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EXPLOITATION OF A ROUGH APPROXIMATION OF THE OUTRANKING RELATION

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ABSTRACT

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Exploitation d'une approximation de la relation de surclassement

Résumé

Etant donné un ensemble A d'actions évaluées selon une famille de critères, nous considérons une information préférentielle en forme d'une table des comparaisons par paires (PCT) comprenant des actions d'un sous-ensemble $B \subseteq A \times A$ décrit par des relations de préférences graduées et par une relation globale de surclassement. En appliquant l'approche de la théorie des ensembles approximatifs à l'analyse de la PCT, nous obtenons une approximation de la relation de surclassement en termes d'une relation graduée de dominance. Des règles de décision générées à partir de cette approximation sont ensuite appliquées à un ensemble $M \subseteq A$ d'actions. En résultat, nous obtenons une relation de surclassement à quatre valeurs de vérité sur l'ensemble M , qui constitue un modèle de préférences sur cet ensemble. La définition d'une procédure adéquate à l'exploitation d'un tel modèle en vue d'obtention d'une recommandation est, dans ce contexte, un problème ouvert. Nous proposons une procédure d'exploitation pour des problèmes du choix et du rangement multicritère, et prouvons que c'est la seule procédure qui satisfait quelques propriétés désirables.

Exploitation of a rough approximation of the outranking relation

Abstract

Given a finite set A of actions evaluated by a family of criteria, we consider a preferential information in the form of a pairwise comparison table (PCT) including pairs of actions from a subset $B \subseteq A \times A$ described by graded preference relations on particular criteria and a comprehensive outranking relation. Using the rough set approach to the analysis of the PCT, we obtain a rough approximation of the outranking relation by a graded dominance relation. Decision rules derived from this approximation are then applied to a set $M \subseteq A$ of potential actions. As a result, we obtain a four-valued outranking relation on set M . The definition of a suitable exploitation procedure in order to obtain a recommendation within this context is an open problem. We propose an exploitation procedure for choice and ranking problems and prove that it is the only one which satisfies some desirable properties.

1. Introduction

A rough set approach to multicriteria decision analysis has been proposed by Greco, Matarazzo and Slowinski (1996). This methodology operates on a pairwise comparison table (PCT) (Greco, Matarazzo and Slowinski, 1995 and 1997), including pairs of actions described by graded preference relations on specific criteria and by a comprehensive preference relation. It builds up a rough approximation of the comprehensive preference relation using graded dominance relations. Furthermore, some decision rules in the “if ...then...” form are derived from the rough approximation of the preference relation. If the comprehensive preference relation is an outranking relation, the application of these decision rules to a set of actions gives a four-valued outranking relation (Tsoukias and Vincke, 1992, 1995), i.e., a binary relation which, with respect to any pair of actions (a,b), characterises the proposition “a is at least as good as b” as true, contradictory, unknown or false. Finally, in order to obtain a recommendation (Roy, 1993) for the decision problem at hand a suitable exploitation procedure of the four-valued outranking relation should be applied. This paper is focused on this exploitation procedure. More precisely, we consider multicriteria ranking and choice problems, and we propose an exploitation procedure, called scoring procedure, which we characterise by proving that it is the only one ensuring some desirable properties.

The paper is structured as follows. In section 2, we introduce the rough approximation of a preference relation and the generation of decision rules. In section 3, we describe the four-valued outranking relation. In section 4, we introduce the application of decision rules, showing how it defines a four-valued outranking relation. Furthermore, the scoring procedure is presented. Section 5 proposes a characterisation of this scoring procedure. Section 6 groups conclusions.

2. Rough set analysis of a preferential information

2.1 Pairwise Comparison Table

In order to represent preferential information provided by the decision maker (DM) in form of a pairwise comparison of some actions, we shall use a pairwise comparison table introduced in Greco, Matarazzo and Slowinski (1995 and 1997).

Let A be a finite set of actions (feasible or not), considered by the DM as a basis for exemplary pairwise comparisons. Let also C be the set of criteria (condition attributes) describing the actions.

For any criterion $q \in C$, let V_q be its domain and T_q a finite set of binary relations defined on V_q such that, $\forall (v'_q, v''_q) \in V_q \times V_q$, exactly one binary relation

$t \in T_q$ is verified. For interesting applications it should be $\text{card}(T_q) \geq 2$, $\forall q \in C$.

Furthermore, let T_d be a set of binary relations defined on set A (comprehensive pairwise comparisons) such that at most one binary relation $t \in T_d$ is verified, $\forall (x,y) \in A \times A$.

The *pairwise comparison table* (PCT) is defined as information table $S_{\text{PCT}} = \langle B, C \cup \{d\}, T_C \cup T_d, g \rangle$ where $B \subseteq A \times A$ is a non-empty *sample of pairwise comparisons*, $T_C = \bigcup_{q \in C} T_q$, d is a decision corresponding to the comprehensive pairwise comparison (comprehensive preference relation), and $g: B \times (C \cup \{d\}) \rightarrow T_C \cup T_d$ is a total function such that $g[(x,y), q] \in T_q \forall (x,y) \in A \times A$ and $\forall q \in C$, and $g[(x,y), d] \in T_d, \forall (x,y) \in B$. It follows that for any pair of actions $(x,y) \in B$ one and only one binary relation $t \in T_d$ is verified. Thus, T_d induces a partition of B . In fact, information table S_{PCT} can be seen as decision table, since the set of considered criteria C and decision d are distinguished.

In this paper, we consider S_{PCT} related to the choice and ranking problems (Roy, 1985) and assume that the exemplary pairwise comparisons provided by the DM can be presented in terms of *binary graded preference relations* (for a substantially equivalent representation see Moreno and Tsoukias (1996)):

$$T_q = \{ p_q^h, h \in H_q \}$$

where $H_q = \{ h \in \mathbb{Z} : h \in [-p_q, r_q] \}$ and $p_q, r_q \in \mathbb{Z}^+, \forall q \in C$ and $\forall (x,y) \in A \times A$

- $x p_q^h y, h > 0$, means that action x is preferred to action y by degree h with respect to criterion q ,

- $x p_q^h y, h < 0$, means that action x is not preferred to action y by degree h with respect to criterion q ,

- $x p_q^0 y$ means that x is similar (asymmetrically indifferent) to y with respect to criterion q .

Of course, $x p_q^0 x \forall x \in A$ and $\forall q \in C$, i.e. p_q^0 is reflexive, and

$$[x p_q^h y, h \geq 0] \Leftrightarrow [y p_q^k x, k \leq 0].$$

Therefore, $\forall (x,y), (w,z) \in A \times A$ and $\forall q \in C$:

- if $x p_q^h y$ and $w p_q^k z, k \geq h \geq 0$, then w is preferred to z not less than x is preferred to y with respect to criterion q ;

- if $x p_q^h y$ and $w p_q^k z, k \leq h \leq 0$, then w is not preferred to z not less than x is not preferred to y with respect to criterion q .

