,y_j\in X_j^2$.

Consequently, when decision aid is grounded on a given MCAP, the numerical values attributed to the importance parameters of this MCAP are sufficient to specify the three borderlines. Hence, the numerical values for importance parameters must be chosen in the light of the three resulting borderlines.

2.2.4 Opposition-incomparability result

Considering the comprehensive relational system (I,P,R) which underlies the sets $\Delta_j^H(x_j, y_j)$ (H=P,S,V), we have the following result:

Result 2.4: $\forall j\in F$, $\forall x_j, y_j\in X_j^2$ such that $x_jP_jy_j$,

$$O'_j(x_j,y_j)=\varnothing \iff \nexists\ \underline{x}_{-j},\underline{y}_{-j}\in X_j^2 \text{ such that } (\underline{x}_{-j},x_j)R(\underline{y}_{-j},y_j)$$

As a consequence, we have (see 3.1, 3.2 and 3.3):

$$\forall j\in F,\ \forall x_j,y_j\in X_j^2 \text{ such that } x_jP_jy_j,\ \Delta_j^S(x_j,y_j)=\Delta_j^V(x_j,y_j) \iff R=\varnothing$$

3. CHARACTERIZATION OF THE SETS $\Delta_j^H(x_j, y_j)$ IN SOME BASIC MCAPs

For a given MCAP, the sets $\Delta_j^H(x_j, y_j)$ ($\forall$ H$\in${P,S,V}) are completely defined when numerical values are attributed to importance parameters. Conversely, the knowledge of all the sets $\Delta_j^H(x_j, y_j)$ defines an MCAP in a unique way. In fact, the sets $\Delta_j^H(x_j, y_j)$ contain the comprehensive preference induced by any couple of performance vectors. There is a one-to-one correspondence between an MCAP (with given numerical values for its importance parameters) and the sets $\Delta_j^H(x_j, y_j)$.

So as to highlight the link between importance parameters and the shape of the sets $\Delta_j^H(x_j, y_j)$ (or their respective borderlines), we will characterize these sets below in the case of basic simple MCAPs.

[8] if not, the following inclusions still hold by substituting to C_j, E_j and O_j their topological closure.

3.1 Weighted sum

The weighted sum aggregation uses importance coefficients k_j ($k_j \geq 0$) such that:

$$\begin{cases} \underline{x}P\underline{y} \Leftrightarrow \sum_{i=1}^{n} k_i x_i > \sum_{i=1}^{n} k_i y_i \\ \underline{x}I\underline{y} \Leftrightarrow \sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} k_i y_i \end{cases}$$

This MCAP implies that all criteria are true criteria. It follows that the sets $\Delta_j^H(x_j, y_j)$ are characterized by:

$$\begin{cases} \Delta_j^P(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \sum_{i \neq j} k_i(x_i - y_i) > k_j(y_j - x_j)\} \\ \Delta_j^S(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \sum_{i \neq j} k_i(x_i - y_i) \geq k_j(y_j - x_j)\} \\ \Delta_j^V(x_j,y_j) = \Delta_j^S(x_j,y_j) \end{cases}$$

In this MCAP, the importance is taken into account essentially through the ability of criteria to convince. The ability to equilibrate is very weak: the only couples $(\underline{x}_{-j},\underline{y}_{-j})$ contained in $E_j(x_j,y_j)$ and in $E_j'(x_j,y_j)$ are those verifying $\sum_{i \neq j} k_i(x_i - y_i) = k_j(y_j - x_j)$. No ability to oppose is attached to the criteria ($O_j(x_j,y_j)=O_j'(x_j,y_j)=\varnothing$)

3.2 Lexicographic aggregation

In this MCAP, the importance attached to each criterion is characterized by its position in a hierarchy such that g_1 is the most important criterion and g_n the least important one. More precisely, in this MCAP, we have:

$$\begin{cases} x_1 > y_1 \Rightarrow \underline{x}P\underline{y} \text{ whatever } \underline{x}_{-j},\underline{y}_{-j} \\ x_1 = y_1, \dots, x_{i-1} = y_{i-1}, x_i > y_i \Rightarrow \underline{x}P\underline{y} \text{ whatever } x_{i+1}, \dots, x_n, y_{i+1}, \dots, y_n \\ x_1 = y_1, \dots, x_n = y_n \Rightarrow \underline{x}I\underline{y} \end{cases}$$

This implies[9] that all criteria are true criteria ($q_j=0$). In this MCAP, the importance parameter takes the form of a permutation on the set of criteria. The sets $\Delta_j^H(x_j, y_j)$ are characterized by:

