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MULTI-OBJECTIVE FUZZY LINEAR PROGRAM: THE MOFAC METHOD

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Programmation linéaire floue multi-objective: la méthode MOFAC

Résumé Les sous-ensembles flous, considérés dans le cadre de la théorie de Dempster, sont bien adaptés à la modélisation des imprécisions inhérentes à la réalité industrielle. Alors que l'approche stochastique requiert la détermination des informations probabilistes, les sous-ensembles flous peuvent être compris comme des distributions de probabilités imprécises. Nous rappelons, tout d'abord, que la méthode de comparaison de deux nombres flous, dites de la compensation des aires, est cohérente avec le cadre théorique considéré. Dans le même contexte, nous dérivons un indice de comparaison.

Le sujet principal de ce cahier est l'introduction d'une nouvelle méthode de résolution des problèmes de programmation linéaire multi-objective sous imprécision. MOFAC (qui constitue l'abréviation de Multi-Objective Fuzzy Area Compensation) suppose que l'information pertinente consiste en probabilités imprécises. Finalement, nous réalisons une comparaison de notre méthode avec deux autres méthodes très connues – STRANGE et FLIP – afin de préciser les caractéristiques de l'approche proposée.

Multi-Objective Fuzzy Linear Programming: the MOFAC method

Abstract Fuzzy sets, in Dempster's framework, is well-suited for modelling imprecision inherent to industrial reality. Whilst stochastic approach implies the difficult determination of probability informations, fuzzy sets can be understood as imprecise probability distributions. We first recall that the area compensation method for the comparison of two fuzzy numbers is consistent with this framework. We derive then a comparison index in the same interpretative context.

The main topic of this paper is the introduction of a new method for solving multi-objective linear programming problems with imprecision: MOFAC (which stands for Multi-Objective Fuzzy Area Compensation) assumes that the relevant information consists of imprecise probabilities. Finally, we perform a comparison with well-known methods, STRANGE and FLIP, in order to exhibit the main features of the proposed approach.

Introduction

One of the possible goals for implementing fuzzy sets in mathematical models is to enhance the latter as far as the imprecision is concerned. The literature related to the conjunction of imprecision-minded fuzzy sets and mathematical programming models is particularly wide. For comprehensive surveys, we refer the reader to (Rommelfanger and Słowiński 1997) or, alternatively, to (Lai and Hwang 1992, Sakawa 1993). On the other hand, Roubens and Teghem's (1991) paper on the comparison of fuzzy and stochastic methodologies deserves to be read; the authors draw further prospects in both domains.

Imprecision is the information characteristic arising from partial knowledge. For example, the sentence "we leave Belgium this week-end" gives a piece of incomplete information about the departure day: it is either Saturday or Sunday. Imprecision is always represented by a disjunctive set, i.e. only one value inside this set is allowed. A typical example of imprecision is an error interval, which encodes the set of possible values of one uni-valued variable.

Imprecision is quite inherent to the industrial reality. On the one hand, it makes no sense to consider that industrial data are as precisely measured as it is assumed in academic models. A lot of uncontrollable parameters can influence the value of variables, while precise measurement would require loss of time or intrusive experiments. On the other hand, the current evolution of the market prescribes very fast modification of the production range. Therefore, new products or new production methodologies are to be scheduled with very few information about the data. Before giving any precise value to data, one should remember this maxim: "future is no more what it has been".

Stochastic mathematical programming is a rather well-known field of research, but very few has been actually done in the determination of the probability informations required for those models. As a matter of fact, the difficulties of stochastic programming is not only computational. For example, when you want to optimize the assignment of your money amount to portfolio, you need precise probability distribution about the value the shares can reach in the future. It is somehow counter-intuitive to ask so precise information to the DM, especially if new shares have appeared on the market. If he turns himself to stochastic models, it is probably because he lacks of knowledge.

Three main difficulties arise in Multi-Objective Fuzzy Linear Programming (MOFLP). The first issue is concerned with a meaningful treatment of the fuzzy objective(s); and the second one is a consistent interpretation of the inequality constraints (for a survey, see Rommelfanger and Słowiński 1997). And last but not least, in case of several objectives, one has to select among the different crisp Multi-Objective Programming approaches (see Vanderpooten and Vincke 1989, Słowiński 1997).

As far as the interpretation of fuzziness is concerned, it has been shown that fuzzy membership functions can be viewed as *imprecise probability distribution* (Dubois and Prade 1987, Fortemps and Roubens 1996). In other words, we consider in this paper the difficult problem of a meaningful and consistent treatment of fuzzy numbers in a mathematical model with imprecision. According to (Fortemps and Roubens 1996), the area compensation method for comparing two fuzzy numbers is recalled, in Section 1, to build a ranking relation between all the fuzzy numbers. Section 2 is devoted to derive a new comparison index in the same interpretative context. This index will be used in the treatment of inequality satisfaction. Therefore, it makes sense to apply these two methods to mathematical programming models where the relevant information consists of imprecise probabilities.

In Section 3, we exemplify the use of the area compensation methods on fuzzy linear programming. We introduce also in Section 4 a new interactive procedure called MOFAC (which stands for Multi-Objective Fuzzy Area Compensation), for solving a Multi-Objective Linear Programming problem. The method is illustrated on a didactic example in Section 5. Finally in Section 6, we perform a comparison with a stochastic programming method (STRANGE, Teghem et al. 1986, Teghem 1990) and a fuzzy one (FLIP, Słowiński 1986, Słowiński 1990).

1 The Area Compensation

The literature concerned with the choice among alternatives characterized by fuzzy numbers is particularly productive, since this problem occurs in a cluster of fields like approximate reasoning, multicriteria decision making and mathematical programming. Some of the proposed methods for the comparison of fuzzy numbers have been reviewed by Bortolan and Degani (1985) and, more recently, by Chen and Hwang (1992).

In this paper, we recall the *area compensation* method (Fortemps and Roubens 1996). The authors merged a revised version of the procedure proposed by Roubens (1990) and the ranking by total integral value by Liou and Wang (1992). This leads to a very intuitive ranking method which enjoys a mathematical interpretation in terms of imprecise probabilistic distribution.