The set of binary relations T_d is defined analogously; however, $xP_d^h y$ means that x is comprehensively preferred to y by degree h .

Since $q \in C$ is a criterion, i.e. there exists a function $c_q: A \rightarrow \mathbb{R}$ such that, $\forall x, y \in A$, $c_q(x) \geq c_q(y)$ means „ x is at least as good as y with respect to q ” (Roy, 1985), then, in order to define the set of graded preference relations T_q , one can use a function $k_q: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying the following properties $\forall x, y, z \in A$:

$$c_q(x) > c_q(y) \Leftrightarrow k_q[c_q(x), c_q(z)] > k_q[c_q(y), c_q(z)],$$

$$c_q(x) > c_q(y) \Leftrightarrow k_q[c_q(z), c_q(x)] < k_q[c_q(z), c_q(y)],$$

$$c_q(x) = c_q(y) \Leftrightarrow k_q[c_q(x), c_q(y)] = 0.$$

The function $k_q[c_q(x), c_q(y)]$ measures the strength of positive (when $c_q(x) > c_q(y)$) or negative (when $c_q(x) < c_q(y)$) preference of x over y with respect to q . Typical representatives of k_q are

$$k_q[c_q(x), c_q(y)] = c_q(x) - c_q(y)$$

and, if $c_q(z) > 0$, $\forall z \in A$,

$$k_q[c_q(x), c_q(y)] = \frac{c_q(x)}{c_q(y)} - 1.$$

The strength of preference represented by k_q is then transformed into a specific binary relation P_q^h using a set of thresholds

$$\Delta_q = \{ \Delta_q^j, j = -p_q - 1, -p_q, \dots, -1, 1, \dots, r_q, r_q + 1: \Delta_q^j < 0 \text{ if } j < 0, \Delta_q^j > 0 \text{ if } j > 0, \Delta_q^j > \Delta_q^{j-1} \forall j > -p_q - 1 \}$$

where

$$\Delta_q^{-p_q-1} = \min_{(x,y) \in A \times A} \{ k_q[c_q(x), c_q(y)] \}$$

and

$$\Delta_q^{r_q+1} = \max_{(x,y) \in A \times A} \{ k_q[c_q(x), c_q(y)] \}.$$

On the basis of the thresholds of the set Δ_q , a set of intervals I_q is obtained:

$$I_q = \{ [\Delta_q^{-p_q-1}, \Delta_q^{-p_q}), (\Delta_q^{-p_q}, \Delta_q^{-p_q+1}), \dots, (\Delta_q^{r_q-1}, \Delta_q^{r_q}), (\Delta_q^{r_q}, \Delta_q^{r_q+1}] \}$$

where “(” and “)” can mean “[” or “]” according to the constraint that if an interval is open (closed) on the right, the next interval (if it exists) is closed (open) on the left.

Then we can state :

$$k_q[c_q(x), c_q(y)] \in (\Delta_q^h, \Delta_q^{h+1}) \Leftrightarrow x P_q^h y \text{ for } h \in H_q \text{ and } h > 0,$$

$$k_q[c_q(x), c_q(y)] \in (\Delta_q^{h-1}, \Delta_q^h) \Leftrightarrow x P_q^h y \text{ for } h \in H_q \text{ and } h < 0$$

and

$$k_q[c_q(x), c_q(y)] \in (\Delta_q^{-1}, \Delta_q^{+1}) \Leftrightarrow x P_q^0 y.$$

The above definition of graded preference relations is not the only one which could be considered for the S_{PCT} . Any system of binary preference relations can be used to express the pairwise comparisons (see Roubens and Vincke, 1985).

2.2 Rough approximation of a preference relation

Let $H_P = \bigcap_{q \in P} H_q$ $\forall P \subseteq C$. Given $x, y \in A$, $P \subseteq C$ and $h \in H_P$, we say that x *positively dominates* y by degree h with respect to the set of criteria P iff $x P_q^f y$ with $f \geq h$, $\forall q \in P$. Analogously, $\forall x, y \in A$, $P \subseteq C$ and $h \in H_P$, x *negatively dominates* y by degree h with respect to the set of criteria P iff $x P_q^f y$ with $f \leq h$ $\forall q \in P$.

Thus, $\forall h \in H_P$, every $P \subseteq C$ generates two binary relations (possibly empty) on A , which will be called *P-positive-dominance of degree h* , denoted by D_{+P}^h , and *P-negative-dominance of degree h* , denoted by D_{-P}^h , respectively. The relations D_{+P}^h and D_{-P}^h satisfy the following properties:

(P1) if $(x, y) \in D_{+P}^h$, then $(x, y) \in D_{+R}^k$, for all $R \subseteq P$ and for every $k \leq h$;

(P2) if $(x, y) \in D_{-P}^h$, then $(x, y) \in D_{-R}^k$, for all $R \subseteq P$ and for every $k \geq h$.

In the following, we consider a PCT where the decision d can have only two values on $B \subseteq A \times A$:

1) x outranks y , which will be denoted by $x S y$ or $(x, y) \in S$,

2) x does not outrank y , which will be denoted by $x S^c y$ or $(x, y) \in S^c$,

where “ x outranks y ” means “ x is at least as good as y ” (Roy, 1985). Let us remember that the minimal property verified by the outranking relation S is reflexivity (see Roy, 1991; Bouyssou, 1996).

We propose to approximate the binary relation S by means of the D_{+p}^h binary dominance relations. Therefore, S is seen as a *rough binary relation* (see. Greco, Matarazzo and Slowinski, 1995 and 1997).

The P -lower approximation of S , denoted by $\underline{P}S$, and the P -upper approximation of S , denoted by $\overline{P}S$, are defined, respectively, as:

$$\underline{P}S = \bigcup_{h \in H_p} \{(D_{+p}^h \cap B) \subseteq S\},$$

$$\overline{P}S = \bigcap_{h \in H_p} \{(D_{+p}^h \cap B) \supseteq S\}.$$

Taking into account property (P1) of the dominance relations D_{+p}^h , $\underline{P}S$ can be viewed as the dominance relation D_{+p}^h which has the largest intersection with B included in the outranking relation S , and $\overline{P}S$ as the dominance relation D_{+p}^h including S which has the smallest intersection with B .

Analogously, we can approximate S^c by means of the D_{+p}^h binary dominance relations:

$$\underline{P}S^c = \bigcup_{h \in H_p} \{(D_{+p}^h \cap B) \subseteq S^c\},$$

$$\overline{P}S^c = \bigcap_{h \in H_p} \{(D_{+p}^h \cap B) \supseteq S^c\}.$$

The interpretation of $\underline{P}S^c$ and $\overline{P}S^c$ is similar to the interpretation of $\underline{P}S$ and $\overline{P}S$. Taking into account property (P2) of the dominance relations D_{+p}^h , $\underline{P}S^c$ can be viewed as the dominance relation D_{+p}^h which has the largest intersection with B included in the negation of S , and $\overline{P}S^c$ as the dominance relation D_{+p}^h including the negation of S which has the smallest intersection with B .