[9] A direct transposition of these formulae to the case of quasi-criteria leads to a comprehensive preference relation which is not transitive

$$\begin{cases} \Delta_1^P(x_1,y_1) = \Delta_1^S(x_1,y_1) = \Delta_1^V(x_1,y_1) = \{(\underline{x}_{-1},\underline{y}_{-1}) \in X_{-1}^2\} \\ \Delta_2^P(x_2,y_2) = \Delta_2^S(x_2,y_2) = \Delta_2^V(x_2,y_2) = \{(\underline{x}_{-2},\underline{y}_{-2}) \in X_{-2}^2 \text{ such that } x_1 \geq y_1\} \\ \Delta_3^P(x_3,y_3) = \Delta_3^S(x_3,y_3) = \Delta_3^V(x_3,y_3) = \{(\underline{x}_{-3},\underline{y}_{-3}) \in X_{-3}^2 \text{ such that } (x_1 = y_1 \text{ and } x_2 \geq y_2) \text{ or } (x_1 > y_1)\} \\ \quad \text{etc.} \end{cases}$$

In this MCAP, the RIC is only taken into account through the ability of criteria to convince ($E_j(x_j,y_j)=O_j(x_j,y_j)=\varnothing$). We can observe that g_1 is a dictator.

3.3 Majority rule

This MCAP is defined by:
$$\begin{cases} \underline{x}S\underline{y} \Leftrightarrow \sum_{i \in C[\underline{x}S\underline{y}]} k_i \geq s \\ \underline{x}P\underline{y} \Leftrightarrow \underline{x}S\underline{y} \text{ and not}[\underline{y}S\underline{x}] \end{cases}$$

where: s = majority level, $s \in]½,1]$
$C[\underline{x}S\underline{y}]$ = coalition of criteria such that $x_iS_iy_i$
k_j = weight of criterion g_j, $\sum_{i=1}^{n} k_j = 1$

In this MCAP, it holds:

$$\begin{cases} \Delta_j^S(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \sum_{i \neq j \text{ and } x_i \geq y_i - q_i} k_i \geq s - k_j\} \\ \Delta_j^V(x_j,y_j) = \Delta_j^S(x_j,y_j) \\ \Delta_j^P(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in \Delta_j^S(x_j,y_j) \text{ such that } \sum_{y_i \geq x_i - q_i} k_i < s - k_j\} \end{cases}$$

In this aggregation procedure, the importance of criteria is taken into account both through the ability of criteria to convince and to equilibrate (no ability to oppose is attached to criteria: $O_j(x_j,y_j)=\varnothing$).

3.4. Electre I MCAP extended to semi-criteria[10]

In this MCAP, the importance parameters are of two types: for each criterion g_j, there is a weight k_j and a veto threshold v_j ($v_j \geq q_j$). $\underline{x}S\underline{y}$ holds iff:

i) the majority rule holds for a given majority level s,

ii) $\nexists i \in F$ such that $y_i > x_i + v_i$ (non-veto condition).

[10] see [Roy & Bouyssou 93] or [Vincke 92].

In this MCAP, it holds:

$$\begin{cases} \Delta_j^S(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \sum_{i\neq j \text{ and } x_i \geq y_i - q_i} k_i \geq s - k_j \text{ and } y_i \leq x_i + v_i \, \forall i \neq j\} \\ \Delta_j^V(x_j,y_j) = \begin{cases} X_{-j}^2 & \text{if } x_j > y_j + v_j \\ \Delta_j^S(x_j,y_j) \cup V(X_{-j}^2) & \text{otherwise} \\ \quad \text{with } V(X_{-j}^2) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in X_{-j}^2 \text{ such that } \exists i \in F \text{ verifying } x_i > y_i + v_i\} \end{cases} \\ \Delta_j^P(x_j,y_j) = \{(\underline{x}_{-j},\underline{y}_{-j}) \in \Delta_j^S(x_j,y_j) \text{ such that } \sum_{y_i \geq x_i - q_i} k_i < s - k_j \text{ or } \exists i \in F \text{ such that } y_i > x_i + v_i\} \end{cases}$$

In this MCAP, the importance of criteria is taken into account through all three aspects: ability to convince, to equilibrate and to oppose. This is due to the fact that, in general:

$$\begin{cases} C_j(x_j,y_j) \neq \varnothing \\ E_j(x_j,y_j) \neq \varnothing \\ O_j(x_j,y_j) \neq \varnothing \end{cases}$$

4. CONCERNING SOME FUNDAMENTAL QUESTIONS

4.1. How to give sense to the relation "*at least as important as*" between two criteria

Let us consider the sets $\Delta_j^H(x_j, y_j)$ ($H \in \{P,S,V\}$) corresponding to an MCAP and a particular importance parameter w_j. If, when w_j increases (decrease respectively) the sets $\Delta_j^H(x_j, y_j)$ do not decrease, it seems reasonable to state that greater (lesser respectively) is w_j, and greater is the importance given to criterion g_j in the considered MCAP. This consideration shows that variations of the value assigned to an importance parameter result in variations of the importance of the corresponding criterion in a constant direction. Unfortunately, this link is not very useful for comparing the respective importance of two criteria.