1.1 Notations

We consider the set $\mathfrak{R}$ of normalized convex fuzzy numbers. Recall that a fuzzy number A is a fuzzy set on the real line $\mathbb{R}$ defined by the membership function $\mu_A(x), x \in \mathbb{R}$ with

$$\begin{cases} \max_x \mu_A(x) = 1 \\ \mu_A(y) \geq \min\{\mu_A(x), \mu_A(z)\}, \forall x \geq y \geq z \in \mathbb{R} \end{cases}$$

The α -level sets $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ are convex subsets of $\mathbb{R}$ whose lower and upper limits are represented by $\underline{a}_\alpha = \inf_{x \in \mathbb{R}} \{x : \mu_A(x) \geq \alpha\}$ and $\bar{a}_\alpha = \sup_{x \in \mathbb{R}} \{x : \mu_A(x) \geq \alpha\}$. For the sake of simplicity, we suppose that both limits are finite. The following results can be adapted to the case of infinite-support fuzzy numbers, but it is not really relevant to the field of mathematical programming.

1.2 The Idea of Area Compensation

Roubens (1990) proposed an intuitive comparison method based on the area compensation determined by the membership functions of two fuzzy numbers. And Fortemps and Roubens (1996) revised it in order to explicitly obtain a ranking as well as a defuzzification procedure. Let us first define the area in favour of $A \geq B$ w.r.t. the left and right parts of the fuzzy numbers:

$$S_L(A \geq B) = \int_{U(A,B)} [\underline{a}_\alpha - \underline{b}_\alpha] d\alpha \quad S_R(A \geq B) = \int_{V(A,B)} [\bar{a}_\alpha - \bar{b}_\alpha] d\alpha \quad (1)$$

where

$$U(A, B) = \{\alpha | 0 \leq \alpha \leq 1, \underline{a}_\alpha \geq \underline{b}_\alpha\} \quad V(A, B) = \{\alpha | 0 \leq \alpha \leq 1, \bar{a}_\alpha \geq \bar{b}_\alpha\} \quad (2)$$

The different areas are drawn on Figure 1.

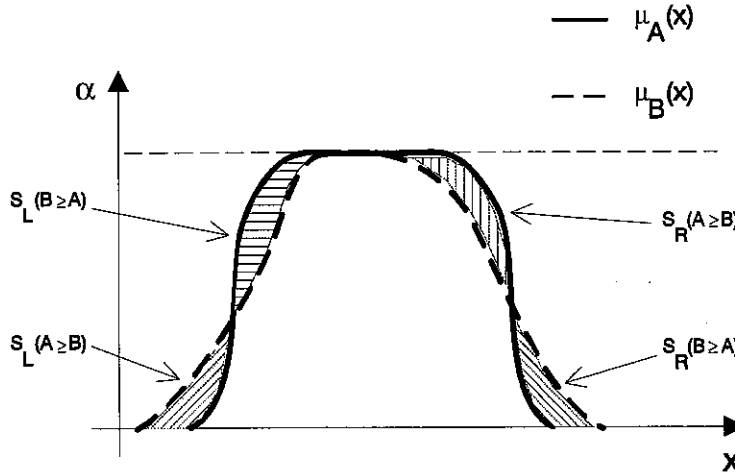


Figure 1: Comparing A to B

$C(A \geq B)$, the global degree to which A is larger than B , is naturally computed as the sum of all arguments in favour of $A \geq B$ minus the sum of the antagonistic ones. The magnitude of $C(A \geq B)$ represents how much the coalition in favour of A wins against the one in favour of B .

$$C : \mathfrak{R} \times \mathfrak{R} \rightarrow \mathbb{R}$$

$$(A, B) \rightarrow C(A \geq B)$$

$$C(A \geq B) = \frac{1}{2} \left\{ \begin{array}{l} S_L(A \geq B) + S_R(A \geq B) \\ -S_L(B \geq A) - S_R(B \geq A) \end{array} \right\} \quad (3)$$

so that $C(B \geq A) = -C(A \geq B)$. We consider that

$$A \geq B \text{ iff } C(A \geq B) \geq 0 \quad (4)$$

This relation $C(A \geq B)$ induces a complete ranking of all fuzzy numbers, which corresponds to the defuzzification function $\mathfrak{F} : \mathfrak{R} \rightarrow \mathbb{R}$ defined in the following theorem.

Theorem 1 (Fortemps and Roubens 1996)

$$C(A \geq B) = \mathfrak{F}(A) - \mathfrak{F}(B) \quad (5)$$

where

$$\mathfrak{F}(A) = \frac{1}{2} \int_0^1 (\underline{a}_\alpha + \bar{a}_\alpha) d\alpha \quad (6)$$

Therefore, equation (4) can be rewritten as

$$A \geq B \text{ iff } \mathfrak{F}(A) \geq \mathfrak{F}(B) \quad (7)$$

The function $\mathfrak{F} : \mathfrak{R} \rightarrow \mathbb{R}$ is a mapping from the set of fuzzy numbers to the set of real numbers $\mathbb{R}$, where the ordering is natural. This defuzzification function is the mean of the two areas defined by the vertical axis and respectively by the left and the right slope of the fuzzy number (see Fig. 2). Both areas describe in fact how much the number A is greater than 0: the first one ($\bar{a}_\alpha$) in an optimistic way, the second ($\underline{a}_\alpha$) in a pessimistic one. In the context of imprecise probability distribution, both areas correspond to extreme mean values. Briefly, a fuzzy set can be viewed as describing a set of admissible probability measures. Therefore, the mean value of a fuzzy set can be stated as the interval of mean values w.r.t. all these admissible probability distributions (Dubois and Prade 1987). This interval is denoted by $[E_*(A), E^*(A)]$ for any fuzzy number A , where

$$E_*(A) = \int_0^1 \underline{a}_\alpha d\alpha \quad E^*(A) = \int_0^1 \bar{a}_\alpha d\alpha. \quad (8)$$