2.3 Decision rules

We can derive a generalised description of the preferential information contained in a given PCT in terms of decision rules.

We will consider the following kinds of decision rules:

- 1) D_{++} -decision rule, being a statement of the type: $x D_{+p}^h y \Rightarrow x S y$;
- 2) D_{+-} -decision rule, being a statement of the type: *not* $x D_{+p}^h y \Rightarrow x S^c y$;

3) D_{+} -decision rule, being a statement of the type: $not\ x\ D_{+P}^h\ y \Rightarrow xSy$;

4) D_{-} -decision rule, being a statement of the type: $x\ D_{+P}^h\ y \Rightarrow xS^c y$.

Speaking about decision rules we will simply understand all the four kinds of decision rules together.

If: (P3) [(P5)] there is at least one pair $(w,z) \in B$ such that $wD_{+P}^h z$ [$wD_{+P}^h z$] and wSz [$wS^c z$], and

(P4) [(P6)] there is no $(v,u) \in B$ such that $vD_{+P}^h u$ [$vD_{+P}^h u$] and $vS^c u$ [vSu]

then $x\ D_{+P}^h\ y \Rightarrow xSy$ [$x\ D_{+P}^h\ y \Rightarrow xS^c y$] is accepted as a D_{++} -decision rule [D_{-} -decision rule].

Analogously, if:

(P7) [(P9)] there is at least one pair $(w,z) \in B$ such that $not\ wD_{+P}^h z$ [$not\ wD_{+P}^h z$] and $wS^c z$ [wSz], and

(P8) [(P10)] there is no $(v,u) \in B$ such that $not\ vD_{+P}^h u$ [$not\ vD_{+P}^h u$] and vSu [$vS^c u$]

then $not\ x\ D_{+P}^h\ y \Rightarrow xS^c y$ [$not\ x\ D_{+P}^h\ y \Rightarrow xSy$] is accepted as a D_{+-} -decision rule [D_{+} -decision rule].

A D_{++} -decision rule [D_{+} -decision rule] $x\ D_{+P}^h\ y \Rightarrow xSy$ [$not\ x\ D_{+P}^h\ y \Rightarrow xS^c y$] will be called *minimal* if there is not any other rule $x\ D_{+R}^k\ y \Rightarrow xSy$ [$not\ x\ D_{+R}^k\ y \Rightarrow xS^c y$] such that $R \subseteq P$ and $k \leq h$ [$k \geq h$]. A D_{+-} -decision rule [D_{-} -decision rule] $not\ x\ D_{+P}^h\ y \Rightarrow xS^c y$ [$x\ D_{+P}^h\ y \Rightarrow xS^c y$] will be called *minimal* if there is not any other rule $not\ x\ D_{+R}^k\ y \Rightarrow xS^c y$ [$x\ D_{+R}^k\ y \Rightarrow xS^c y$] such that $R \subseteq P$ and $k \leq h$ [$k \geq h$].

Let us observe that, since each decision rule is an implication, the minimal decision rules represent the implications such that there is no other implication with an antecedent at least of the same weakness and a consequent of at least the same strength.

Theorem 2.1. (Greco, Matarazzo, Slowinski, 1996). If

- 1) $x\ D_{+P}^h\ y \Rightarrow xSy$ is a minimal D_{++} -decision rule, then $\underline{PS} = D_{+P}^h \cap B$,
- 2) $x\ D_{+P}^h\ y \Rightarrow xS^c y$ is a minimal D_{-} -decision rule, then $\underline{PS}^c = D_{+P}^h \cap B$,
- 3) $not\ x\ D_{+P}^h\ y \Rightarrow xS^c y$ is a minimal D_{+-} -decision rule, then $\overline{PS} = D_{+P}^h \cap B$,
- 4) $not\ x\ D_{+P}^h\ y \Rightarrow xSy$ is a minimal D_{+} -decision rule, then $\overline{PS}^c = D_{+P}^h \cap B$.

3. Four-valued outranking

The basic idea of the four-valued outranking model of preferences (Tsoukias and Vincke, 1995, 1997) is connected with the search of “positive reasons” and “negative reasons” supporting a hypothesis of the truth of a comprehensive outranking relation for an ordered pair (x,y) of actions. The combination of presence and absence of the positive and the negative reasons creates four possible situations for the outranking:

- 1) *true outranking*, denoted by $xS^T y$, if there exist sufficient positive reasons to establish xSy and there are not sufficient negative reasons to establish $xS^c y$;
- 2) *contradictory outranking*, denoted by $xS^K y$, if there exist sufficient positive reasons to establish xSy and sufficient negative reasons to establish $xS^c y$;
- 3) *unknown outranking*, denoted by $xS^U y$, if there do not exist sufficient positive reasons to establish xSy and there are not sufficient negative reasons to establish $xS^c y$;
- 4) *false outranking*, denoted by $xS^F y$, if there do not exist sufficient positive reasons to establish xSy and there exist sufficient negative reasons to establish $xS^c y$.

Table 1 summarises the four outranking relations.

Table 1. Four outranking relations

(x,y)	S^T	S^K	S^U	S^F
xSy	1	1	0	0
$xS^c y$	0	1	0	1

By combining the four types of outranking binary relations for ordered pairs (x,y) and (y,x) of actions, we get the following ten preference situations for comparison of x and y :

- 1) *strict preference*, denoted by xPy , if x is strictly better than y , i.e., if $xS^T y$ and $yS^F x$;
- 2) *weak preference*, denoted by xHy , if x could be better than y , but we are not sure because of some evidence against it, i.e., if $xS^T y$ and $yS^K x$;
- 3) *semi-preference*, denoted by xJy , if x could be better than y , but we are not sure due to the lack of all the necessary information, i.e., $xS^T y$ and $yS^U x$;
- 4) *semi-weak preference*, denoted by xLy , if x is possibly better than y , but we have both contradictory information and lack of all the necessary information, i.e., $xS^K y$ and $yS^U x$;

- 5) *indifference*, denoted by xIy , if x and y are strictly equivalent, i.e., $xS^T y$ and $yS^T x$;
- 6) *ambiguity*, denoted by xKy , if x and y could be indifferent, but there exist contradictions in both directions, i.e., $xS^K y$ and $yS^K x$;
- 7) *ignorance*, denoted by xUy , if we cannot establish what holds between x and y , i.e., $xS^U y$ and $yS^U x$;
- 8) *incomparability*, denoted by xRy , if x and y are in strong opposition, i.e., $xS^F y$ and $yS^F x$;
- 9) *weak incomparability*, denoted by xQy , if x could be incomparable to y , but there is some contradictory information, i.e., $xS^K y$ and $yS^F x$;
- 10) *semi incomparability*, denoted by xVy , if x could be in opposition with y , but we are not sure due to the lack of all the necessary information, i.e., $xS^U y$ and $yS^F x$.