In order to give sense to the relation "*at least as important as*" between two criteria, we propose to take the following empirical idea as a starting point.

<u>Basic empirical hypothesis</u>: The assertion "g_j is at least as important as g_i" refers to the following statements:

- the variety of cases in which xPy with $x_jP_jy_j$ and $y_iP_ix_i$ is greater than the variety of similar cases in which g_i is substituted for g_j and the reverse,
- the variety of cases in which xSy with $x_jP_jy_j$ and $y_iP_ix_i$ is greater than the variety of similar cases in which g_i is substituted for g_j and the reverse,
- the variety of cases in which not[yPx] with $x_jP_jy_j$ and $y_iP_ix_i$ is greater than the variety of similar cases in which g_i is substituted for g_j and the reverse.

This empirical hypothesis leads us to propose hereafter a formal condition for g_j to be at least as important as g_i grounded on the sets $\Delta_j^H(x_j, y_j)$ and $\Delta_i^H(x_i, y_i)$. However, we are confronted with difficulties linked to the fact that these two sets are neither defined on the same space (X_{-j}^2 and X_{-i}^2, respectively), nor under the same conditions ($\forall x_j, y_j \in X_j^2$ such that $x_j P_j y_j$ and $\forall x_i, y_i \in X_i^2$ such that $x_i P_i y_i$, respectively). So as to circumvent these difficulties, we will introduce the following additional notations and definitions:

$X_{-ij} = \prod_{k \neq i,j} X_k$

$\Delta_j^H(x_j, y_j / y_i P_i x_i)$ = Restriction of $\Delta_j^H(x_j, y_j)$ to couples $(\underline{x}_{-j}, \underline{y}_{-j})$ verifying $y_i P_i x_i$

Let $D \subseteq X_{-j}^2$; we define $\mathrm{Proj}_i(D)$ as the subset of X_{-ij}^2 obtained by deleting in each element of D the evaluations x_i and y_i.

On such basis, we propose the following condition.

<u>Condition 4.1.</u>

A necessary condition for g_j to be at least as important as g_i is:

$$\forall H \in \{P,S,V\}, \quad \forall\ z_i, t_i \in X_i^2 \text{ such that } z_i P_i t_i, \quad \exists\ z_j, t_j \text{ verifying :}$$
$$\left[\begin{array}{l} z_j P_j t_j \\ \mathrm{Proj}_j[\Delta_i^H(z_i, t_i / y_j P_j x_j)] \subset \mathrm{Proj}_i[\Delta_j^H(z_j, t_j / y_i P_i x_i)] \end{array}\right.$$

for H=P this condition corresponds to the following statement :

$\forall\ \underline{x}, \underline{y} \in X^2$ such that $\underline{x} P \underline{y}$ with $y_j P_j x_j$ and $x_i P_i y_i$,

there exist $\underline{z}, \underline{t}$ such that $\underline{z} P \underline{t}$ with $z_h = x_h$, $t_h = y_h$ $\forall h \neq i,j$

$z_j P_j t_j$ and $t_i P_i z_i$

When applying condition 4.1 to the MCAP studied in section 3, it appears that:

- In the weighted sum aggregation, criterion g_j is at least as important as criterion g_i only if: $k_j L_j \geq k_i L_i$ with $L_j = g_j^* - g_{j*}$
- In the lexicographic aggregation, g_j is at least as important as g_i only if: j<i
- In the aggregation based on the majority rule, criterion g_j is at least as important as criterion g_i only if: $k_j \geq k_i$

Condition 4.1 can lead to refuse conjointly both assertions "g_j is as least as important as g_i" and "g_i is as least as important as g_j". This can be the case in an MCAP using two importance parameters for each criterion. Such a case (and more generally the following considerations) shows how difficult it is to give meaning to the empirical idea that one can have concerning the relative importance of two criteria.

The above condition cannot be considered as a sufficient one because it can be verified while the empirical idea of importance leads to reject this assertion. For example, it is the cases in Electre (see 3.4) with $k_i=k_j$ and $v_i=q_i$ and v_j much greater than q_j. In this case, condition 4.1 holds not only for validating "g_i is as least as important as g_j" andbut also for validating "g_j is as least as important as g_i" (which is contradicts the empirical idea of importance). Let us remark that if the ability to oppose a veto exists for g_j and not for g_i (still with $k_i=k_j$), the condition 4.1 states that "g_j is at least as important as g_i", the reverse being fàlse. The difficulty arises when both criteria can oppose effectively their veto. A suggestion to formulate a necessary and sufficient condition may be found in [Roy 91a] section 5.1.