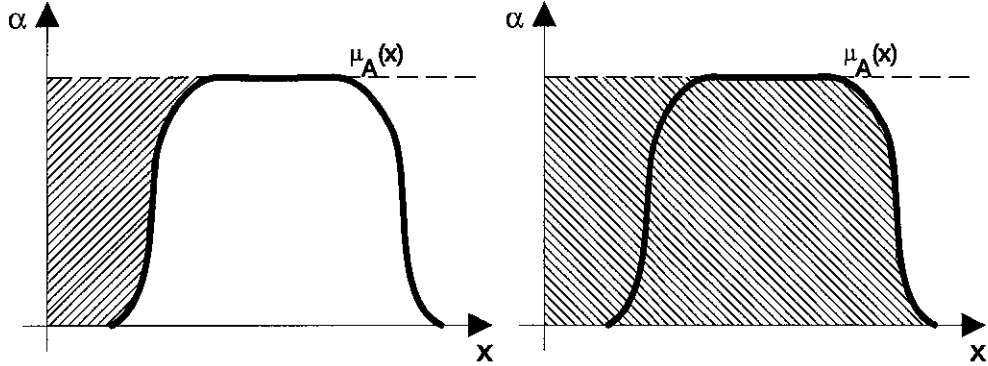


Figure 2: Mapping a fuzzy number A to the real set $\mathbb{R}$: the first picture represents the area limited by the left slope (or $E_*(A)$), the second one is defined by the right slope (or $E^*(A)$). The mapping $\mathfrak{F}$ consists of computing the arithmetic mean of those two areas.

In (Fortemps and Roubens 1996), further details are presented about the properties of this comparison method for fuzzy numbers; in particular, its linearity: $\forall A, B \in \mathfrak{R}, \forall c \in \mathbb{R}$:

$$\begin{aligned} \mathfrak{F}(A) &\in A_0 = \text{cl}\{x : \mu_A(x) > 0\} & (a) \\ \mathfrak{F}(c.A) &= c.\mathfrak{F}(A) & (b) \\ \mathfrak{F}(A + B) &= \mathfrak{F}(A) + \mathfrak{F}(B) & (c) \end{aligned} \quad (9)$$

where “cl” denotes the set closure.

2 Comparison Index

The comparison method that has been designed in Section 1 on the basis of the area compensation results from a defuzzification procedure. In other words, $C(A \geq B)$ evaluates a slack between A and B , this distance being measured on the same scale as A and B .

In contrast, several other comparison methods give an index with value in the unit interval $[0, 1]$. In such a way, different inequalities can be appreciated on a common scale. In this section, we propose such a comparison index related to the area compensation, as well as to the imprecise probability interpretation of fuzzy numbers.

On the other hand, the comparison of two fuzzy numbers A and B can be considered as a multicriteria problem. Indeed, there may simultaneously exist elements in favour of $A > B$, $A = B$ and $A < B$. When comparing two alternatives A and B in a multicriteria framework, some criteria could be in favour of the preference of A over B , the indifference or the reverted preference.

In fact, our aim is to construct a triple index $(I(A > B), I(A = B), I(A < B))$ such that, $\forall A, B \in \mathfrak{R}$, we have

$$I(A > B) + I(A = B) + I(A < B) = 1 \quad (10)$$

where $I(A > B)$ (resp., $I(A = B)$) represents how much A is strictly greater than (resp., equal to) B . Indices associated with non-strict inequalities will also be defined and used extensively in applications.

2.1 Deriving the Comparison Index

Again, we consider the comparison of two fuzzy numbers A and B , as presented in Section 1. We denote by $S(A > B)$ the total area in favour of $A > B$, i.e., $S(A > B) = S_L(A \geq B) + S_R(A \geq B)$ (see Section 1.2). Therefore, we have

$$S(A > B) = \int_0^1 [\bar{a}_\alpha - \min(\bar{a}_\alpha, \bar{b}_\alpha) + \underline{a}_\alpha - \min(\underline{a}_\alpha, \underline{b}_\alpha)] d\alpha \quad (11)$$

In a similar way, it makes sense to consider $S(A = B)$ as the total area claiming that A is equal to B :

$$S(A = B) = \int_0^1 [\min(\bar{a}_\alpha, \bar{b}_\alpha) - \max(\underline{a}_\alpha, \underline{b}_\alpha)] d\alpha \quad (12)$$

We denote by $S(A, B)$ the total area involved in the comparison problem

$$S(A, B) = \int_0^1 [\max(\bar{a}_\alpha, \bar{b}_\alpha) - \min(\underline{a}_\alpha, \underline{b}_\alpha)] d\alpha \quad (13)$$

One could observe that

$$S(A, B) = S(A > B) + S(A = B) + S(A < B) \quad (14)$$

The total area involved in the comparison scheme is partitioned into three separated parts (see Figure 3). It is worth noting that the area marked by a star (*) is taken into account two times in $S(A > B)$ and one time negatively in $S(A = B)$. Therefore, the total contribution of this area to $S(A, B)$ is counted one time only.

We propose to consider the following indices

$$I(A > B) = \frac{S(A > B)}{S(A, B) + S(A = B)} \quad (15)$$

$$I(A = B) = \frac{2S(A = B)}{S(A, B) + S(A = B)} \quad (16)$$

$$I(A < B) = \frac{S(A < B)}{S(A, B) + S(A = B)} \quad (17)$$

and because of relation (14), we have the desired property that the three indices sum up to one (Eqn. (10)).

It appears that $I(A > B)$ may be greater than 1, or smaller than 0, for example when A and B have disjoint supports. In such a case, there doesn't exist a value x such that $\mu_A(x) > 0$ and $\mu_B(x) > 0$, and