The above binary relations can be gathered in a symmetric preference modelling matrix (Table 2).

Table 2. The ten preference situations

$(x,y) \backslash (y,x)$	$yS^T x$	$yS^K x$	$yS^U x$	$yS^F x$
$xS^T y$	xIy	xHy	xJy	xPy
$xS^K y$	yHx	xKy	xLy	xQy
$xS^U y$	yJx	yLx	xUy	xVy
$xS^F y$	yPx	yQx	yVx	xRy

Note that in the classical outranking approach only two relations (S^T and S^F) are used, directly defined with respect to the pair (x,y) and its symmetric counterpart (y,x) . Thus only four relations are obtained: preference (xPy , yPx), indifference (xIy) and incomparability (xRy), displayed in the four corners of the preference matrix, shown in Table 2.

In the main diagonal of the preference matrix, four symmetric relations are grouped: the above mentioned indifference I ($xS^T y$ and $yS^T x$) and incomparability R ($xS^F y$ and $xS^F y$) and the two new relations of ambiguity K ($xS^K y$ and $yS^K x$) and ignorance U ($xS^U y$ and $yS^U x$).

The two hesitations between preference and indifference are all named from preference, and the two hesitations between preference and incomparability are all named from incomparability. All these relations could be considered having a common degree of preference, between that of strict preference relation and that of symmetric relations. Moreover we use “semi” only for hesitations due to unknown states, and “weak” only for hesitations due to contradictory states.

Thus, other five different (strictly, semi, weakly) asymmetric relations and another one (semi-weakly) symmetric relation are built up (see Table 2).

This way of preference modelling allows us to consider three different levels of preference, instead of only two states obtained using the traditional outranking approach (P,I,R) or the classical model (P,I).

By such definitions it is possible to apply the rough approximations of outranking relations S and S^c defined on B , in order to build a preference model on $M \times M$, where $M \subseteq A$, which could further be exploited to get a recommendation with respect to a set of actions from M (choice or ranking). In other words, we are able to move from a descriptive model of decision maker's preferences expressed on B to a prescriptive model on $M \subseteq A$.

4. Application of decision rules and definition of a final recommendation

Given a set D of decision rules, obtained in the way described in section 2, and two actions $v, u \in A$, if

- 1) $x D_{++}^h y \Rightarrow x S y$ is a D_{++} -decision rule and $v D_{++}^h u$, we conclude that $v S u$,
- 2) $not\ x D_{++}^h y \Rightarrow x S^c y$ is a D_{++} -decision rule and $not\ v D_{++}^h u$, we conclude that $v S^c u$,
- 3) $not\ x D_{+-}^h y \Rightarrow x S y$ is a D_{+-} -decision rule and $not\ v D_{+-}^h u$, we conclude that $v S u$,
- 4) $x D_{+-}^h y \Rightarrow x S^c y$ is a D_{+-} -decision rule and $v D_{+-}^h u$, we conclude that $v S^c u$.

According to the four-valued logic, from the application of the decision rules to the pair of actions $(x, y) \in A \times A$ there may arise one of the four following states:

- *true outranking*, denoted by $x S^T y$: this is the case when there exists at least one D_{++} -decision rule and/or at least one D_{+-} -decision rule stating that $x S y$, and no D_{--} -decision rule or D_{+-} -decision rule stating that $x S^c y$;
- *false outranking*, denoted by $x S^F y$: this is the case when there exists at least one D_{--} -decision rule and/or at least one D_{+-} -decision rule stating that $x S^c y$, and no D_{++} -decision rule or D_{+-} -decision rule stating that $x S y$;
- *contradictory outranking*, denoted by $x S^K y$: this is the case when there exists at least one D_{++} -decision rule and/or at least one D_{+-} -decision rule stating that

xSy , and at least one D_{--} -decision rule and/or at least one D_{+-} -decision rule stating that $xS^c y$;

- *unknown outranking*, denoted by $x S^u y$: this is the case when there is no D_{++} -decision rule or D_{+-} -decision rule stating that xSy , and no D_{--} -decision rule or D_{+-} -decision rule stating that $xS^c y$.

Theorem 4.1. (Greco, Matarazzo, Slowinski, 1996) The application of all the decision rules obtained for a given S_{PCT} on any pair of actions $(v,u) \in A \times A$ results in the same outranking relation as obtained by the application of the minimal decision rules only.

From Theorem 4.1, we conclude that the set of all decision rules is completely characterised by the set of the minimal rules. Therefore, only the latter ones are presented to the DM and applied in the decision problem at hand.

In order to define a recommendation with respect to the actions of $M \subseteq A$, we can calculate a particular score based on the outranking relations S and S^c obtained from the application of these rules to the actions of M .

$\forall M \subseteq A$ and $\forall x \in M$, let

- $M(x)^{++} = \{y \in M - \{x\} : \text{there is at least one } D_{++} \text{-decision rule and/or at least one } D_{+-} \text{-decision rule stating that } xSy\}$,
- $M(x)^+ = \{y \in M - \{x\} : \text{there is at least one } D_{++} \text{-decision rule and/or at least one } D_{+-} \text{-decision rule stating that } ySx\}$,
- $M(x)^{-+} = \{y \in M - \{x\} : \text{there is at least one } D_{+-} \text{-decision rule and/or at least one } D_{--} \text{-decision rule stating that } yS^c x\}$,
- $M(x)^- = \{y \in M - \{x\} : \text{there is at least one } D_{+-} \text{-decision rule and/or at least one } D_{--} \text{-decision rule stating that } xS^c y\}$.

To each $x \in M$ we assign a *score*

$$S(x, M) = S^{++}(x, M) - S^+(x, M) + S^{-+}(x, M) - S^-(x, M)$$

where $S^{++}(x, M) = \text{card}[M^{++}(x)]$, $S^+(x, M) = \text{card}[M^+(x)]$, $S^{-+}(x, M) = \text{card}[M^{-+}(x)]$, $S^-(x, M) = \text{card}[M^-(x)]$.

We can use this score to work out a recommendation in the ranking and choice problems. For the ranking problem, $S(x, M)$ establishes a total preorder on M . For choice problems, the final recommendation is $x^* \in M$ such that $S(x^*, M) = \max_{x \in M} S(x, M)$. We call these exploitation procedures *scoring procedures*.

With respect to the conventional scoring procedure, where the balance between arcs leaving an action a (i.e. all arcs between a and y such that aSy) and arcs entering a (i.e. all arcs between a and y such that ySa) is performed, our exploiting procedure enables a higher granularity of the result. Since

negative outranking is *explicitly* represented (not as a complement of positive outranking), the scoring computes the balance between *positive and negative reasons*. Unknown situations are not considered (they carry no information) and contradictory situations are equilibrated (they carry information in both directions). This is compatible with the semantics of the four-valued logic underlying our approach, where the concept of negation is kept separated from the one of complement.