4.2. Importance parameters dependent on or independent of the encoding of criteria

Defining a criterion consists of specifying how to take into account (for decision aid) the particular consequences attached to a given viewpoint. Thus, encoding criterion j amounts to choosing a real value function g_j on the basis of which the preferences, relative to the attached viewpoint, may be argued.

However, there is a certain degree of freedom concerning how to define the function g_j in a convenient way. Hence, there are monotonically increasing tranformations ϕ such that g_j and $\phi(g_j)$ may be viewed as equally convenient in order to define the criterion, i.e., $\phi(g_j)$ could, in the decision context considered, replace g_j so as to describe, conceive and argue comprehensive preferences (the MCAP being unchanged). In this case, ϕ is said to be an admissible transformation for g_j. Let Φ_j be the set of admissible transformations for g_j. Specifying the set Φ_j is necessary in order to specify the signification attached to each of the elements of X_j so as to ground comparisons of preference differences. Let us recall that (X_j,Φ_j) is called (see [Vansnick 90]):

- an absolute scale when Φ_j = {identity},
- a ratio scale when $\Phi_j = \{\phi$ such that $\phi(g_j)=\alpha.g_j$ with $\alpha>0\}$,
- an interval scale when $\Phi_j = \{\phi$ such that $\phi(g_j)=\alpha.g_j+\beta$ with $\alpha>0\}$,
- an ordinal scale when $\Phi_j = \{\phi$ monotonically increasing$\}$.

The sets $\Delta_j^H(x_j, y_j)$ ($H\in\{P,S,V\}$) are subsets of X_j^2. Consequently, X_j does not depend on the ϕ chosen in Φ_j. For all $\phi\in\Phi_j$, the following invariance conditions must hold: $\Delta_j^H(x_j,y_j)=\Delta_j^H(\phi(x_j),\phi(y_j))$, $\forall H\in\{P,S,V\}$, $\forall x_j,y_j$. Considering a given MCAP, there is no guarantee that these invariance conditions hold, $\forall x_j,y_j$ and $\forall\phi\in\Phi_j$, without adapting the values of importance parameters to the encoding of X_j, i.e., to the chosen ϕ in Φ_j. This leads us to distinguish two cases for admissible transformations ($\in\Phi_j$) adopted in a given context.

1- If these equalities hold without changing the values of importance parameters, the characterization of the sets $\Delta_j^H(x_j, y_j)$ is, in the MCAP under consideration, invariant when the encoding of g_j is changed (i.e., when we substitute $\phi(g_j)$ for g_j). Considering the definition proposed for importance parameters, it appears that these parameters are independent of the encoding of g_j; such parameters are said to be intrinsic to the significance axis of g_j.

2- In the opposite case, it is necessary to adapt the value of some importance parameters in order to keep the sets $\Delta_j^H(x_j, y_j)$ unchanged when the encoding of g_j is transformed. In this case, the importance parameters are said to be dependent on the encoding of the criterion.

Such terminology, when applied to the characterizations of the sets $\Delta_j^H(x_j, y_j)$ for the MCAP studied in section 3, leads to the following results:

- The coefficients k_j of a weighted sum aggregation are dependent on the encoding of criteria.
- The ranking of criteria that specifies the relative importance of criteria in the lexicographic aggregation is intrinsic to the significance axis of criteria.
- The coefficients k_j of the MCAP based on the majority rule are independent of the encoding of criteria.
- In the MCAP Electre I, the importance coefficients k_j are intrinsic while the veto thresholds are dependent on the encoding of criteria.

Hence, the reader should be aware that it is inconsistent to assign values to importance parameters that are dependent on the encoding of the considered criterion without explicitly taking this encoding into account. Moreover, we should keep in mind that some MCAP are appropriate only if Φ_j is restricted to a limited type of transformation (for example, the weighted sum is convenient only if Φ_j, $j \in F$, are restricted to ϕ such that $\phi(g_j)=\alpha.g_j+\beta$ with $\alpha>0$).

4.3. Importance invariant by translation

The formalism introduced in section 2.3 leads us to highlight an interesting property: the invariance of the importance of a criterion by translation.

Definition 4.2: the importance of criterion g_j is invariant by translation if and only if:

$$\forall\ \alpha \in \mathbb{R},\ \forall\ x_j, y_j \in X_j,\quad \Delta_j^H(x_j, y_j) = \Delta_j^H(x_j + \alpha, y_j + \alpha),\qquad \forall\ H \in \{P, S, V\}$$

When the importance of a criterion g_j is invariant by translation, a given difference between the performances of two alternatives has the same influence on the comprehensive preferences whatever the value of the performances. This definition may be restricted to a subset of X_j^2.

