CAHIER DU LAMSADE

A FIRST ORDER, FOUR-VALUED
WEAKLY PARACONSISTENT LOGIC

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Résumé

Dans ce cahier, on présente une logique quadrivalente du premier ordre, appelée DDT, qui est une extension de la logique de Belnap et utilise une négation faible et definit une sémantique adéquate pour le calcul des prédicats. Cette logique a une structure algébrique simple, celle du plus petit bi-treillis entrelacé non trivial. Cette structure permet de définir tous les connecteurs logiques en imposant une proprieté de monotonie qui préserve l'ordre du bi-treillis. Cette logique est faiblement para-consistent. Dans cette perspective la négation ne coincide pas avec le complément et les raisons pour lesquelles une formule peut être considerée comme vraie ne sont pas le complémentaire des raisons pour lesquelles elle peut être considerée comme fausse. De cette manière on établit deux sources d'incertitude. Cette logique peut donc être considerée comme une logique de l'incertain. On présente une extension de la théorie des ensembles approximatifs et on montre qu'il s'agit d'une semantique pour le fragment bivalué de la logique DDT. On montre aussi comment utiliser cette logique pour la modélisation des préférences, raison première pour laquelle elle fut crée.

A first order, four-valued, weakly paraconsistent logic

Abstract

A first order four valued logic, named DDT, is presented in the paper as an extension of Belnap's logic using a weak negation and establishing an appropriate semantic for the predicate calculus. The logic uses a simple algebraic structure, that is the smallest non trivial interlaced bilattice on the four truth values. Such a structure enables to correctly define all the logical connectives (mainly the negations) imposing a continuity property so that the order of the lattice is always preserved. The logic is weakly paraconsistent. Under this perspective the negation does not coincide with the complement and the reasons for which a sentence is considered to be true are not complementary to the reasons for which is considered to be false. Such a property enables to distinguish two basic sources of uncertainty in reasoning, that is lack or excess of information. The logic is a language for reasoning under uncertainty. An extension of rough sets theory is introduced in the paper as it could be used as the semantics for the two valued fragment of the logic. The use of the logic as language for preference modeling purposes is discussed in the paper since the logic has been specially created for such reason.

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Introduction

Contradictory information is a common situation in real life and in every day human reasoning. Moreover humans normally are able to act either under such "contradictory" situations or in "absence" of information. Under this perspective is known that classic logic fails to be a good representation of human reasoning since any inconsistency allows the deduction of everything and absence of information simply is not considered. Classic logic enables to deduce automatically all the possible theorems from a given set of sentences. The introduction of new information (under the form of a new sentence) will change nothing (if the sentence is consistent with the already given set) or will destroy the conclusions (if it is inconsistent). The problem of reasoning under inconsistency has been faced in paraconsistent logics (see da Costa, 1974; Rescher and Brandom, 1980). The problem of revision and updating in a knowledge base has been faced in the non monotonic reasoning and belief revision literature (see Ginsberg, 1987; Gärdenfors, 1988). Very few attempts appear in literature comparing the two approaches (see Besnard and Laerens, 1994; Benferhat 1994).

The paper aims to present a first order extension of Belnap's four valued logic (see Belnap 1976 and 1977) with strong connectives and which is weakly paraconsistent together with some applications. The paper is organized as follows. In section 1 the basic idea of the four valued logic is presented and its underlying algebraic properties are discussed. In section 2 the first order extension of Belnap's logic with strong connectives is defined. In section 3 some relevant properties of the logic are presented. In section 4 an extension of the rough sets theory is introduced as a semantic for the two valued fragment of the logic. In section 5 a brief account of the application of the logic in preference modeling is presented. The open problems are discussed at the end of the paper.

1 Four truth values

The four values introduced by Belnap in his two seminal papers (Belnap 1976 and 1977) have a clear epistemic nature. Actually these truth values represent different states where an agent (natural or artificial) may find himself/herself when asked to answer a query. Giving a sentence ϕ, the agent may have been told that "ϕ holds", that "ϕ does not hold", both or nothing. The problem is how the agent should react in any of these

cases, independently of the ontology of ϕ since (s)he is obliged to provide an answer. The logic presented tries to model the epistemic nature of reasoning without introducing epistemic operators as the modal ones of knowledge and belief. The basic idea is to characterize some basic states in which an agent may find himself/herself through a four valued valuation of his/her language. The four values established are:

- true (t): there is evidence that is true and there is no evidence that is false;
- false (f): just evidence that is false and there is no evidence that is true;
- both (k): there is evidence both that is true and that is false;
- unknown (u): there is no evidence either that is true or that is false;

and we define the four corresponding epistemic states as the "true" one, the "false" one, the "contradictory" one and the "unknown" one. The logic we develop will therefore be a calculus over epistemic states and not on the ontology of the language.

1.1 Lattices and Bilattices

Let me introduce first some basic definitions and notations (see Ginsberg 1988, Fitting, 1991) limited to complete lattices.

Definition 1.1 *A complete lattice L is a triplet $\langle T, \sqcup, \sqcap \rangle$ where T is a partially ordered set, $\sqcup$ and $\sqcap$ being the joint and the meet operators, respectively.*

Therefore we have:

$$\exists \succeq \; \subseteq T \times T : \quad \forall x, y \in T : x \sqcup y = glb_{\succeq}(x, y) \quad x \sqcap y = lub_{\succeq}(x, y)$$

Definition 1.2 *A complete bilattice B is a five-tuple $\langle T, \sqcup, \sqcap, +, \bullet \rangle$ where T is a twice partially ordered set, $\sqcup$ and $\sqcap$ being the joint and the meet operators of the first partial order, respectively, and $+$ and $\bullet$ being the joint and meet operators of the second partial order, respectively.*

Therefore we have:

$$\exists \succeq_1, \succeq_2 \; \subseteq T \times T : \quad \forall x, y \in T :$$

$$x \sqcup y = glb_{\succeq_1}(x, y) \quad x \sqcap y = lub_{\succeq_1}(x, y)$$

$$x + y = glb_{\succeq_2}(x, y) \quad x \bullet y = lub_{\succeq_2}(x, y)$$

I restrict the field by considering a particular class of bilattices known as "interlaced bilattices" (see Fitting, 1991).