The difference between the conventional scoring procedure and our scoring procedure can also be captured in the following way. In conventional scoring procedure, value 1 is assigned to each arc present (xSy holds) and 0 to each arc absent (xSy does not hold). Given any two actions a and b , the balance in favour of a will give either 1 (strict or weak preference: asymmetric part of the outranking relation in favour of a), or 0 (indifference or incomparability: symmetric part of the outranking relation) or -1 (inverse strict or weak preference: asymmetric part of the outranking relation in favour of b). In our case, given $x \in A$, we may assign instead:

- value 1 to a “positive arc in favour of x ”, i.e., xSy , and to a “negative arc in favour of x ”, i.e. $yS^c x$,
- value -1 to a “positive arc against x ”, i.e., ySx and to a “negative arc against x ”, i.e. $xS^c y$,
- value 0 to the absence of arcs.

With respect to the four outranking relations we have:

- the true outranking gives a value of 1: in fact, $xS^T y$ means that there is a positive arc in favour of x (xSy , i.e., a value of 1) and that there is not a negative arc against x (not $xS^c y$, i.e., a value of 0),
- the contradictory outranking gives a value of 0: in fact, $xS^K y$ means that there is a positive arc in favour of x (xSy , i.e., a value of 1) and that there is a negative arc against x ($xS^c y$, i.e., a value of -1),
- the unknown outranking gives a value of 0: in fact, $xS^U y$ means that there is not a positive arc in favour of x (not xSy , i.e., a value of 0) and that there is not a negative arc against x (not $xS^c y$, i.e., a value of 0),
- the false outranking gives a value of -1: in fact, $xS^F y$ means that there is not a positive arc in favour of x (not xSy , i.e., a value of 0) and that there is a negative arc against x (not $xS^c y$, i.e., a value of -1).

The final result is that $\forall x, y \in A$ the balance in favour of x will give five possibilities: 2 (strict preference: case 1 of section 3; strong asymmetric part of the four-valued outranking in favour of a), 1 (doubtful preference: cases 2, 3, 9, 10 of section 3; weakly asymmetric part of the four-valued outranking in favour of a), 0 (indifference and/or incomparability: cases 4,5,6,7 and 8 of section 3; symmetric part of the four-valued outranking), -1 (inverse doubtful preference; weakly asymmetric part of the four-valued outranking in favour of b), -2

(inverse strict preference; strong asymmetric part of the four-valued outranking in favour of b). The following Table 3 shows these five possibilities.

Table 3. The value of the score in favour of x from the ten preference situations

$(x,y)\backslash(y,x)$	$yS^T x$	1	$yS^K x$	0	$yS^U x$	0	$yS^P x$	-1
$xS^T y$ 1	xIy	0	xHy	1	xJy	1	xPy	2
$xS^K y$ 0	yHx	-1	xKy	0	xLy	0	xQy	1
$xS^U y$ 0	yJx	-1	yLx	0	xUy	0	xVy	1
$xS^P y$ -1	yPx	-2	yQx	-1	yVx	-1	xRy	0

5. A characterisation of the scoring procedure

The use of a score-based procedure in presence of a four-valued outranking relation is a problem which goes beyond the exploitation of rough approximations (see Tsoukiàs and Vincke, 1997). For this reason we start with some general remarks concerning the use of such procedures.

We want also to stress that such procedures are not the only possibility when four-valued outranking relations have to be exploited. Moreover, the reader may notice that the use of the score, as defined in this paper, conceals the difference between uncertainty due to contradictions and uncertainty due to lack of information, since it gives a single value (1 or -1) to any situation of hesitation. However, in our opinion, any exploitation procedure results in a loss of information since it reduces the rich form of knowledge contained in the outranking relation to a poorer one which is the final choice or ranking. In favour of the scoring procedure play its intuitive nature (it is easy to understand by decision makers), its clear and straightforward characterization (as it will be demonstrated in the following) and its easiness in implementation. In other words, we sacrifice some richness of the information to the operability of the final result.

Finally, we wish to point out that the scoring procedure is consistent with the interpretation given in section 3 concerning the three different levels of preference allowed by the four-valued outranking relation.

5.1 Some previous results

The scoring procedure proposed in the previous section can be considered as an extension to the four-valued logic of the well-known Copeland ranking and choice method (see Goodman, 1954 ; Fishburn, 1973).

These procedures have been characterised by Rubinstein (1980) and Henriot (1985) and, with respect to valued binary relations, by Bouyssou (1992a and b).

The ranking procedure was also used in the Multiple Criteria Decision Making method PROMETHEE II (Brans and Vincke, 1985).

In this subsection we remember synthetically the results of Bouyssou, while in the following subsection we extend them to the four-valued outranking relation.

A valued (binary) outranking relation on A is a function R associating an element of $[0,1]$ with each ordered pair of actions $(a,b) \in A \times A$ with $a \neq b$. Let $R(A)$ be the set of all valued relations on A and 2^A the set of all nonempty subsets on A . A *ranking method* (RM), denoted by $\geq$, is a function assigning a ranking $\geq(M,R)$ on $M \subseteq A$ to any valued relation $R \in R(A)$ and to any $M \subseteq A$. A *choice function* (CF) on A is a function

$$C: 2^A \times R(A) \rightarrow 2^A$$

such that $C(M,R) \subseteq M$, for each $M \in 2^A$ and $R \in R(A)$.

The following properties of ranking and choice exploitation procedure are considered (Bouyssou, 1992a and b):

1) *strong monotonicity*: an exploitation procedure is strongly monotonic if it responds in the right direction to a modification of R . More formally,

1a) $RM \geq$ is strongly monotonic if $\forall a,b \in M \subseteq A$ and $\forall R \in R(A)$

$$a \geq(M,R)b \Rightarrow a >(M,R')b,$$

where $>(M,R)$ is the asymmetric part of $\geq(M,R)$ and R' is identical to R except that $R(a,c) < R'(a,c)$ or $R(c,a) > R'(c,a)$ for some $c \in M - \{a\}$;

1b) a CF C is strongly monotonic if $\forall R \in R(A)$ and all $M \in 2^A$

$$a \in C(M,R) \Rightarrow \{a\} = C(M,R')$$

where R' is defined as previously.