Two particular cases of non-invariance of the importance of a criterion by translation should be emphasized:

- The condition $\forall\ \alpha\geq 0,\ \forall\ x_j,y_j\in X_j^2,\ \ \Delta_j^H(x_j,y_j) \subseteq \Delta_j^H(x_j+\alpha,y_j+\alpha)\ ,\ \forall\ H=\{P, S, V\}$ characterizes the situation in which the importance of criterion g_j increases with its performance.
- Reversing the inclusions in the preceding condition leads to characterize the situation in which the importance of criterion g_j decreases with its performance.

When the importance parameters of a criterion are intrinsic to its significance axis, if invariance by translation holds for a specific encoding, this property still holds for any other admissible encoding. On the other hand, if certain importance parameters are dependent on the encoding, this property of invariance may be verified for certain encodings but becomes false for others.

4.4. Link between the notions of importance and compensation

The following considerations and definitions will demonstrate the strong relationship which exists between the notions of relative importance of criteria and compensatoriness of preferences. This last notion is, in its current use as well as in multicriteria methods, linked to the possibility of making substitutions between performances on different criteria. These substitutions allow us to compensate for a disadvantage on one or several criteria by a sufficient advantage on other criteria.

When an MCAP uses such ideas in order to solve conflicts between criteria, this MCAP is usually said to be of a compensatory nature; in the opposite case, this MCAP is said to be non-compensatory. Consequently, a preference structure induced by an MCAP has a more or less compensatory nature.

More precisely, we will say that there is no possibility of compensation relative to criterion g_j if, in each indifference situation $\underline{x}I\underline{y}$ with $x_jP_jy_j$, when we destroy the indifference situation by substituting u_j for x_j and v_j for y_j (with $u_jP_jv_j$), it is not possible to re-establish it by modifying performances on other criteria (g_i, $i\neq j$).

The following definition formalizes this notion:

Definition 4.3: There is no possibility of compensation relative to criterion g_j iff
$\forall\ x_j,y_j\in X_j^2$ verifying $x_jP_jy_j$, $\forall(\underline{x}_{-j},\ \underline{y}_{-j}) \in E_j(x_j,\ y_j)$,
if there exist $u_j,v_j\in X_j^2$ verifying $u_jP_jv_j$ and $(\underline{x}_{-j},\ \underline{y}_{-j}) \notin E_j(u_j,\ v_j)$,
then $E_j(u_j,\ v_j)=\emptyset$

On the contrary, there are some possibilities of compensation between the criteria g_j and g_k if there exist $\underline{x}$ and $\underline{y}$ such that $\underline{x}I\underline{y}$ for which it is possible to destroy this indifference situation by decreasing the difference of performances between their k^{th} component and to re-establish it by increasing the difference of performances on their j^{th} component.

The following definition formalizes this conception:

Definition 4.4: There are some possibilities of compensation between the criteria g_j and g_k iff: $\exists\ x_j, y_j$ verifying $x_jP_jy_j$, $\exists(\underline{x}_{-j}, \underline{y}_{-j}) \in E_j(x_j, y_j)$,

$\exists\ (\underline{x}^k_{-j}, \underline{y}^k_{-j}) \in X^2_{-j}$ deduced from $(\underline{x}_{-j}, \underline{y}_{-j})$ by decreasing the k^{th} performance of $\underline{x}_{-j}$ and/or by increasing the k^{th} performance of $\underline{y}_{-j}$, such that $(\underline{x}^k_{-j}, \underline{y}^k_{-j}) \notin E_j(x_j, y_j)$

and

$\exists\ u_j, v_j \in X^2_{-j}$ verifying $u_jP_jv_j$ such that $(\underline{x}^k_{-j}, \underline{y}^k_{-j}) \in E_j(u_j, v_j)$.

In the MCAP weighted sum (see 3.1), there are possibilities of compensation between any pair of criteria; in the three others (see 3.2 to 3.4), there is no possibility of compensation relatively to criterion g_j ($\forall j \in F$).

Definition 4.4 can easily be extended to possibilities of compensation between coalitions of criteria.