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Definition 1.3 *An interlaced bilattice is a complete bilattice such that meets and joints of one order are monotone with respect to the other order of the bilattice. That is:*

$$\forall x, y, z, w \in T, \quad x \succeq_1 y \text{ and } z \succeq_1 w, \text{ then } x \bullet z \succeq_1 y \bullet w$$

$$\forall x, y, z, w \in T, \quad x \succeq_2 y \text{ and } z \succeq_2 w, \text{ then } x \sqcap z \succeq_2 y \sqcap w$$

The concept of monotonicity is introduced as a basic condition for a bilattice to be interlaced. Interlacity is the minimum property of a bilattice in order not to be just two lattices stuck together. However the concept of monotonicity will be also used in order to define basic transformations of a lattice (and a bilattice). In Scott's work (1972, 1982) on "approximation" lattices (mathematically equivalent to complete lattices) the concept of "continuity" is introduced as a necessary property of a function in order to be accepted as a transformation on the lattice. In the discrete case (as in this case) continuity reduces to monotonicity. Such a property is important as monotonic transformations are the only ones that preserve the order in a lattice. I will give therefore the definitions of the basic unary transformations of an interlaced bilattice (keeping in mind monotonicity).

Definition 1.4 *Given an interlaced bilattice B:*
$\mathcal{N}_1 : T \mapsto T$ *is a monotone transformation on $\succeq_1$* ***iff***

$$\forall x, y, \quad x \succeq_1 y \Leftrightarrow \mathcal{N}_1(x) \succeq_1 \mathcal{N}_1(y)$$

$\mathcal{N}_2 : T \mapsto T$ *is a monotone transformation on $\succeq_2$* ***iff***

$$\forall x, y, \quad x \succeq_2 y \Leftrightarrow \mathcal{N}_2(x) \succeq_2 \mathcal{N}_2(y)$$

$\mathcal{I} : T \mapsto T$ *is an interlaced monotone transformation on $\succeq_1$ and $\succeq_2$* ***iff***

$$\forall x, y, \quad x \succeq_2 y \Leftrightarrow \mathcal{I}(x) \succeq_1 \mathcal{I}(y)$$

$$\forall x, y, \quad x \succeq_1 y \Leftrightarrow \mathcal{I}(x) \succeq_2 \mathcal{I}(y)$$
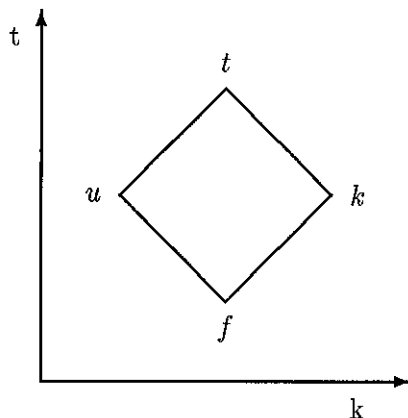
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1.2 Lattice representation of four truth values

Figure 1. The smallest non trivial interlaced bilattice Λ.

Using Scott's results on approximation lattices (see Scott 1972 and 1982) Belnap (1976 and 1977) ordered the four truth values on two lattices the one named "information" lattice and the other "truth" lattice. Not surprisingly these two lattices form the smallest non trivial interlaced bilattice (see Ginsberg 1988 and Fitting, 1991). Such a bilattice is shown in figure 1 and denoted as the bilattice Λ. Following the information order (the k one) we read $x \succeq_k y$ as "y approximates the information at least as x". The glb_k is the value u and the lub_k is the value k. Following the truth order (the t one) we read $x \succeq_t y$ as "y is true at least as x". The glb_t is the value f and the lub_t is the value t. In this context negations are monotone transformations on a lattice with the duality property, that is $\mathcal{H}$ is a negation iff it is a monotone transformation on the bilattice (see definition 1.4) and, $\forall x, \in B \quad \mathcal{H}(\mathcal{H}(x)) = x$ (duality property). Actually imposing the monotonicity of negation is the only way to preserve the structure of the bilattice and its interlaced property.

Belnap developed his propositional logic using as a negation a monotone transformation on the k lattice and as basic binary connectives the conjunction which corresponds to the meet on the t lattice and the disjunction which corresponds to the joint on the t lattice. He then defined implication as a two-valued binary connective such that "$x \to y$" is true iff $x \succeq_t y$ and false otherwise.

Such a logic however lacks any specific semantics and is too weak in order

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to use it as calculus (at least in the domain of preference modeling where I was intended to use it). Following the pioneering work of Dubarle (1963) I therefore tried to develop a more strong logic which could allow a first order calculus and strong connectives enough to represent both four valued and two valued sentences.

The basic extensions done to the propositional logic introduced by Belnap are the following.

1. Introduce a weak negation $\not\neg$ which is an interlaced monotone transformation of the Λ with duality. We therefore have the usual strong negation $\neg$ as defined by Belnap and a weak negation. The truth tables of the two negations are shown in table 1 (on the use of two negations, see also Fages and Ruet, 1994).

α	t	k	u	f
$\neg\alpha$	f	k	u	t
$\not\neg\alpha$	k	t	f	u

Table 1. The truth tables of the two negations.

2. Define implication "$\rightarrow$" as follows:
$\alpha\rightarrow\beta =_{def} \neg\not\neg\neg\not\neg\alpha\vee\beta$
The reasons for such a definition will become clearer in the next section.