2) *neutrality*: an exploitation procedure is neutral if it does not discriminate between actions just because of their labels. More formally,

2a) a $RM \geq$ is neutral if for all permutations σ on A , $\forall R \in R(A)$ and $\forall a,b \in M \subseteq A$

$$a \geq(M,R)b \Leftrightarrow \sigma(a) \geq(\sigma(M),R^\sigma)\sigma(b)$$

where R^σ is defined by $R^\sigma(\sigma(a), \sigma(b)) = R(a,b) \quad \forall a,b \in A$;

2b) a CF C is neutral if for all permutations σ on A , $\forall R \in R(A)$ and $\forall M \in 2^A$

$$a \in C(M,R) \Leftrightarrow \sigma(a) \in C(\sigma(M),R^\sigma).$$

3) *independence of circuits*: a circuit of length q in a digraph is an ordered collection of arcs $(u_1, u_2, \dots, u_q)$ such that for $i=1, 2, \dots, q$, the initial extremity of u_i is the final extremity of u_{i-1} and the final extremity of u_i is the initial extremity of u_{i+1} , where u_0 is interpreted as u_q and u_{q+1} as u_1 . A circuit is

elementary if and only if each node being the extremity of one arc in the circuit is the extremity of exactly two arcs in the circuit. A transformation on an elementary circuit consists of adding the same quantity to the value of the arcs in the circuit. A transformation on an elementary circuit is admissible if all the transformed valuations are still between 0 and 1. An exploitation procedure is independent of circuits if its results do not change after an admissible transformation of R . More formally,

3a) a $RM \geq$ is independent of circuits if $R, R' \in R(A)$, R' is obtained from R through an admissible transformation on an elementary circuit of length 2 or 3 and $\forall a, b \in M \subseteq A$

$$a \geq (M, R) b \Rightarrow a \geq (M, R') b;$$

3b) a CF C is independent of circuits if $\forall M \in 2^A$ and $\forall R, R' \in R(A)$, such that R' is obtained from R through an admissible transformation on an elementary circuit of length 2 or 3 on M ,

$$C(M, R) = C(M, R').$$

The property of independence of circuits makes an explicit use of the cardinal properties of the valuations $R(a, b)$. This is not the case of the neutrality and monotonicity (Bouyssou, 1992a and b).

Given $R \in R(A)$ and $M \subseteq A$, a net flow $S_{NF}(x, M, R)$ can be associated to each $x \in M$ as follows:

$$S_{NF}(x, M, R) = \sum_{b \in M - \{x\}} (R(x, b) - R(b, x)).$$

More specifically, the $RM \geq$ such that

$$a \geq (M, R) b \text{ iff } S_{NF}(a, M, R) \geq S_{NF}(b, M, R)$$

is called *net flow ranking method*, and the CF C such that

$$C(M, R) = \{a \in M: S_{NF}(a, M, R) \geq S_{NF}(b, M, R) \forall b \in M\}.$$

is called *net flow choice method*.

Theorem 5.1. (Bouyssou 1992a). The net flow method is the only RM that is neutral, strongly monotonic and independent of circuits.

Theorem 5.2. (Bouyssou 1992b). The net flow method is the only CF that is neutral, strongly monotonic and independent of circuits.

5.2 Properties of the exploitation procedures for the four-valued outranking

In order to characterise the scoring procedure we consider a four-valued outranking relation as a function R_{4v} associating an element of $\{S^T, S^U, S^I, S^D\}$,

$S^K, S^F\}$ with each ordered pair of actions $(a,b) \in A$. Now, $RM \geq$ and CF C are defined analogously for a four-valued outranking relation, i.e., for $R_{4v}(A)$ being the set of all possible four-valued relations on A , $RM \geq$ is a function assigning a ranking $\geq(M, R_{4v})$ on $M \subseteq A$ to any $R_{4v} \in R_{4v}(A)$ and to any $M \subseteq A$, and CF C on A is a function

$$C: 2^A \times R_{4v}(A) \rightarrow 2^A$$

such that $C(M, R_{4v}) \subseteq M$, for each $M \in 2^A$ and each $R_{4v} \in R_{4v}(A)$.

Moreover, the property of neutrality maintains the same formulation as in the exploitation procedure for the valued outranking relation, i.e.

- a $RM \geq$ is neutral if for all permutations σ on A , $\forall M \subseteq A$, $\forall R_{4v} \in R_{4v}(A)$ and $\forall a, b \in M$

$$a \geq (M, R_{4v}) b \Leftrightarrow \sigma(a) \geq (\sigma(M), R_{4v}^\sigma) \sigma(b)$$

- a CF C is neutral if for all permutations σ on A , $\forall M \in 2^A$ and $\forall R_{4v} \in R_{4v}(A)$

$$a \in C(M, R_{4v}) \Leftrightarrow \sigma(a) \in C(\sigma(M), R_{4v}^\sigma)$$

where for any permutation σ and $\forall a, b \in A$, R_{4v}^σ is defined by

$$R_{4v}^\sigma(\sigma(a), \sigma(b)) = R_{4v}(a, b).$$

Instead, the strong monotonicity and the independence of circuits properties have a formal definition which is slightly different from the previous definition and requires some new concepts.

A *4v-transformation* on the pair $(a,b) \in A \times A$ consists of changing the outranking relation S^X into the outranking relation S^Y , where $S^X, S^Y \in \{S^T, S^U, S^K, S^F\}$, and it is denoted by

$$aS^Xb \rightarrow aS^Yb.$$

Let us denote by $S^X \rightarrow S^Y$ the class of all the transformations $aS^Xb \rightarrow aS^Yb$ with $(a,b) \in A \times A$ and $S^X, S^Y \in \{S^T, S^U, S^K, S^F\}$.

Let T be the set of all 4v-transformations on the pairs $(a,b) \in A \times A$. We introduce an equivalence binary relation E on T . More specifically,

$$[aS^Xb \rightarrow aS^Yb] E [aS^Wb \rightarrow aS^Zb]$$

means that the transformation $[aS^Xb \rightarrow aS^Yb]$ has the same “strength” as the transformation $[aS^Wb \rightarrow aS^Zb]$, where $S^X, S^Y, S^W, S^Z \in \{S^T, S^U, S^K, S^F\}$.

We define the following *equivalence classes* for E :

- 1) $E^0 = (S^T \rightarrow S^T) \cup (S^F \rightarrow S^F) \cup (S^U \rightarrow S^U) \cup (S^K \rightarrow S^K) \cup (S^K \rightarrow S^U) \cup (S^U \rightarrow S^K)$,
i.e. the class of the transformations from an outranking S^X to an outranking S^Y of the same strength;

- 2) $E^1 = (S^U \rightarrow S^T) \cup (S^K \rightarrow S^T) \cup (S^F \rightarrow S^U) \cup (S^F \rightarrow S^K)$, i.e. the class of the transformations from an outranking S^X to an outranking S^Y having a greater strength;
- 3) $E^1 = (S^T \rightarrow S^U) \cup (S^T \rightarrow S^K) \cup (S^U \rightarrow S^F) \cup (S^K \rightarrow S^F)$, i.e. the class of the transformations from an outranking S^X to an outranking S^Y having a weaker strength;
- 4) $E^2 = (S^F \rightarrow S^T)$, i.e. the class of the transformation from an outranking S^X to an outranking S^Y having a far greater strength (from total absence of outranking to sure presence of outranking);
- 5) $E^2 = (S^T \rightarrow S^F)$, i.e. the class of the transformation from an outranking S^X to an outranking S^Y having a far weaker strength (from sure presence of outranking to total absence of outranking).