Let us recall that the first formal definition of a totally non-compensatory (I,P) preference structure was given by [Fishburn 76] ($R=\varnothing$):

An (I,P) preference structure Ψ is totally non-compensatory (TNC) iff:

$$\forall x,y,u,v \in X^4,\ [P(x,y)=P(u,v) \text{ and } P(y,x)=P(v,u)] \implies [xPy \Leftrightarrow uPv]$$

with $P(x,y)=\{i \in F \ / \ x_iP_iy_i\}$

An (I,P) preference structure is TNC if two pairs of alternatives that have the same preferential profile are linked with the same comprehensive preference relation. In other words, a TNC MCAP considers preferences restricted to a single criterion g_j ($\forall j \in F$) to be ordinal information when building comprehensive preferences[11]. The reader will easily verify that in a TNC (I,P) preference structure, there is no possibility of compensation relative to any criterion (according to definition 4.3).

[Bouyssou 86] proposes a definition of a minimally compensatory (I,P) preference structure: An (I,P) preference structure Ψ is minimaly compensatory iff:

[11] This idea has been generalized by [Bouyssou & Vansnick 86]. See also [Vansnick 86].

$$\exists\ x,y,u,v \in X^4 \quad \text{such that} \quad \left.\begin{array}{r} P(x,y) = P(u,v) \\ P(y,x) = P(v,u) \\ x_i=y_i \text{ and } u_i=v_i\ \forall i \in I(x,y) \end{array}\right] \quad \text{with} \quad [\ xSy \text{ and } vPu\ ]$$

$$\text{with} \quad \left[\begin{array}{l} P(x,y) = \{\ i \in F\ /\ x_i P_i y_i\ \} \\ I(x,y) = \{\ i \in F\ /\ i \notin P(x,y) \text{ and } i \notin P(y,x)\ \} \end{array}\right.$$

This definition formally expresses phenomena similar to those taken into account in definition 4.4. However, it does not specify between which criteria the compensation occurs. Moreover, the reader will easily verify that, if there are some possibilities of compensation between two criteria g_j and g_k (definition 4.4), then the corresponding preference structure is minimally non-compensatory.

The analysis of the concept of compensatoriness of preference structures and its relations with the notion of RIC is too vast a subject to be developed fully here. Definitions 4.3 and 4.4 lead us to think that the theoretical framework proposed in this paper constitutes a suitable basis on which to ground further research.

5. OPERATIONAL INTEREST OF THE PROPOSED FORMALISM

So as to give meaning to importance parameters of any type, it is necessary to refer to a preference structure Ψ (see 1.2). In other words, assigning numerical values to such parameters requires questioning such a structure. The way to do this differs according to whether we adopt a descriptivist or a constructivist approach[12].

Adopting the *descriptivist approach* involves aknowledging that way in which any two performance vectors $(\underline{x}_{-j},x_j)$ and $(\underline{y}_{-j},y_j)$ are compared on the comprehensive level is stable and well-defined in the mind of the DM prior to being questioned on his/her preference structure Ψ^*. The preference model built for decision aid generates a preference structure Ψ, which is intended to give an exact an image as possible of Ψ^*. Towards this end, the numerical values of importance parameters have to be chosen in such a way that the sets $\Delta_j^H(x_j, y_j)$, $H \in \{P,S,V\}$ induced from the adopted model match the pre-existing sets $\Delta_j^{*H}(x_j, y_j)$, $H \in \{P,S,V\}$. Hence, adjustment techniques should be used to assign values to importance parameters. Several authors (see [Mousseau 92] for a review) talk in terms of estimating the numerical value of certain parameters, such as weight w_j. This supposes that the sets $\Delta_j^H(x_j, y_j)$ and $\Delta_j^{*H}(x_j, y_j)$, $H \in \{P,S,V\}$ (or more simply their respective borderlines) admit of a commom analytical representation.

[12] For more details on this distinction, see [Roy 93].

The *constructivist approach*[13], on the contrary, aknowledges that preferences are not entirely pre-formed in the decision-maker's mind and that the very nature of the work involved in the modelling process (and, a fortiori, in decision-aid) is to specify and even to modify pre-existing elements. The preference structure Ψ, induced by the MCAP adopted for decision aid, cannot be viewed here as the faithful image of a pre-existing preference structure Ψ^*, as it is considered locally ill-defined and unstable. Thus, Ψ is here nothing other than the result of a set of rules deemed appropriate for comparing any performance vectors $(\underline{x}_{-j},x_j)$ and $(\underline{y}_{-j},y_j)$. The importance parameters appear as keys which allow us to differentiate the role played by each criterion in the preference model selected for use. The numerical values attributed to them correspond to a working hypothesis accepted for decision-aid. They must be seen as suitable values with which it seems reasonable and instructive to work. So as to judge if this is the case with a given set of values, it is important to analyse the shape and the variety of the sets $\Delta_j^H(x_j, y_j)$, $H\in\{P,S,V\}$ which result from it. One should check if these values fit with the opinion of the DM (or of various stakeholders) regarding the fact that such and such $(\underline{x}_{-j},\underline{y}_{-j})$ belongs or does not belong to one of the preceding sets. The analysis of these sets does not aim at discovering a unique set of values for importance parameters that fits the DM's opinion, but it can determine a domain containing an appropriate set of values for reasoning, investigating and communicating among the stakeholders in the decision-making process.