The resulting logic is a functionally complete propositional logic as has already shown by Dubarle (1963) and it corresponds to a Boolean algebra on the bilattice Λ. Ruet (1993) demonstrated also the soundness and completeness of a practically equivalent logic.

2 A first order four valued logic

2.1 Syntax

An alphabet of the first order language $\mathcal{L}$, henceforth called DDT, consists of (for a preliminary version, see Doherty et al., 1992):
- a denumerable set of *individual variables* (possibly subscripted):
$x_1, x_2 \cdots y_1, y_2 \cdots z_1, z_2 \cdots$

- the *logical connectives* "∨" (or), "∧" (and), "→" (implication), "∼" (complementation), "⫫" (weak negation) and "¬' (strong negation),
- the *unary operators* "T" (true), "F" (false), "U" (unknown), "K" (both), "△ (presence of truth),
- the *quantifiers* "∀ (for all) and "∃" (exists),
- the constants $\mathcal{T}, \mathcal{K}, \mathcal{U}, \mathcal{F}$,
- the symbols "(" and ")" serving as punctuation marks,
- a countable set of *predicate constants* (i, p, q, r, ...) of positive arity, including "=" for identity.

We use greek letters $\alpha, \beta, \gamma, \cdots$ to represent general formula of the language.

Well-formed formula are defined the usual way.
If α, β are wff, then $\neg\alpha$, $\not\sim\alpha$, $T\alpha$, $\alpha\wedge\beta$, $\alpha\vee\beta$ etc. are wff.

In the following we give the truth tables of the principal connectives. In Table 2 are provided the truth tables of the negations and their combinations. In table 3 the truth tables of the three basic binary operations, that is the conjunction, the disjunction and the implication, are provided. In table 4 the truth tables of the strong unary operators are presented.

α	$\not\sim\alpha$	$\sim\not\sim\alpha$	$\sim\alpha$	$\neg\alpha$	$\neg\not\sim\alpha$	$\neg\sim\not\sim\alpha$	$\neg\sim\alpha$
t	k	u	f	f	k	u	t
k	t	f	u	k	f	t	u
u	f	t	k	u	t	f	k
f	u	k	t	t	u	k	f

Table 2. The truth tables of ¬, ⫫ and ∼ and their combinations.

From an algebraic point of view these eight combinations represent one of the Sylow subgroups of the group of all permutations of four elements and precisely the one preserving complementarity between t and f and between k and u. Under such a property it is easy to observe that the "complementation" can be defined through the other two negations. The following identities are true (the demonstrations are trivial from the truth table).

$\sim\alpha \equiv \neg\not\sim\neg\not\sim\alpha \equiv \not\sim\neg\not\sim\neg\alpha$
$\neg\neg\alpha \equiv \alpha$

$$\sim\sim\alpha \equiv \alpha$$
$$\not\sim\not\sim\alpha \equiv \alpha.$$

This is not a surprising result. Actually only $\neg$ and $\not\sim$ are negations (fulfilling monotonicity and duality) while $\sim$ (which is not a monotone transformation) should be viewed as an abbreviation of $\neg\not\sim\neg\not\sim$ which represents on its turn the complement on the bilattice. It is easy to observe that the negation corresponding to the monotone transformation on the t lattice can be defined as the sequence $\not\sim\neg\not\sim$. Moreover the implication introduced corresponds to the conventional strong monotonic implication. In fact $\alpha\rightarrow\beta$ should be read as *"either the complement of α or β"*.

We introduce now the truth tables for the basic binary operators.

$\wedge$	u	f	t	k
u	u	f	u	f
f	f	f	f	f
t	u	f	t	k
k	f	f	k	k

$\vee$	u	f	t	k
u	u	u	t	t
f	u	f	t	k
t	t	t	t	t
k	t	k	t	k

$\rightarrow$	t	k	u	f
t	t	k	u	f
k	t	t	u	u
u	t	k	t	k
f	t	t	t	t

Table 3. The truth tables of $\wedge$, $\vee$ and $\rightarrow$.

A two valued fragment of the language, named DDT^2, can be created introducing some strong unary operators. Their truth tables are as follows:

1. $T\alpha =_{def} \alpha \wedge \sim\neg\alpha$.

2. $F\alpha =_{def} \sim\alpha \wedge \neg\alpha$.

3. $U\alpha =_{def} \sim\not\sim\alpha \wedge \neg\not\sim\alpha$.

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4. $K\alpha =_{def} \not\nabla\alpha \wedge \not\nabla\neg\alpha$.

5. $\triangle\alpha =_{def} T\alpha \vee K\alpha$.

6. $\triangle\neg\alpha =_{def} F\alpha \vee K\alpha$.

The truth tables for the defined operators are presented in table 4. Actually it is easy to verify that $\triangle\alpha \equiv T(\alpha\vee\not\nabla\alpha)$ and $\triangle\neg\alpha \equiv T(\neg\alpha\vee\not\nabla\neg\alpha)$.

α	$T\alpha$	$K\alpha$	$U\alpha$	$F\alpha$	$\triangle\alpha$	$\triangle\neg\alpha$
t	t	f	f	f	t	f
k	f	t	f	f	t	t
u	f	f	t	f	f	f
f	f	f	f	t	f	t

Table 4. Truth Tables for the strong unary operators.

2.2 Semantics

The logic introduced deals with uncertainty. A set A may be defined, but the membership of an object a to the set may be not sure either because the information is not sufficient or because the information is contradictory.

In order to distinguish these two principal sources of uncertainty the knowledge about the *"membership"* of a in A and about the *"non-membership"* of a in A are evaluated independently since they are not necessarily complementary. Under this perspective from a given knowledge we have two possible entailments, one, positive, about membership and one, negative, about non-membership. Therefore any predicate is defined by two sets, its positive and its negative extension in the universe of discourse. Since the negative extension do not correspond necessarily to the complement of the positive extension of the predicate we can expect that the two extensions possibly overlap (due to the independent evaluation) and that there exist parts of the universe of discourse that do not belong to any of the two extensions. The four truth values capture these situations.