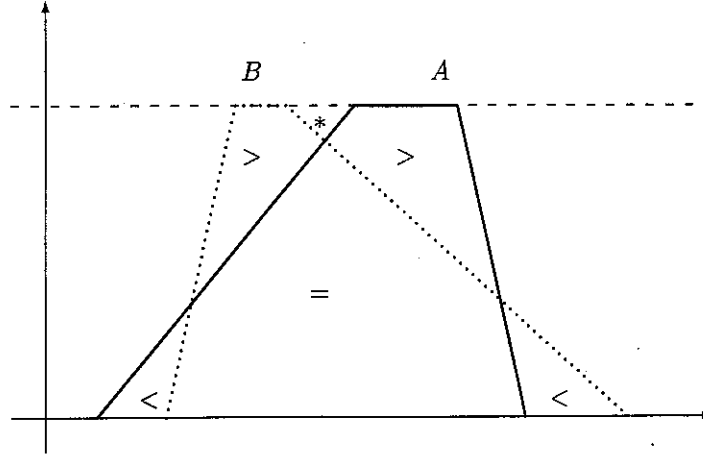


Figure 3: The partition of the areas

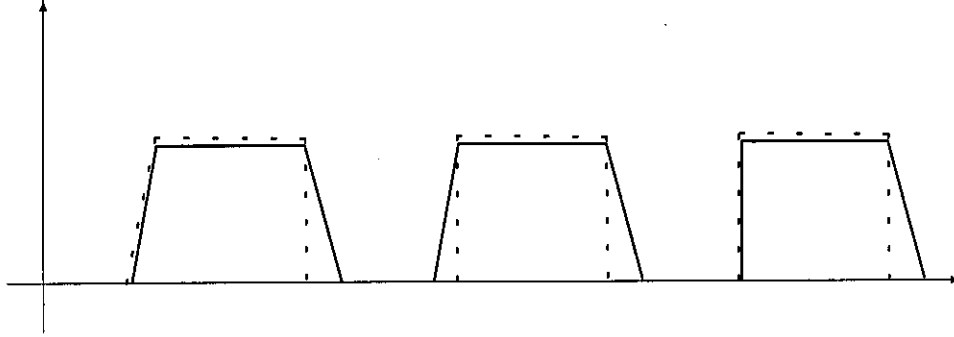


Figure 4: Three typical cases for the comparison of two fuzzy numbers (A in solid lines and B in dashed lines)

$|S(A = B)|$ measures how much A is different from B . Even if this is not a usual behaviour for a comparison index, we decide to keep the current definition of $I(A > B)$ for three main reasons. First, it leads to easy computation (see below). Then, it satisfies property (10). Finally, it encompasses the following thresholded index I^* , which can always be used instead of I .

$$I^* = \begin{cases} I & \text{if } 0 \leq I \leq 1 \\ 0 & \text{if } I < 0 \\ 1 & \text{if } I > 1 \end{cases}$$

See also the Note 2.3 below.

In the definition of our comparison indices, we have decided to concentrate on the shared area as well as on the total area. With such indices, we are able to distinguish between the 3 cases of figure 4. Those three cases are typical. For the middle case, which is the most symmetrical one, the common area is the area of B and neither A nor B may be declared greater than the other. The two other cases can be obtained either by increasing the common area (the left slope of B reaches the one of A) or by reducing it. In both cases, A will be declared greater than B , but with a different degree, since the common area $S(A = B)$ is different.

For the weak inequality index, we propose to consider

$$I(A \geq B) = I(A > B) + \frac{1}{2}I(A = B) \quad (18)$$

so that $I(A \geq B) + I(A \leq B) = 1$. On the other hand, the difference index is defined as

$$I(A \neq B) = 1 - I(A = B) \quad (19)$$

$$= I(A > B) + I(A < B) \quad (20)$$

In the case where $I(A = B)$ is negative, we have obviously $I(A \neq B) > 1$.

In the next section, we will state some properties of these comparison indices as well as cope with the computational issues. At the same time, we will present a probabilistic interpretation of these indices.

2.2 Properties – Interpretation

As a very first property, one would like to have intuitive results for the comparison of any fuzzy number with itself. Since $S(A > A) = S(A < A) = 0$ and $S(A = A) = S(A, A)$, we have

$$I(A > A) = 0 \quad I(A = A) = 1 \quad I(A \geq A) = \frac{1}{2} \quad I(A \neq A) = 0 \quad (21)$$

Indeed, it is desirable that our indices do not declare that A is strictly greater than itself, but that A is equal to A .

There is no doubt that the indices $I(A > B)$, $I(A = B)$ and $I(A \neq B)$ are quite hard to compute, since they involve the integration of minimum function over membership functions. On the other hand, this difficulty disappears for the evaluation of the weak comparison index $I(A \geq B)$, which is quite easy to determine. Hopefully, this index is particularly relevant to Mathematical Programming, since most of the constraints are modelled by weak inequalities.

Proposition 1

$$\forall A, B \in \mathfrak{R}, \quad I(A \geq B) = \frac{E^*(A) - E_*(B)}{[E^*(A) - E_*(A)] + [E^*(B) - E_*(B)]} \quad (22)$$

where $E^*(A)$ (resp., $E_*(A)$) is the upper (resp., lower) mean of A (see Equation (8)).

Recalling that the imprecise probability distribution underlying the fuzzy number A is characterized by an interval mean value $[E_*(A), E^*(A)]$, $I(A \geq B)$ can be understood as the comparison of the relevant intervals. The denominator of the ratio is equal to the sum of the length of both intervals, whilst the numerator is the largest length in favour of $A \geq B$.

The property below establishes a link between the comparison index I and the area compensation method.

Proposition 2

$$\forall A, B \in \mathfrak{R}, \quad I(A \geq B) = \frac{1}{2} + \frac{C(A \geq B)}{[(E^*(A) - E_*(A)) + (E^*(B) - E_*(B))]} \quad (23)$$

In conclusion, the behaviours of $I(A \geq B)$ and $C(A \geq B)$ are equivalent.

Theorem 2

$$\forall A, B \in \mathfrak{R}, \quad I(A \geq B) \geq \frac{1}{2} \Leftrightarrow C(A \geq B) \geq 0 \Leftrightarrow \mathfrak{F}(A) \geq \mathfrak{F}(B) \quad (24)$$

2.3 Note

Other comparison indices could be investigated in order to establish the multicriteria aspects of the comparison of two fuzzy numbers.