Within the context of a four-valued outranking relation,

1'a) a RM $\geq$ is strongly monotonic if $\forall M \subseteq A$ and $\forall a, b \in M$

$$a \geq (M, R_{4v}) b \Rightarrow a > (M, R'_{4v}) b$$

where $>(M, R_{4v})$ is the asymmetric part of $\geq(M, R_{4v})$ and R'_{4v} is identical to R_{4v} except that R'_{4v} is obtained from R_{4v} by means of a 4v-transformation $aS^Xc \rightarrow aS^Yc$ with $(S^X \rightarrow S^Y) \in E^1 \cup E^2$ or $cS^Xa \rightarrow cS^Ya$ with $(S^X \rightarrow S^Y) \in E^{-1} \cup E^{-2}$ for some $c \in M - \{a\}$;

1'b) CF C is strongly monotonic if $\forall M \in 2^A$ and $R_{4v} \in R_{4v}(A)$

$$a \in C(M, R_{4v}) \Rightarrow \{a\} = C(M, R'_{4v})$$

where R'_{4v} is defined as previously.

A 4v-transformation on an elementary circuit consists of performing a 4v-transformation of the same equivalence class in the arcs of the circuit. A 4v-transformation on an elementary circuit is admissible if all the transformed outranking relations belong to the set $\{S^T, S^U, S^K, S^F\}$; e.g., if we have aS^Tb , bS^Uc , cS^Ta , an admissible transformation on the elementary circuit $\{(a,b), (b,c), (c,a)\}$ is aS^Ub , bS^Fc , cS^Ka . Let us point out that the elementary transformation on the arcs are $aS^Tb \rightarrow aS^Ub$, $bS^Uc \rightarrow bS^Fc$, $cS^Ta \rightarrow cS^Ka$. Therefore a RM $\geq$ is independent of circuits if $R_{4v}, R'_{4v} \in R_{4v}(A)$, R'_{4v} being obtained from R_{4v} through an admissible transformation on an elementary circuit and

$$a \geq (M, R_{4v}) b \Rightarrow a \geq (M, R'_{4v}) b.$$

Analogously, a CF C is independent of elementary circuits if, under the same hypotheses, $\forall M \in 2^A$ and $\forall R_{4v}, R'_{4v} \in R_{4v}(A)$

$$C(M, R_{4v}) = C(M, R'_{4v}).$$

Let us remark that the four-valued outranking R_{4v} expresses some possible preference situations without using any numerical evaluation. Therefore, the property of independence of circuits makes no use of cardinal properties of the relations, similarly to the property of neutrality and monotonicity.

5.3 An extension of the previous results to the four-valued outranking

To extend the results of Bouyssou (1992a and b), we associate an element of $\{0, 1/2, 1\}$ with each $(a,b) \in A \times A$ introducing the valued binary outranking relation $\hat{R}_{4v} : A \times A \rightarrow [0,1]$ by stating:

$$\hat{R}_{4v}(a,b) = \begin{cases} 0 & \text{if } aS^Fb \\ 1/2 & \text{if } aS^Ub \text{ or } aS^Kb \\ 1 & \text{if } aS^Tb. \end{cases}$$

This is a reduction to the $[0,1]$ interval of the lattice of the four truth values, where the values S^U and S^K are incomparable (no numerical value is used there). Such a reduction could be judged arbitrary, but the following result shows that $\hat{R}_{4v}$ satisfies some desirable properties allowing us to say that $\hat{R}_{4v}$ is the only value relation which faithfully represents R_{4v} . Let us consider $F: \{S^T, S^U, S^K, S^F\} \rightarrow [0,1]$. From each $R_{4v} \in R_{4v}(A)$ we can obtain one $R \in R(A)$ by stating $R(a,b) = F(R_{4v}(a,b)) \forall (a,b) \in A \times A$.

Let us consider the following properties $\forall (a,b) \in A \times A$:

R1) $F(R_{4v}(a,b)) = 1$ iff aS^Tb ,

R2) $F(R_{4v}(a,b)) = 0$ iff aS^Fb ,

R3) $F(R_{4v}^1(a,b)) - F(R_{4v}^2(a,b)) = F(R_{4v}^3(c,d)) - F(R_{4v}^4(c,d))$ iff aS^Xb according to R_{4v}^1 , aS^Yb according to R_{4v}^2 , cS^Wd according to R_{4v}^3 , cS^Zd according to R_{4v}^4 and

$$[aS^Xb \rightarrow aS^Yb] \in [cS^Wd \rightarrow cS^Zd].$$

Property R1) says that $\forall (a,b) \in A \times A$ the transformation of the four-valued outranking R_{4v} in the valued outranking R should give the maximum value, i.e., $R(a,b) = 1$, iff aS^Tb . Analogously, property R2) says that, $\forall (a,b) \in A \times A$, the same transformation should give the minimum value, i.e., $R(a,b) = 0$, iff aS^Fb . Finally, property R3) says that, if 4v-transformations $S^X \rightarrow S^Y$ and $S^W \rightarrow S^Z$ are of the same strength, then we should have $F(S^X) - F(S^Y) = F(S^W) - F(S^Z)$.

Theorem 5.3. Properties R1), R2) and R3) are satisfied if and only if

$$F(R_{4v}(a,b)) = \hat{R}_{4v}(a,b).$$

Proof. Necessity. Let us suppose that aS^Tb according to R_{4v}^1 , aS^Ub according to R_{4v}^2 , cS^Ud according to R_{4v}^3 , cS^Fd according to R_{4v}^4 . Since

$$[aS^Tb \rightarrow aS^Ub] \in [cS^Ud \rightarrow cS^Fd]$$

then, for property R3), we must have $F(S^T)-F(S^U) = F(S^U)-F(S^F)$ and, recalling properties R1) and R2), $2F(S^U)=1$, i.e., $F(S^U)=1/2$. Furthermore, let us suppose that eS^Kf according to R_{4v}^1 , eS^Uf according to R_{4v}^2 , gS^Th according to R_{4v}^3 , gS^Th according to R_{4v}^4 . Since

$$[eS^Kf \rightarrow eS^Uf] \in [gS^Th \rightarrow gS^Th]$$

and $F(R_{4v}^3(e,f))-F(R_{4v}^4(e,f))=0$ (because, of course, $F(S^T)-F(S^T)=0$), then, for property 3), we must have $F(S^K)-F(S^U)=0$, and therefore, for the previous result, $F(S^K)=F(S^U)=1/2$.