More formally we have:

A *similarity type* ρ is a finite set of predicate constants R, where each R has an arity $n_R \leq \omega$. Every alphabet uniquely determines a class of *formulas*. Relative to a given similarity type ρ, $R(x_1, \ldots, x_m)$ is an atomic formula iff

$x_1, \ldots, x_m$ are individual variables, $R \in \rho$, and $n_R = m$. Similarly, $(x = y)$ is an atomic formula iff x and y are variables. The definitions of $\mathcal{L}[\rho]$ formulas, free variables, etc. are defined in the usual way. In this paper, formulas are denoted by the letters $\alpha, \beta, \gamma, \cdots$, possibly subscripted.

A *structure* or *model* M for similarity type ρ consists of a non-empty domain $|M|$ and, for each predicate symbol $R \in \rho$, an ordered pair $R^M = \langle R^{M^+}, R^{M^-} \rangle$ of sets (not necessarily a partition) of n_R-tuples from $|M|$. Actually an individual can be both in the two sets or in no one of them.

A *variable assignment* is a mapping from the set of variables to objects in the domain of the model. Capital letters from the beginning of the alphabet are used to range over variable assignments.

The truth definition for DDT is defined via two semantic relations, $\models_t$ (true entailment) and $\models_f$ (false entailment), by simultaneous recursion as in the following definition (due to the structure introduced the case of "not true entailment" $\not\models_t$ does not coincide with the false entailment and the case of "not false entailment" $\not\models_f$ does not coincide with the true entailment). Each formula is univoquely defined through its model which, however, is a couple of sets, the "positive" and "negative" extensions of the formula.

Definition 2.1 *Let M be a model structure and A a variable assignment.*

1. $M \models_t R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in R^{M^+}$.

2. $M \models_f R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in R^{M^-}$.

3. $M \not\models_t R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in |M| \setminus R^{M^+}$.

4. $M \not\models_f R(x_1, \ldots, x_n)[A]$ *iff* $\langle A(x_1), \ldots, A(x_n) \rangle \in |M| \setminus R^{M^-}$.

5. $M \models_t (x = y)[A]$ *iff* $A(x) = A(y)$.

6. $M \models_f (x = y)[A]$ *iff* $A(x) \neq A(y)$.

7. $M \models_t \neg\alpha[A]$ *iff* $M \models_f \alpha[A]$.

8. $M \models_f \neg\alpha[A]$ *iff* $M \models_t \alpha[A]$.

9. $M \not\models_t \neg\alpha[A]$ *iff* $M \not\models_f \alpha[A]$.

10. $M \not\models_f \neg\alpha[A]$ *iff* $M \not\models_t \alpha[A]$.

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11. $M \models_t \not\sim\alpha[A]$ iff $M \models_t \alpha[A]$.

12. $M \models_f \not\sim\alpha[A]$ iff $M \not\models_f \alpha[A]$.

13. $M \not\models_t \not\sim\alpha[A]$ iff $M \not\models_t \alpha[A]$.

14. $M \not\models_f \not\sim\alpha[A]$ iff $M \models_f \alpha[A]$.

15. $M \models_t (\alpha\vee\beta)[A]$ iff $M \models_t \alpha[A]$ or $M \models_t \beta[A]$.

16. $M \models_f (\alpha\vee\beta)[A]$ iff $M \models_f \alpha[A]$ and $M \models_f \beta[A]$.

17. $M \not\models_t (\alpha\vee\beta)[A]$ iff $M \not\models_t \alpha[A]$ and $M \not\models_t \beta[A]$.

18. $M \not\models_f (\alpha\vee\beta)[A]$ iff $M \not\models_f \alpha[A]$ or $M \not\models_f \beta[A]$.

19. $M \models_t (\alpha\wedge\beta)[A]$ iff $M \models_t \alpha[A]$ and $M \models_t \beta[A]$.

20. $M \models_f (\alpha\wedge\beta)[A]$ iff $M \models_f \alpha[A]$ or $M \models_f \beta[A]$.

21. $M \not\models_t (\alpha\wedge\beta)[A]$ iff $M \not\models_t \alpha[A]$ or $M \not\models_t \beta[A]$.

22. $M \not\models_f (\alpha\wedge\beta)[A]$ iff $M \not\models_f \alpha[A]$ and $M \not\models_f \beta[A]$.

23. $M \models_t \forall x\alpha[A]$ iff $M \models_t \alpha[A']$ for all A' differing with A at most at x.

24. $M \not\models_t \forall x\alpha[A]$ iff $M \not\models_t \alpha[A']$ for all A' differing with A at most at x.

25. $M \models_f \forall x\alpha[A]$ iff $M \models_t \alpha[A']$ for an A' differing with A at most at x.

26. $M \not\models_f \forall x\alpha[A]$ iff $M \not\models_t \alpha[A']$ for an A' differing with A at most at x.

It is now possible to introduce an evaluation function $v(\alpha)$ mapping $\mathcal{L}$ to the set of truth values $\{t, k, u, f\}$ as follows:
- $v(\alpha) = t$ iff $M \models_t \alpha[A]$ and $M \not\models_f \alpha[A]$
- $v(\alpha) = k$ iff $M \models_t \alpha[A]$ and $M \models_f \alpha[A]$
- $v(\alpha) = u$ iff $M \not\models_t \alpha[A]$ and $M \not\models_f \alpha[A]$
- $v(\alpha) = f$ iff $M \not\models_t \alpha[A]$ and $M \models_f \alpha[A]$

Recalling that the truth values are ordered on the bilattice Λ it is easy to verify that the evaluation function previously defined fulfills the following

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properties:

- $v(\alpha \wedge \beta) = \min_t(v(\alpha), v(\beta))$
- $v(\alpha \vee \beta) = \max_t(v(\alpha), v(\beta))$
- $v(\alpha \rightarrow \beta) = t$ **iff** $v(\alpha) \preceq_t v(\beta)$
- $v(\alpha \equiv \beta) = t$ **iff** $v(\alpha) = v(\beta)$

where the subscript t indicates the "truth" dimension of the bilattice Λ.