A priori it could appear relevant to define indices $I'(A > B)$, $I'(A = B)$ and $I'(A < B)$ which still satisfy relation 10:

$$I'(A > B) + I'(A = B) + I'(A < B) = 1$$

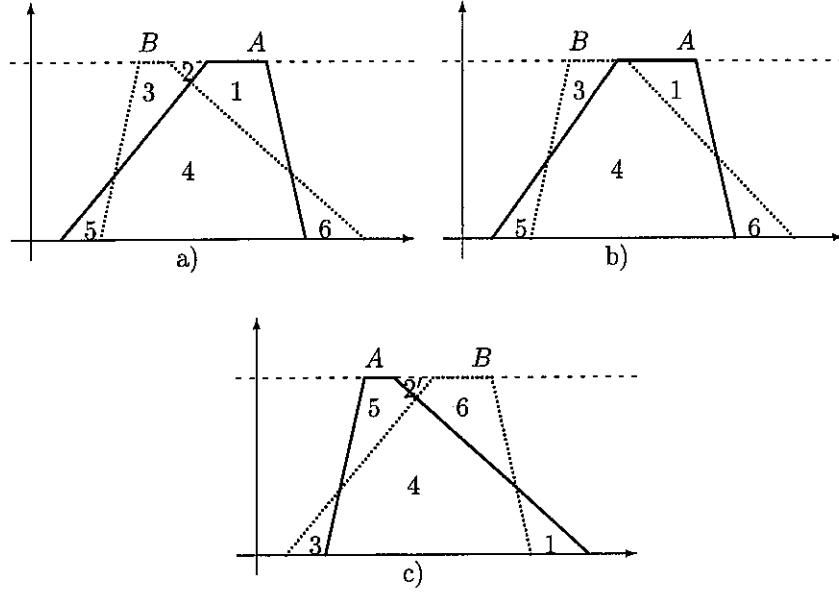


Figure 5: Three cases for the non negative-indices

but requiring the three indices to be non-negative (whilst in Section 2.1, $I(A = B)$ can be negative).

A possibility — among others — is to consider the positive part of the areas:

$$S'(A > B) = \int_0^1 \left[[\bar{a}_\alpha - \max(\underline{a}_\alpha, \bar{b}_\alpha)]^+ + [\min(\bar{b}_\alpha, \underline{a}_\alpha) - \underline{b}_\alpha]^+ + [\underline{a}_\alpha - \bar{b}_\alpha]^+ \right] d\alpha \quad (25)$$

$$S'(A = B) = \int_0^1 \left[[\min(\bar{a}_\alpha, \bar{b}_\alpha) - \max(\underline{a}_\alpha, \underline{b}_\alpha)]^+ \right] d\alpha \quad (26)$$

With respect to the three cases of figure 5, the three indices $S'(A > B)$, $S'(A = B)$ and $S'(A < B)$ correspond¹ to the total areas of regions (1) + (2) + (3), (4) and (5) + (6) + (2'), respectively. One of the two regions (2) and (2') has necessarily to be empty.

The total area $S'(A, B)$ considered in these comparison indices is the same as the total area $S(A, B)$ of Section 2.1:

$$S'(A, B) = S'(A > B) + S'(A = B) + S'(A < B) = S(A, B)$$

If we define

$$I'(A > B) = \frac{S'(A > B)}{S'(A, B)} \quad I'(A = B) = \frac{S'(A = B)}{S'(A, B)} \quad I'(A \geq B) = \frac{S'(A > B) + S'(A = B)}{S'(A, B)}$$

we have then

$$I'(A \geq B) = \frac{\int_0^1 [\bar{a}_\alpha - \underline{b}_\alpha]^+ d\alpha}{\int_0^1 (\max(\bar{a}_\alpha, \bar{b}_\alpha) - \min(\underline{a}_\alpha, \underline{b}_\alpha)) d\alpha} \quad (27)$$

These indices satisfy the following properties:

$$\begin{aligned} I'(A = B) = 0 &\Leftrightarrow \text{either } I'(A > B) = 1, I'(A < B) = 0 \\ &\quad \text{or } I'(A > B) = 0, I'(A < B) = 1 \\ I'(A > B) = 1 &\Leftrightarrow \bar{b}_\alpha \leq \underline{a}_\alpha, \forall \alpha \\ I'(A \geq B) = 1 &\Leftrightarrow \underline{b}_\alpha \leq \underline{a}_\alpha, \forall \alpha \\ &\quad \bar{b}_\alpha \leq \bar{a}_\alpha, \forall \alpha \end{aligned}$$

¹The values of $S(A > B)$, $S(A = B)$ and $S(A < B)$ from Section 2.1 correspond the total areas of (1) + 2(2) + (3), (4) - (2) - (2') and (5) + (6) + 2(2'), respectively.

When $\bar{a}_\alpha - \underline{b}_\alpha \geq 0$, $\forall \alpha$, i.e., when $\underline{b}_1 \leq \bar{a}_1$ (see Figure 5a. and b.), the numerator of Equation (27) is the same as the one of relation (22): in this particular case, it is equal to $E^*(A) - E_*(B)$.

However, the indices are definitely more difficult to use than those proposed in Section 2.1 and, in general, they cannot be related to the results of Section 1.

We arrive at the same conclusion with other tentatives to define such non-negative indices.

3 Multi-Objective Fuzzy Linear Programming: the MOFAC method

There is no need to recall that fuzzy linear programming gives birth to a large literature. Recent surveys of this domain have been made by Lai and Hwang (1992, Chapter 4) and Rommelfanger and Słowiński (1997).

Different models have been investigated, involving imprecise resources, technological coefficients or objective coefficients. Basically, the common procedure consists of a “defuzzification” of the imprecise objective and of the constraints to give rise to an associated deterministic problem. Differences appear in the way the defuzzification is performed.

In case the objective involves fuzzy coefficients, it may be transformed into one crisp criterion through the use of any ranking procedure (see, e.g., Tanaka et al. 1984). Another approach leads to several crisp criteria: Rommelfanger et al. (1989) consider two extreme cases, the “worst” and the “best” ones, and solve the associated multi-objective linear programming (MOLP) problem.

As to fuzziness in the constraints, Ramik and Rimanek (1985) propose to use the set-inclusion concept. Therefore, they transform a fuzzy constraint into several crisp ones, depending on the fuzzy number shape. Tanaka et al. (1984) define an optimism degree α for the DM and compare fuzzy numbers with respect to their α -section. Namely, they transform a fuzzy inequality into two crisp ones corresponding to the boundaries of the α -cut intervals.