Sufficiency. Properties R1) and R2) are trivially verified. With respect to property 3), $\forall (a,b) \in A \times A$ we have:

- 1) $\hat{R}'_{4v}(a,b) = \hat{R}_{4v}(a,b)$ if $R'_{4v}(a,b)$ is obtained from $R_{4v}(a,b)$ by means of a 4v-transformation $aS^Xb \rightarrow aS^Yb$ with $(S^X \rightarrow S^Y) \in E^0$;
- 2) $\hat{R}'_{4v}(a,b) = \hat{R}_{4v}(a,b) + 1/2$ if $R'_{4v}(a,b)$ is obtained from $R_{4v}(a,b)$ by means of a 4v-transformation $aS^Xb \rightarrow aS^Yb$ with $(S^X \rightarrow S^Y) \in E^1$;
- 3) $\hat{R}'_{4v}(a,b) = \hat{R}_{4v}(a,b) - 1/2$ if $R'_{4v}(a,b)$ is obtained from $R_{4v}(a,b)$ by means of a 4v-transformation $aS^Xb \rightarrow aS^Yb$ with $(S^X \rightarrow S^Y) \in E^{-1}$;
- 4) $\hat{R}'_{4v}(a,b) = \hat{R}_{4v}(a,b) + 1$ if $R'_{4v}(a,b)$ is obtained from $R_{4v}(a,b)$ by means of a 4v-transformation $aS^Xb \rightarrow aS^Yb$ with $(S^X \rightarrow S^Y) \in E^2$;
- 5) $\hat{R}'_{4v}(a,b) = \hat{R}_{4v}(a,b) - 1$ if $R'_{4v}(a,b)$ is obtained from $R_{4v}(a,b)$ by means of a 4v-transformation $aS^Xb \rightarrow aS^Yb$ with $(S^X \rightarrow S^Y) \in E^{-2}$. #

Moreover, in order to prove the extension of the basic results recalled in section 5.1, the following notation and Lemma 5.1 are useful.

$\forall a,b \in A$ let us introduce the marginal positive $s(R_{4v})^+(a,b)$, negative $s(R_{4v})^-(a,b)$, and resultant $s(R_{4v})(a,b)$ score of a in relation to b ,

- $s(R_{4v})^+(a,b)=1$ if aS^Tb or aS^Kb and $s(R_{4v})^-(a,b)=0$ otherwise,
- $s(R_{4v})^-(a,b)=1$ if aS^Fb or aS^Ub and $s(R_{4v})^+(a,b)=0$ otherwise,
- $s(R_{4v})(a,b) = s(R_{4v})^+(a,b) - s(R_{4v})^-(a,b)$.

In consequence, we have

$$s(R_{4v})(a, b) = \begin{cases} -1 & \text{if } aS^F b \\ 0 & \text{if } aS^U b \text{ or } aS^K b \\ 1 & \text{if } aS^T b. \end{cases}$$

Lemma 5.1. The following relation holds:

$$S(x, M) = \sum_{b \in M - \{x\}} (s(R_{4v})(x, b) - s(R_{4v})(b, x)).$$

Proof. In fact, we have that

$$\begin{aligned} & \sum_{b \in M - \{x\}} (s(R_{4v})(x, b) - s(R_{4v})(b, x)) = \\ &= \sum_{b \in M - \{x\}} (s(R_{4v})^+(x, b) - s(R_{4v})^-(x, b) - s(R_{4v})^+(b, x) + s(R_{4v})^-(b, x)) = \\ &= \sum_{b \in M - \{x\}} s(R_{4v})^+(x, b) - \sum_{b \in M - \{x\}} s(R_{4v})^-(x, b) - \sum_{b \in M - \{x\}} s(R_{4v})^+(b, x) + \sum_{b \in M - \{x\}} s(R_{4v})^-(b, x). \end{aligned}$$

Since $XS^T b$ or $XS^K b$ implies that there exists at least one decision rule for which xSb , then we have

$$\sum_{b \in M - \{x\}} s(R_{4v})^+(x, b) = S^{++}(x).$$

Analogously, we obtain

$$\sum_{b \in M - \{x\}} s(R_{4v})^-(x, b) = S^-(x),$$

$$\sum_{b \in M - \{x\}} s(R_{4v})^+(b, x) = S^+(x),$$

$$\sum_{b \in M - \{x\}} s(R_{4v})^-(b, x) = S^{-+}(x)$$

and, therefore, we have the thesis. #

Lemma 5.2. The following relation between the overall score $S(x, M)$ and the net flow $S_{NF}(x, M, \hat{R}_{4v})$ holds:

$$S(x, M) = 2 S_{NF}(x, M, \hat{R}_{4v})$$

$\forall M \subseteq A$ and $\forall x \in M$.

Proof. Let us observe that, $\forall a, b \in A$, we have $s(R_{4v})(a, b) = 2 \hat{R}_{4v}(a, b) - 1$.
Therefore, on the base of Lemma 5.1, we obtain:

$$\begin{aligned}
S(x, M) &= \sum_{b \in M - \{x\}} (s(R_{4v})(x, b) - s(R_{4v})(b, x)) = \\
&= \sum_{b \in M - \{x\}} ((2 \hat{R}_{4v}(x, b) - 1) - (2 \hat{R}_{4v}(b, x) - 1)) = \\
&= 2 \sum_{b \in M - \{x\}} (\hat{R}_{4v}(x, b) - \hat{R}_{4v}(b, x)) = 2 S_{NF}(x, M, \hat{R}_{4v}). \#
\end{aligned}$$

Lemma 5.2 shows that the overall score $S(x, M)$ is a strictly positive monotonic transformation of the net flow $S_{NF}(x, M, \hat{R}_{4v})$. Therefore, we conclude that the ranking and the choice obtained from $S(x, M)$ are the same as those obtained from $S_{NF}(x, M, \hat{R}_{4v})$.

Lemma 5.3. Given $R_{4v}, R'_{4v} \in R_{4v}(A)$, if R'_{4v} is obtained from R_{4v} by an admissible 4v-transformations on an elementary circuit, then $\hat{R}'_{4v}$ is obtained from $\hat{R}_{4v}$ by an admissible transformation on an elementary circuit.

Proof. It is a direct consequence of property R3). #

Theorem 5.4. With respect to a four-valued outranking relation established by a set of decision rules, the scoring procedure based on $S(x, M)$ is the only RM which is neutral, strongly monotonic and independent of circuits.

Proof. Due to Lemma 5.2 and to Lemma 5.3, Theorem 5.1 implies also Theorem 5.4. #

Theorem 5.5. With respect to a four-valued outranking relation established by a set of decision rules, the scoring procedure based on $S(x, M)$ is the only CF which is neutral, strongly monotonic and independent of circuits.

Proof. Due to Lemma 5.2 and to Lemma 5.3, Theorem 5.2 implies also Theorem 5.5. #

6. Conclusions

We have been using the rough set approach to the analysis of preferential information concerning multicriteria choice and ranking problems. This information is given by a decision maker as a set of pairwise comparisons among some reference actions using the outranking relation. The outranking relation is approximated by means of a special form of dominance relation and decision rules are derived from these approximations. They represent the preference model of the decision maker. In result of application of these rules to a new set of potential actions, we get a four-valued outranking relation.

In this paper, we dealt with the problem of obtaining a recommendation from the above four-valued outranking relation. With this aim we proposed an exploitation procedure for ranking and choice problems based on a specific net flow score. Furthermore, we proved that this procedure is the only one which is neutral, strongly monotonic and independent of circuits.

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