From the above definitions, it is easy to see that when $M \models_t \alpha[A]$ the formula α *can be "true" or "contradictory"* which in any case implies that there is a presence of truth in α. Such a consequence relation introduces a kind of "ambiguity" since it does not allow to assign a truth value univoquely (actually we need the "false consequence relation"). We can therefore define a *"strong consequence"* relation which may correspond to the case where the formula α, in a variable assignment A, has exactly the truth value "true". This is typical of two valued valuations.

Definition 2.2 (Strong Consequence.) *A formula α is* true *in a model M iff $M \models_t \alpha[A]$ and $M \not\models_f \alpha[A]$ for all variable assignments A and we write $M \models \alpha[A]$. A formula α is* satisfiable *iff α is true in a model M for some M. A set of formulas Γ is said to be a* strong consequence *or* strongly entails *a formula α (written $\Gamma \models \alpha$) when for all models M and variable assignments A, if $M \models \beta_i[A]$, for all $\beta_i \in \Gamma$, then $M \models \alpha[A]$.*

See Thomason and Horty (1988) and Fenstad et. al. (1987) for an account of related logics and their applications.

3 Some properties of DDT

3.1 Axioms of DDT

The following formulas hold in the DDT logic.

1. $\not\sim \alpha \equiv (\alpha \vee \mathcal{K}) \rightarrow (\alpha \wedge \mathcal{K})$.

2. $T \not\sim \alpha \equiv K(\alpha \wedge \neg \alpha)$.

3. $\not\sim \alpha \wedge \not\sim \beta \rightarrow \not\sim (\alpha \wedge \beta)$.

4. $\alpha \wedge \not\sim \beta \rightarrow \not\sim (\alpha \wedge \beta)$.

5. $\not\sim \alpha \wedge \beta \rightarrow \not\sim (\alpha \wedge \beta)$.

6. $(\alpha{\rightarrow}\beta){\rightarrow}((\alpha{\rightarrow}(\beta{\rightarrow}\gamma)){\rightarrow}(\alpha{\rightarrow}\gamma))$.

7. $\alpha{\rightarrow}((\alpha{\rightarrow}\beta){\rightarrow}\beta))$.

8. $\alpha{\wedge}\beta{\rightarrow}\beta$.

9. $\alpha{\wedge}\beta{\rightarrow}\alpha$.

10. $\alpha{\rightarrow}(\beta{\rightarrow}(\alpha{\wedge}\beta))$.

11. $\alpha{\rightarrow}\beta{\vee}\alpha$.

12. $\beta{\rightarrow}\beta{\vee}\alpha$.

13. $\alpha{\vee}{\sim}\alpha$.

14. $\neg\forall x\phi(x) \equiv \exists x\neg\phi(x)$.

15. $\neg\exists x\phi(x) \equiv \forall x\neg\phi(x)$.

16. $\not\sim\forall x\phi(x) \equiv \forall x\neg\not\sim\neg\phi(x)$.

17. $\not\sim\exists x\phi(x) \equiv \exists x\neg\not\sim\neg\phi(x)$.

The following formulas do not hold in DDT.

1. $\alpha{\wedge}\neg\alpha{\rightarrow}\beta$.

2. $\beta{\rightarrow}\alpha{\vee}\neg\alpha$.

3.2 Paraconsistency

DDT is a paraconsistent logic. From the previous section we see that the *"reduction ad absurdum"* law does not hold in this logic and this is sufficient to characterize it. However it is possible to make two observations.

1. The same law may be valid if we substitute the "strong negation" by the "complementation". The following therefore holds:
   - $\alpha{\wedge}{\sim}\alpha{\rightarrow}\beta$.
   - $\beta{\rightarrow}\alpha{\vee}{\sim}\alpha$.

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2. In the two valued fragment of the DDT logic, that is using formula containing the strong unary operators $T, K, U, F, \triangle$ the law is again valid. The following therefore holds:

- $T\alpha \wedge T\neg\alpha \rightarrow \beta$.
- $T\alpha \wedge \neg T\alpha \rightarrow \beta$.
- $T(\alpha \wedge \neg\alpha) \rightarrow \beta$.
- $\beta \rightarrow T\alpha \vee \neg T\alpha$.

Since the logic contains not paraconsistent fragments I will call it a *"weakly paraconsistent logic"*.

4 Rough sets semantics for DDT[2]

4.1 About rough sets

Rough sets have been introduced by Pawlak (1982) as a new approach concerning the treatment of uncertain information and more specially the capability of distinguishing objects described in a more or less accurate way (see also Pawlak, 1991).

Following Pawlak, given $U \neq \emptyset$ a set or universe of objects and $P \subseteq R$ (R being a family of equivalence relations), $P \neq \emptyset$, then we define an *"indiscernability"* relation $IND(P)$ as

$$IND(P) = \bigcap_U P \text{ intersection of all the equivalence relations belonging to} P$$

and $U/IND(P)$ or U/P as the family of all the equivalence classes of the equivalence relation $IND(P)$ on U.

We denote the couple (U, R) as a knowledge base $\mathcal{B}$. Let $\mathcal{B} = (U, R)$ a knowledge base, then

$$IND(\mathcal{B}) = \{IND(P): \ \emptyset \neq P \subseteq R\}$$

is the family of all equivalence relations defined in $\mathcal{B}$.