By the way, it is nowadays undoubted that the multiobjective paradigm commands attention. In most decision problems, several, generally conflicting, criteria are to be considered. Typically, they arise from various decision makers or from various viewpoints. A single criterion procedure is sometimes possible, when conversion rates between heterogeneous issues can be identified. But, in general, the estimation of a global utility function is not straightforward and the utility approach not always pertinent (Vanderpooten and Vincke 1989).

Uncertainty models and the multiobjective paradigm have been combined. For a recent state-of-the-art, we refer the reader to (Słowiński and Teghem 1990b).

3.1 Problem Statement

We consider the following multiobjective linear program (MOLP):

$$(P) \quad \begin{cases} \text{“maximize”} & \begin{cases} Z_1 = \bar{C}_1 \bar{x} \\ \vdots \\ Z_k = \bar{C}_k \bar{x} \end{cases} \\ \text{such that} & \begin{cases} \bar{A}_1 \bar{x} \leq B_1 \\ \vdots \\ \bar{A}_m \bar{x} \leq B_m \\ \bar{x} \geq \bar{0} \end{cases} \end{cases} \quad (28)$$

where $\bar{x}$ is a column vector of n crisp positive decision variables, $\bar{C}_1, \dots, \bar{C}_k$ are row vectors of fuzzy cost coefficients, and $\bar{A}_i$ and B_i ($i = 1, \dots, m$) are the row vector of fuzzy coefficients and the fuzzy right-hand side of the i -th constraint, respectively. In this Section, capital letters are used to denote fuzzy numbers, whilst small letters represent crisp variables or values.

As stressed in (Słowiński 1990) and in (Lai and Hwang 1992), the key question in fuzzy MOLP is the comparison of fuzzy numbers, either in the constraints or in the objectives. The comparison principle should meet two basic requirements: it has to be meaningful and computationally efficient.

We have already shown that the area compensation principle induces a comparison that supports the imprecise probability interpretation of a fuzzy set. We still have to prove that it allows a transformation of a fuzzy MOLP into a crisp (also called *deterministic*) multiobjective linear programming problem.

3.2 Problem Defuzzification

The objectives are naturally considered by means of the area compensation method. In fact, the comparison procedure based on the index $C(A \geq B)$ induces a ranking on the set of fuzzy numbers related to the natural ranking on $\mathbb{R}$ by the defuzzification function $\mathfrak{F}$. Therefore, to find the “optimal” solution of a problem with respect to a fuzzy criterion — in the sense of the area compensation principle — amounts to find the optimal solution of the same problem with respect to the defuzzified criterion.

We can thus replace the fuzzy criterion

$$Z_i = \sum_{j=1}^n C_{ij} x_j$$

by the defuzzified criterion; and recalling that $\mathfrak{F}$ is linear, we obtain

$$z_i = \mathfrak{F} \left(\sum_{j=1}^n C_{ij} x_j \right) = \sum_{j=1}^n \mathfrak{F}(C_{ij}) x_j = \sum_{j=1}^n c_{ij} x_j \quad (29)$$

where c_{ij} denotes the defuzzified coefficient $\mathfrak{F}(C_{ij})$.

The constraints require a specific, but consistent, treatment. In view of future relaxation or strengthening of the constraints, we need to adopt the same scale to measure all constraint satisfactions. This leads us to prefer the comparison index I . Supporting the same interpretation as $\mathfrak{F}$, the comparison index basically ranges over the unit interval.

Considering the constraint $\bar{A}_i \bar{x} \leq B_i$, we can write

$$C(\bar{A}_i \bar{x} \leq B_i) \geq 0 \Leftrightarrow I(\bar{A}_i \bar{x} \leq B_i) \geq \sigma_i \quad (30)$$

or, equivalently,

$$\frac{E^*(B_i) - E_*(\bar{A}_i \bar{x})}{E^*(B_i) - E_*(B_i) + E^*(\bar{A}_i \bar{x}) - E_*(\bar{A}_i \bar{x})} \geq \sigma_i \quad (31)$$

if we denote by σ_i the satisfaction requirement of this constraint, typically $\sigma_i = 1/2$. Since the mean operator E is linear and that every decision variable is to be positive, we may write

$$\frac{E^*(B_i) - \sum_j E_*(A_{ij}) x_j}{E^*(B_i) - E_*(B_i) + \sum_j [E^*(A_{ij}) - E_*(A_{ij})] x_j} \geq \frac{1}{2} \quad (32)$$

or

$$\sum_j [\sigma_i E^*(A_{ij}) + (1 - \sigma_i) E_*(A_{ij})] x_j \leq (1 - \sigma_i) E^*(B_i) + \sigma_i E_*(B_i) \quad (33)$$

It is worth noting that, with $\sigma_i = 1/2$, this comes down to the initial constraint $\bar{A}_i \bar{x} \leq B_i$ defuzzified via the function $\mathfrak{F}$. But, with other values of σ_i , a stronger (if σ_i increases) constraint or a weaker (if σ_i decreases) one can be obtained. In particular, when $\sigma_i = 1$, we reach the worst case:

$$\sum_j E^*(A_{ij}) x_j \leq E_*(B_i)$$

Namely, the upper mean value of $\bar{A}_i \bar{x}$ is constrained to be smaller than the lower mean value of B_i .

In summary, as soon as satisfaction levels σ_i have been specified for the constraints ($i = 1, \dots, m$), the MOFAC method transforms the fuzzy MOLP into a crisp one, using either the defuzzification function $\mathfrak{F}$ for the objectives, or the comparison index I for the constraints. This defuzzification is straightforward

because the problem is linear. Finally, we obtain

$$(P_{\bar{\sigma}}) \left[\begin{array}{l} \text{"maximize"} \\ \text{such that} \end{array} \left\{ \begin{array}{l} z_1 = \bar{c}_1 \bar{x} \\ \vdots \\ z_k = \bar{c}_k \bar{x} \\ \bar{a}_1^{\sigma_1} \bar{x} \leq b_1^{\sigma_1} \\ \vdots \\ \bar{a}_m^{\sigma_m} \bar{x} \leq b_m^{\sigma_m} \\ \bar{x} \geq 0 \end{array} \right. \right. \quad (34)$$

where $a_{ij}^{\sigma_i}$ is obtained as the weighted mean $[\sigma_i E^*(A_{ij}) + (1 - \sigma_i) E_*(A_{ij})]$, $b_i^{\sigma_i} = [(1 - \sigma_i) E^*(B_i) + \sigma_i E_*(B_i)]$ and $c_{ij} = \mathcal{F}(C_{ij})$.