As shown in the examples of section 3, the shape and the variety of the sets $\Delta_j^H(x_j, y_j)$, $H\in\{P,S,V\}$ vary significantly according to the type of MCAP considered. Whatever the approach, descriptivist or constructivist, the preceding remark highlights the fact that the meaning of importance parameters (and consequently the appropriate manner for giving a numerical value to them) depends on the MCAP. The discussions presented in section 4 show the danger of questioning the DM directly on values assigned to importance parameters. These discussions have proved that assigning values to certain importance parameters relying on the metaphor of weight results in serious misunderstandings (see 4.1 and 4.2). Even when this metaphor is pertinent, its evocative power is different when used with intrinsic weights (referring to voting power), or with contingent weights (such as scaling constants for example). Specifically, this metaphor is of no use in evaluating parameters such as veto thresholds in the Electre methods (see [Roy 91b]), or coefficients k_{ij} that account for certain forms of depedencies between criteria by adding terms of the form $k_{ij}.g_i(a).g_j(a)$ to an additive aggregation $\sum_{k\in F} k_j.g_j(a)$ (see [Keeney & Raiffa 76]). In order to determine suitable values for such parameters, it is possible to base an argument on the fact a couple $(\underline{x}_{-j}, \underline{y}_{-j})$ belongs (or not) to such and such $\Delta_j^H(x_j, y_j)$.

[13] An overview of behavioral decision research within this approach can be found in [Paynes et al. 92].

Conclusion

The foregoing considerations show that the notion of relative importance of criteria is more complex than it is commonly assumed to be. The formal definition that we have proposed (see section 1.3) seems sufficiently general when criteria are true or semi-criteria. The examples from section 3 and the considerations developed in section 4 allow us to think that the sets $\Delta_j^H(x_j, y_j)$ $H \in \{P,S,V\}$ (certain results have been established in section 2) constitute useful keys in order to study the significance of an MCAP and to clarify other notions closely linked to the notion of importance.

Importance parameters of any type give an account of a value system. Consequently, the values assigned to such parameters will always have a subjective character. As is true for any other subjective data, these values can be grasped only through communicating with certain stakeholders in the decision process. The definitions, results and comments presented in this paper should enable us to develop better means of conceptualizing this type of communication in order to understand and decode it with greater insight within the framework of the MCAP selected for use.

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APPENDIX A :By definition[14], a familly F of n criteria $g_1, g_2, ..., g_n$ is consistent iff F verifies the three following axioms (these axioms express natural requirements in a formal way):

- *Exhaustivity*: if $\forall j \in F$, $g_j(a)=g_j(b)$, then $\left[\begin{array}{l} cHa \Rightarrow cHb \\ aHc \Rightarrow bHc \end{array}\right.$ $\forall H \in \{P,I,R\}$
- *Cohesion*: if, $\forall j \in F\backslash\{k\}$ $\left[\begin{array}{l} g_j(b^k)=g_j(b),\ g_k(b^k)\geq g_k(b) \\ g_j(a)=g_j(a_k),\ g_k(a)\geq g_k(a_k) \end{array}\right.$ one of the inequalities being strict,

  then $bPa \Rightarrow b^kPa_k$, $bIa \Rightarrow b^kSa_k$

  moreover, if $\left[\begin{array}{l} g_k(a)=g_k(b) \\ a_k I_k b^k \end{array}\right.$ then $\left[\begin{array}{l} bHa \Rightarrow b^kHa_k \\ aHb \Rightarrow a_kHb^k \end{array}\right.$ $\forall H \in \{P,I,R,S\}$
- *Non-redundancy*: $\forall j \in F$, $F\backslash\{j\}$ is either not exhaustive or not cohesive.

APPENDIX B : Proofs

Proof of Result 2.1:

The first two inclusions are direct consequences of:

$xPy \Rightarrow xSy \Rightarrow non[yPx]$

It follows from the exhaustivity and cohesion axioms that $\forall i \in F$, $g_i(a)=g_i(b)$ $\forall i \neq j$ and $aP_jb \Rightarrow aSb$ (see [Roy & Bouyssou 93]). Hence, the last inclusion is true.

Proof of result 2.2:

This property is an immediate consequence of the cohesion axiom which formally expresses a certain form of monotony of preferences: the higher the performance, the better the alternative.

Proof of result 2.3:

The justification of this property comes from the cohesion axiom.