Given a $\mathcal{B} = (U, R)$ for each subset $X \subseteq U$ we associate two sets:

$$X_R = \bigcup\{Y \in U/R: \ Y \subseteq X\}$$

$$X^R = \bigcup\{Y \in U/R: \ Y \cap X \neq \emptyset\}$$

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the R-lower and R-upper approximation of X by the description R of U, respectively. In other words, given a set of objects X difficult to be described, it is possible to approximate it using the description U/R with two sets:

- the lower approximation which are the elements of U which surely are in X (following the classification U/R);

- the upper approximation which are the elements of U which possibly are in X (following the classification U/R);.

We finally define as $B(X) = X^R \setminus X_R$ the R-boundary region of X, that is the set of elements on which there is a doubt about their belonging in X. Some properties of the approximation sets are presented in the following (see Pawlak, 1991), keeping in mind that X^c denotes the complement of a set X (and unless differently specified the complement will coincide with the negation).

1. $X_R \subseteq X \subseteq X^R$.

2. $\emptyset^R = \emptyset_R = \emptyset$, $U_R = U^R = U$.

3. $(X \cup Y)^R = X^R \cup Y^R$.

4. $(X \cap Y)_R = X_R \cap Y_R$.

5. $X \subseteq Y$ implies $X_R \subseteq Y_R$.

6. $X \subseteq Y$ implies $X^R \subseteq Y^R$.

7. $(X \cup Y)_R \supseteq X_R \cup Y_R$.

8. $(X \cap Y)^R \subseteq X^R \cap Y^R$.

9. $(X^c)_R = (X^R)^c$.

10. $(X^c)^R = (X_R)^c$.

4.2   Semantics for DDT^2

I will now try to extend the basic idea of the rough sets theory to the two valued fragment of DDT. A basic implicit hypothesis in rough sets is that the universe U is *completely* described by a set of equivalence relations R. When a new set X has to be described there are no elements of U with unknown properties (under the points of view represented by R) so that the membership of an element to X can be doubtful only because of conflicting information, but not because of lack of information.

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I will relax therefore this hypothesis allowing that the relations in R do not completely describe U, that is there are elements of U for which there is no sufficient information for a given classification.

Example 4.1 *Suppose that $\mathcal{B} = (U, R)$ is given with:*

$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$

$R = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_7, x_8\}\}$

It can be observed that the definition of R leaves the elements x_5, x_6 undefined. Suppose we want to describe, using R, the set $X = \{x_1, x_2, x_3, x_5\}$. Following the definitions from the rough set theory we have:

$X_R = \{x_1, x_2\}$, *the lower approximation,*

$X^R = \{x_1, x_2, x_3, x_4\}$, *the upper approximation.*

Following the properties of the rough sets theory we can also conclude that:

$\neg X_R = \{x_5, x_6, x_7, x_8\}$, *the lower approximation of the negation,*

$\neg X^R = \{x_3, x_4, x_5, x_6, x_7, x_8\}$, *the upper approximation of the negation.*

On the other hand if we calculate the upper and lower approximations of the set $\neg X = \{x_4, x_6, x_7, x_8\}$ using the usual definitions, we obtain:

$\neg X_R = \{x_7, x_8\}$, *the lower approximation of the negation,*

$\neg X^R = \{x_3, x_4, x_7, x_8\}$, *the upper approximation of the negation,*

which contradicts the previous result. Actually is the incomplete description of U by R that generates the difference.

The relaxation of the completeness assumption, while leaves the definition of lower and upper approximations the same, modifies the theorems of the theory. The following properties hold under the relaxed theory:

1. $X_R \subseteq X \subseteq (\neg X_R)^c$, $X^R \subseteq (\neg X_R)^c$.

2. $\emptyset^R = \emptyset_R = \emptyset$, $U_R = U^R = U$.

3. $(X \cup Y)^R = X^R \cup Y^R$.

4. $(X \cap Y)_R = X_R \cap Y_R$.

5. $(X \cup Y)_R \supseteq X_R \cup Y_R$.

6. $(X \cap Y)^R = X^R \cap Y^R$.

7. $X^{R^R} = X^R$, $X_{R_R} = X_R$.

The following properties do not hold however:

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I $X \subseteq X^R$.

II $(X^c)_R = (X^R)^c$.

III $(X^c)^R = (X_R)^c$.

Where $\neg X$ is the negation of the set X, $(X)^c$ is the complement of the set X and $[x]$ is a category (an equivalence class of R) including the element x.

Proofs

1a By definition of X_R.

1b,1c Since U/R is not a complete description of U there may be elements of X which are not in U/R. Therefore it is not the case that $X \subseteq X^R$. If $x \in X$ then either $\exists [x] : [x] \cap X \neq \emptyset$ (therefore $x \in X^R$) or $x \in U \backslash U/R$.

In the first case $x \in X^R$ and $[x] \cap \neg X = \emptyset$ therefore it is not the case $[x] \subseteq \neg X$ and therefore $x \in (\neg X_R)^c$ (and this demonstrates also 1c).

In the second case if $x \in U \setminus U/R$ then by definition $x : x \notin \neg X_R$, therefore $x \in (\neg X_R)^c$ (and this completes the demonstration of 1b).

2 See Pawlak (1991).

3 $x \in (X \cup Y)^R$ iff $[x] \cap (X \cup Y) \neq \emptyset$ iff
$([x] \cap X) \cup ([x] \cap Y) \neq \emptyset$ iff $([x] \cap X) \neq \emptyset$ or $([x] \cap Y) \neq \emptyset$ iff
$x \in X^R$ or $x \in Y^R$ iff $x \in (X^R \cup Y^R)$.