Remark that we allow different satisfaction requirements for the different constraints.

4 Interactive Procedure

In the literature (see, e.g., Steuer 1985, Vanderpooten and Vincke 1989, Słowiński 1997), the authors proposed various iterative methods to determine efficient solutions of a MOLP problem. They consist of successive unicriterion optimization of a weighted aggregated objective. At each step, new efficient solutions are computed and in general the DM is asked to relax objective requirements and/or to modify the weight vector. This kind of procedures ends up with a set of efficient solutions for a given MOLP problem.

Since in our model, the decision maker is allowed to specify lower bounds ($\bar{\sigma}$) for the safety indices, new MOLP problems are iteratively generated. They are solved through the same procedure.

Many procedures for MOLP are available and can be applied to problem $(P_{\bar{\sigma}})$. We decided to choose a very basic approach to solve the MOLP problem, relying in some sense on Gonzales et al.'s (1985) work. The interactive procedure proposed is the following.

After the formulation of MOLP problem $(\mathcal{P})$ and the definition of the fuzzy coefficients, the associated deterministic problem $(P_{\bar{\sigma}})$ is built for a uniform vector $\bar{\sigma} = \{\frac{1}{2}, \dots, \frac{1}{2}\}$. In practice, the DM may stipulate another initial $\bar{\sigma}$ -vector.

By individually optimizing each objective of this MOLP problem, one obtains the extremal efficient solutions. In general, k efficient solutions are found. Let S be the set of the currently considered efficient solutions. $S' = \emptyset$ (resp. S'') is the set of new efficient solutions (resp., efficient facets).

From each k -tuple of efficient solutions of S , a new efficient solution is obtained by the optimization of the aggregation of the k objectives with the weights determined as in (Gonzales et al. 1985), i.e., the hyperplane $\mathcal{H}$ going through the k selected points of S is determined and the coefficients of the hyperplane equation are used as weights. Optimizing with these weights, an efficient solution $\bar{y}$ is searched for in the direction orthogonal to the hyperplane $\mathcal{H}$. If $\bar{y}$ belongs to the hyperplane, then the feasible part of $\mathcal{H}$ is an efficient facet F , i.e., a set of efficient solutions, add this facet to S'' . Otherwise, add $\bar{y}$ to S' .

Once all the solutions $\bar{y}$'s have been generated — at the first iteration, only one solution $\bar{y}$ is obtained —, the set S' is added to S . The DM is then asked to remove solutions from S until the cardinality of S ($|S|$) is less than, or equal to, $k + 1$.

At this step, the set S contains compromise solutions and S'' is a set of efficient facets. Check for each facet F in S'' , if S contains at least one point of F . If not, the facet F has been disqualified by preferred compromise solution in S ; thus, remove F from S'' . If $|S| < k$, then stop. Otherwise, if the set S has been modified, loop to the computation of new efficient solutions (with $S' = \emptyset$).

The decision maker may perform the final choice among the sets S and S'' . If he is fully satisfied by one of these solutions, stop. Otherwise, he may revise the requirements on the constraint satisfaction levels encoded in the vector $\bar{\sigma}$. Graphical tools as well as the value of the slack variables can help the DM in this interactive step.

This leads to a new deterministic MOLP problem $(P_{\bar{\sigma}'})$ which can be treated as previously. To speed up the resolution of this new problem, one can consider the weight vectors used to obtain the solutions of S in the resolution of $(P_{\bar{\sigma}})$. This gives a very efficient starting point for the procedure, since the previous interactive phases can be summarized through this kind of information.

It stems from the procedure description that the DM intervenes at two stages. First, he is asked to specify the “safety parameters” $\bar{\sigma}$, determining which constraints require a stronger satisfaction or a weaker one. Moreover, when new efficient solutions are considered for problem $(P_{\bar{\sigma}})$, the DM guides the selection of which solution $\bar{x} \in S$ has to be replaced by $\bar{y}$ ’s, implicitly revising the weights used in the aggregation of the criteria.

5 Example

We consider the following problem (Słowiński and Teghem 1990a):

$$(P) \left[\begin{array}{l} \text{“maximize”} \\ \text{such that} \end{array} \left\{ \begin{array}{l} Z_1 = C_{11}x_1 + C_{12}x_2 \\ Z_2 = C_{21}x_1 + C_{22}x_2 \\ A_{11}x_1 + A_{12}x_2 \leq B_1 \\ A_{21}x_1 + A_{22}x_2 \leq B_2 \\ x_1 \leq 5.5 \\ x_1, x_2 \geq 0 \end{array} \right. \right. \quad (35)$$

The possibility distribution of the uncertain coefficients are depicted on Figure 6.

5.1 First Step

In the first step of the interactive procedure, vector $\bar{\sigma}$ is fixed to $(\frac{1}{2}, \frac{1}{2})$. The associated deterministic problem is

$$(P_{\bar{\sigma}}) \left[\begin{array}{l} \text{“maximize”} \\ \text{such that} \end{array} \left\{ \begin{array}{l} z_1 = 23.5x_1 + 9.99x_2 \\ z_2 = -4.6x_1 + 7.8x_2 \\ -4x_1 + 12.3x_2 \leq 30.25 \\ 4.5x_1 + 4.3x_2 \leq 23 \\ x_1 \leq 5.5 \\ x_1, x_2 \geq 0 \end{array} \right. \right. \quad (36)$$

The two extremal efficient solutions obtained by individual optimization of each objective are

	z_1	z_2	x_1	x_2	s_1	s_2
$\bar{u}$	30.03	-5.88	5.11	0.00	50.70	0.00
$\bar{v}$	6.14	4.80	0.00	2.46	0.00	12.42

where s_1 and s_2 are the slack variables associated to the first and second inequality constraint of the problem, respectively. For example, $s_1 = 30.25 + 4x_1 - 12.3x_2$.