H=P: if $(\underline{x}_{-j},\underline{y}_{-j}) \in \Delta_j^P(x_j, y_j)$, i.e. xPy,

then, $\forall \underline{u}_{-j},\underline{v}_{-j} \in X_{-j}^2$ verifying $\left[\begin{array}{l} \underline{u}_{-j}\Delta_{-j}\underline{x}_{-j} \\ \underline{y}_{-j}\Delta_{-j}\underline{v}_{-j} \end{array}\right.$, we have $\left[\begin{array}{l} \underline{u}\Delta\underline{x} \\ \underline{y}\Delta\underline{v} \end{array}\right.$ with $x_j=u_j$ and $y_j=v_j$

then as a consequence of the cohesion axiom, it holds uPv

i.e., $(\underline{u}_{-j},\underline{v}_{-j}) \in \Delta_j^H(x_j, y_j)$ ($H \in \{P,S,V\}$) ■

Proof of result 2.4:

$\Rightarrow$ Let us suppose that $O_j'(x_j,y_j)=\varnothing$, then $O_j(x_j,y_j)=\varnothing$

i.e., $\Delta_j^V(x_j,y_j)\backslash\Delta_j^S(x_j,y_j)=\varnothing$

$\nexists$ $\underline{x}_{-j},\underline{y}_{-j} \in X_{-j}^2$ verifying:

not$[(\underline{y}_{-j},y_j)P(\underline{x}_{-j},x_j)]$ and not$[(\underline{x}_{-j},x_j)S(\underline{y}_{-j},y_j)]$

i.e., not$[(\underline{y}_{-j},y_j)P(\underline{x}_{-j},x_j)]$ and not$[(\underline{x}_{-j},x_j)P(\underline{y}_{-j},y_j)]$ and not$[(\underline{x}_{-j},x_j)I(\underline{y}_{-j},y_j)]$

Thus $\nexists$ $\underline{x}_{-j},\underline{y}_{-j} \in X_{-j}^2$ verifying $(\underline{x}_{-j},x_j)R(\underline{y}_{-j},y_j)$ ■

[14] for more details, see [Roy & Bouyssou 93] chap. 2.

$\Leftarrow$ Let us suppose that $\nexists\ \underline{x}_{-j},\underline{y}_{-j}\in X_{-j}^2$ such that $(\underline{x}_{-j},x_j)R(\underline{y}_{-j},y_j)$
then, there are only three preferential situations between $(\underline{x}_{-j},x_j)$ and $(\underline{y}_{-j},y_j)$:
$(\underline{x}_{-j},x_j)P(\underline{y}_{-j},y_j)$, $(\underline{x}_{-j},x_j)I(\underline{y}_{-j},y_j)$ and $(\underline{y}_{-j},y_j)P(\underline{x}_{-j},x_j)$
however, $(\underline{x}_{-j},x_j)S(\underline{y}_{-j},y_j) \Leftrightarrow (\underline{x}_{-j},x_j)P(\underline{y}_{-j},y_j)$ or $(\underline{x}_{-j},x_j)I(\underline{y}_{-j},y_j)$
then it holds $(\underline{x}_{-j},x_j)S(\underline{y}_{-j},y_j) \Rightarrow$ not$[(\underline{y}_{-j},y_j)P(\underline{x}_{-j},x_j)]$
then $\Delta_j^S(x_j,y_j)=\Delta_j^V(x_j,y_j)$, i.e., $O_j(x_j,y_j)=\varnothing$
thus $O_j'(x_j,y_j)=\varnothing$ ∎

APPENDIX C : Illustration of the concept of borderline

Let us illustrate the interpretation of the borderline through a bi-criteria example in which each criterion is evaluated on a discrete scale containing five levels ($AP_iBP_iCP_iDP_iE$, $i=1,2$). Table 1 shows an example of the borderline of $\Delta_1^P(x_1, y_2)$ corresponding to x_1=A and y_1=B.

	x_2 = A	x_2 = B	x_2 = C	x_2 = D	x_2 = E
y_2 = A	∈	∈	∈ (shaded)	∉	∉
y_2 = B	∈	∈	∈ (shaded)	∉	∉
y_2 = C	∈	∈	∈	∈ (shaded)	∉
y_2 = D	∈	∈	∈	∈	∈ (shaded)
y_2 = E	∈	∈	∈	∈	∈

Table 1: Borderline of $\Delta_1^P(x_1=A,\ y_1=B)$.

∈	element of $\Delta_1^P(A, B)$.
∉	non-element of $\Delta_1^P(A, B)$.
∈ (shaded)	element of the borderline of $C_1'(A,B)$.

The reader will easily verify that if he knew all the shaded cells, he could deduce all the other elements of $\Delta_1^P(A,B)$.