4 $x \in (X \cap Y)_R$ iff $[x] \subseteq (X \cap Y)$ iff
$[x] \subseteq X$ and $[x] \subseteq Y$ iff $x \in X_R$ and $x \in Y_R$ iff $x \in (X_R \cap Y_R)$.

5 $[x] \subseteq X$ or $x \subseteq Y$. Then $[x] \subseteq X \cup Y$ and therefore $x \in (X \cup Y)_R$.

6 $x \in (X \cap Y)^R$ iff $[x] \cap (X \cap Y) \neq \emptyset$ iff
$[x] \cap X \neq \emptyset$ and $[x] \cap Y \neq \emptyset$ iff $x \in X^R$ and $x \in Y^R$.

∎

Given a structure M of similarity type ρ, let $S(x_1, \cdots x_m)$ be an atomic formula, such that $x_1 \cdots x_m$ are individual variables, $S \in \rho$ with arity m.

Let also $\mathcal{B} = (U, R)$ be a knowledge base in M such that $U \in M$ and R is a set of equivalence relations on U such that $\bigcup_i \{U/R_i\} \subseteq U$. Let also the DDT^2 be the language adopted (in this case strong entailment coincides with regular entailment).

Definition 4.1

$$\mathcal{B} \models \exists\, x \, \triangle S(x) \;\; \textit{iff } \exists\; Y \in U/R, \; x \in Y \; : \; Y \cap S \neq \emptyset$$

$$\mathcal{B} \models \exists\, x \, \triangle \neg S(x) \;\; \textit{iff } \exists\; Y \in U/R, \; x \in Y \; : \; Y \cap \neg S \neq \emptyset$$

$$\mathcal{B} \models \exists\, x \, \neg\triangle S(x) \;\; \textit{iff } \exists\; Y \in U/R, \; x \in Y \; : \; Y \cap S = \emptyset \textit{ or } x \in U \setminus U/R$$

where S and $\neg S$ are the extensions of the formula $S(x)$ and $\neg S(x)$, respectively.

It is easy to demonstrate the following corollaries.

Corollary 4.1

$$\mathcal{B} \models \exists\, x \, TS(x) \;\; \textit{iff } \exists\; Y \in U/R, \; x \in Y : Y \subseteq S$$

Proof
$\mathcal{B} \models \exists\, x \, TS(x)$ iff $\mathcal{B} \models \exists\, x \, \triangle S(x) \wedge \neg \triangle \neg S(x)$
Therefore $\exists Y \in U/R, \; x \in Y \; : \; Y \cap S \neq \emptyset$ **and** $(Y \in U/R \; : \; Y \cap \neg S = \emptyset$ **or** $x \in U \setminus U/R)$. There are two possibilities:
1. $\exists Y \in U/R, \; x \in Y \; : \; Y \cap S \neq \emptyset$ **and** $Y \in U/R : Y \cap \neg S = \emptyset$
which implies $Y \subseteq S$
2. $\exists Y \in U/R, \; x \in Y \; : \; Y \cap S \neq \emptyset$ **and** $x \in U \setminus U/R)$
But the second possibility is always false. Therefore $Y \subseteq S$.

■

Corollary 4.2

$$\mathcal{B} \models \exists\, x \, KS(x) \;\; \textit{iff } \exists\; Y \in U/R, \; x \in Y : Y \cap S \neq \emptyset \textit{ and } Y \cap \neg S \neq \emptyset$$

Proof
$\mathcal{B} \models \exists\, x \, KS(x)$ **iff** $\mathcal{B} \models \exists\, x \, \triangle S(x) \wedge \triangle \neg S(x)$
Therefore just apply the definition.

■

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Corollary 4.3

$$\mathcal{B} \models \exists\, x\; US(x) \;\; \textit{iff}\;\; U \setminus U/R \cap S \neq \emptyset$$

Proof

$\mathcal{B} \models \exists\, x\; US(x)$ iff $\mathcal{B} \models \exists\, x\; \neg\triangle S(x) \wedge \neg\triangle\neg S(x)$

Therefore $\exists Y \in U/R$, $x \in Y$: $(Y \cap S = \emptyset$ **or** $x \in U \setminus U/R)$ **and** $(Y \cap \neg S = \emptyset$ **or** $x \in U \setminus U/R)$

The only possible combination is that $x \in U \setminus U/R$.

Therefore $U \setminus U/R \cap S \neq \emptyset$.

■

Corollary 4.4

$$\mathcal{B} \models \exists\, x\; FS(x) \;\; \textit{iff}\;\; \exists\, Y \in U/R,\; x \in Y : Y \subseteq \neg S$$

Proof

$\mathcal{B} \models \exists\, x\; FS(x)$ iff $\mathcal{B} \models \exists\, x\; \triangle S(x) \wedge \neg\triangle\neg S(x)$

It is now sufficient to apply the same demonstration as for $TS(x)$ inverting S and $\neg S$.

Therefore $Y \subseteq \neg S$.

■

The following proposition follows.

Proposition 4.1 *Given a formula $S(x)$, a knowledge base $\mathcal{B} = (U, R)$, S^t, S^k, S^u, S^f denoting the extensions of $TS(x)$, $KS(x)$, $US(x)$, $FS(x)$, respectively. Then:*

- $S^t = S_R = S^R \cap (\neg S^R)^c$
- $S^k = S^R \cap \neg S^R$
- $S^u = (S^R)^c \cap (\neg S^R)^c$
- $S^f = \neg S_R = \neg S^R \cap (S^R)^c$

Proof

Immediate using the corrolaries previously demonstrated.

■

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Hence the "true" extension of the predicate $S(x)$ under the knowledge base $\mathcal{B} = (U, R)$ is its lower approximation using the description R and the "false" extension is the lower approximation of the negation of $S(x)$. The "contradictory" extension is the intersection among the upper approximations of the predicate and its negation and the "unknown" extension is the intersection of the complements of the two upper approximations (which coincides with the part of U which cannot be described by R).

Example 4.2 *Using the data of the previous example, it is easy to verify that:*
$$X^t = X_R = \{x_1, x_2\}$$
$$X^k = X^R \cap \neg X^R = \{x_3, x_4\}$$
$$X^u = (X^R)^c \cap (\neg X^R)^c = \{x_5, x_6\}$$
$$X^f = \neg X_R = \{x_7, x_8\}$$

We have therefore a one to one correspondence among the concepts introduced in the extended rough set theory (accepting the incompleteness axiom) and the concepts introduced by the DDT^2 language. Under this perspective the recent applications of rough sets in decision analysis (see Pawlak and Słowiński, 1994; Greco et al., 1995) can profit of the preference modeling theory based on the DDT language and vice versa (see Tsoukiàs and Vincke, 1995a,b).

5 Preference modeling applications

Preference modeling problems have been the original stimulus for the development of the DDT logic. Preference modeling is normally used in decision aid situations where uncertainty and/or ambiguity are very common. Moreover when the decision problem has different dimensions, a preference aggregation problem arises and then the problem of uncertainty is even stronger. Finally decision aid calls for more or less *immediate action*. In other words decision makers have to make a decision (whatever that means) into a precise time horizon, provided a specific amount of resources and information - knowledge. Therefore no one cares what may be the definitely optimal choices, while a locally satisfactory solution is searched (see Simon, 1979).

On the other hand, from a decision point of view, the distinction between uncertainty due to lack of information from uncertainty due to contradictory information is of a capital importance since it generates different operational attitudes. In the first case uncertainty may be reduced (if possible) gathering

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for more information (or for the relevant one), while in the second case some conflicts, inconsistencies or contradictions have to be solved. When decision makers are not sure it is always useful to know why.

The DDT language capture in a very clear and intuitive way this kind of reasoning. It has been therefore used as the basic formalism under which a non conventional theory about preference modeling could be developed (see also Kacprzyk and Roubens, 1988). In Tsoukiàs and Vincke (1995a) a new preference structure, named **PC**, is introduced and axiomatized, while in Tsoukiàs and Vincke (1995b) a semantical investigation, from a decision point of view, of the **PC** preference structure is conducted.

The basic ideas of this new theory are the following.

Conventional preference models use the well known $\langle P, I, R \rangle$ preference structure where (see also Roubens and Vincke, 1985):
- P is strict preference;
- I is indifference;
- R is incomparability

Between such crisp and sure situations some hesitation may occur due to two basic reasons: lack of relevant information when an element x is compared to an element y or vice versa and/or contradictory information when an element x is compared to an element y or vice versa. Therefore between each couple of relations two more relations can be introduced (one for each reason of uncertainty) and precisely:
- between P and I, the relations K and H;
- between P and R, the relations V and Q;
- between R and I, the relations U and J;
plus the relation L between R and I (while R, I, U and J are symmetric relations, L is not symmetric since it corresponds to lack of information when x is compared to y and contradictory information when y is compared to x; therefore it is completely uncertain, but for not symmetric reasons).

These ten relations constitute the **PC** preference structure which can be defined using a characteristic relation S (a reflexive large preference relation of the type *"at least as good as"*) and the DDT language (or the DDT^2 one). In Tsoukiàs and Vincke (1995a) is demonstrated that such a preference structure is a *maximal well-founded fundamental relational system of preferences* under the following three axioms.

A1 any preference structure on a set A should be a f.r.s.p. (fundamental relational system of preferences), that is should define a partition on $A \times A$ for any given A; in other words the preference relations in-

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cluded in the preference structure should be exhaustive for all possible situations and not redundant;

A2 the preference structure should follow the axiom of "independence from irrelevant alternatives"; in a more general version the evaluation, if a specific ordered couple belongs (and in which way) to a specific relation, should depend on information concerning only this ordered couple;

A3 the preference structure should be "well-founded" in the sense that any binary relation in it should be univoquely defined by its properties.

As a consequence some theoretical and operational problems in the field of multicriteria decision aid can find elegant and definite solutions (see Roy and Vincke, 1984; Papadopoulos, 1995; Tsoukiàs and Vincke, 1996).

6   Conclusions

A first order, four valued logic is presented in the paper as an extension of Belnap's logic. The logic is equipped with a weak negation (preserving interlaced monotonicity on the bilattice of truth values) and a strong monotonic implication. A two valued fragment, named DDT^2, is presented in the paper enabling to define strong two-valued sentences. A semantic is introduced based on the idea that the evaluation of the negative extension of a predicate is independent from the evaluation of the positive extension, that is the complement of a predicate does not coincide with the extension of its negation and that the universe of discourse may contain elements which do not belong to any of the two extensions. The resulting four possibilities correspond to the four truth values of the logic and define four possible extensions of any predicate. A double entailment relation is used in order to define such concepts and a strong entailment is introduced so as to have a correspondence with the evaluation function of the logic.

Moreover a rough sets semantic is introduced for the DDT^2 language (the two valued fragment of DDT). In such semantics an incompleteness axiom is introduced in the rough sets theory in the sense that the universe of discourse U of a knowledge base $\mathcal{B} = (U, R)$ is not always completely described by the relations R, therefore leaving space for unknown elements besides the contradictory ones. Under such extension of the rough sets theory a one to one correspondence of the two valued sentences of the DDT

language and the concepts of upper and lower approximation is possible. Finally the application of the DDT language in preference modeling and decision aid is outlined in the paper. The interested readers can refer to the quoted literature.

An open question remains the possibility to use the logic in order to perform non monotonic reasoning. The strong implication introduced in the paper is pure monotonic. Other weaker implications can be defined however, enabling eventually different levels of non-monotonicity using the paraconsistent property of the logic. This defines the main research direction in the future.

Acknowledgments

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