The equation of the hyperplane (here a line) containing those two points in the criteria space is

$$z_1 + 2.237z_2 - 16.879 = 0$$

Therefore, to find the next efficient solution, we optimize the aggregated criterion $z = z_1 + 2.237z_2$, subject to the constraints of $(P_{\bar{\sigma}})$. It provides us with the following efficient solution:

	z_1	z_2	x_1	x_2	s_1	s_2
$\bar{w}$	20.23	3.71	2.11	3.14	0.00	0.00

The decision maker decides to keep the three solutions $S = \{\bar{u}, \bar{v}, \bar{w}\}$. But, no new efficient solution is obtained, since both facets defined by $\{\bar{u}, \bar{w}\}$ and $\{\bar{v}, \bar{w}\}$ are efficient.

The decision maker is now allowed to modify the requirement levels.

5.2 Second Step

He decides to consolidate the satisfaction of the first constraint by increasing the value of σ_1 . The new level vector is

$$\bar{\sigma}' = (0.93, 0.50)$$

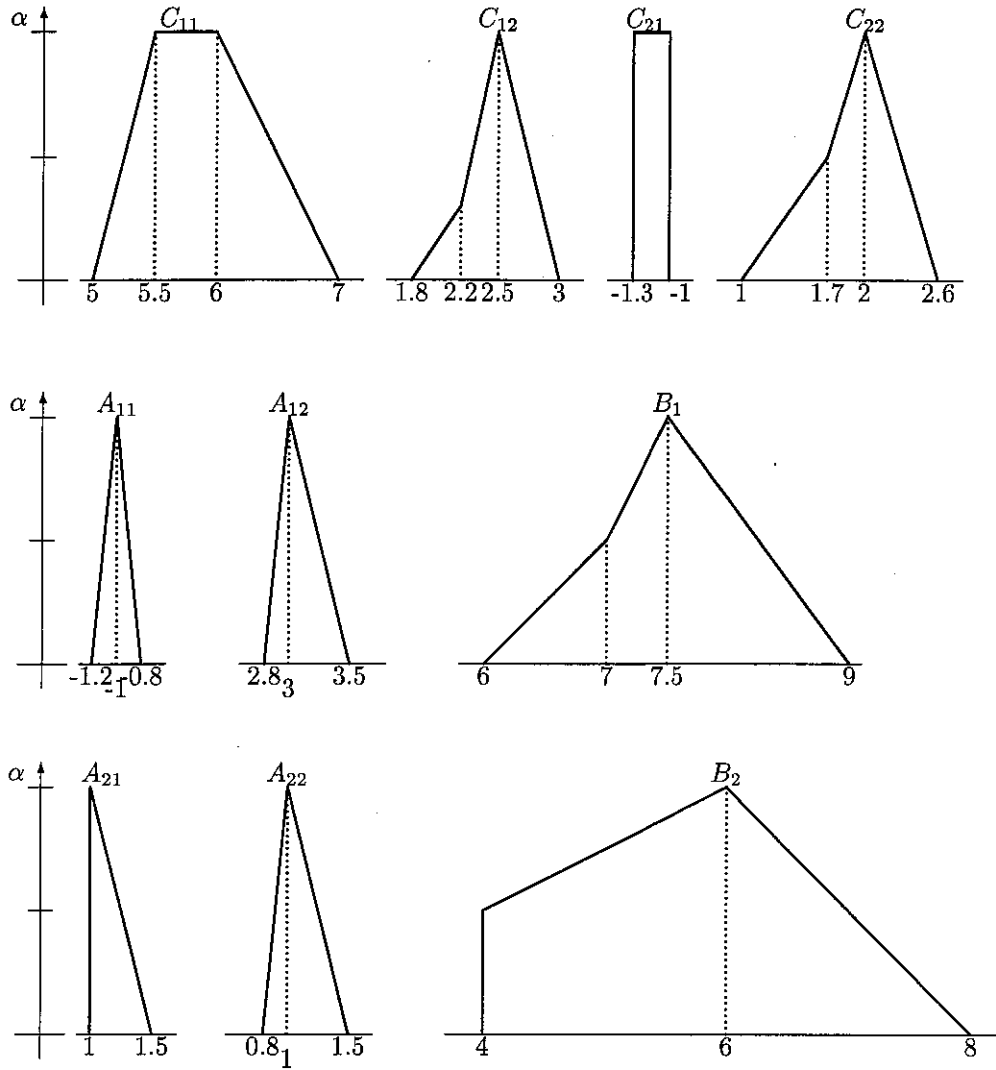


Figure 6: Possibility distributions of uncertain coefficients (Słowiński and Teghem 1990a)

which leads to the MOLP problem

$$(\mathcal{P}_{\bar{\sigma}'}) \left[\begin{array}{l} \text{"maximize"} \\ \text{such that} \end{array} \left\{ \begin{array}{l} z_1 = 23.5x_1 + 9.99x_2 \\ z_2 = -4.6x_1 + 7.8x_2 \\ -3.656x_1 + 12.902x_2 \leq 27.885 \\ 4.5x_1 + 4.3x_2 \leq 23 \\ x_1 \leq 5.5 \\ x_1, x_2 \geq 0 \end{array} \right. \right. \quad (37)$$

The DM also decides to re-use the same optimization weight sets as in the previous step, namely $(1, 0)$, $(0, 1)$ and $(1, 2.237)$.

	z_1	z_2	x_1	x_2	s_1	s_2
$\bar{u}$	30.03	-5.88	5.11	0.00	46.57	0.00
$\bar{v}$	5.40	4.21	0.00	2.16	0.00	13.71
$\bar{w}$	21.17	2.78	2.40	2.84	0.00	0.00

Here again, two efficient facets are determined and the proposition is made for the DM to revise his satisfaction requirements.

5.3 Third Step

He is now willing to increase the value of criterion z_1 . When considering the unicriterion optimization of z_1 in the previous step, it can be observed that the second constraint was saturated. Therefore, to enhance the value of z_1 , we need to relax the σ_2 requirement. For example, we adopt

$$\bar{\sigma}'' = (0.93, 0.30)$$

and solve the MOLP problem

$$(\mathcal{P}_{\bar{\sigma}''}) \quad \left[\begin{array}{l} \text{"maximize"} \\ \text{such that} \end{array} \left\{ \begin{array}{l} z_1 = 23.5x_1 + 9.99x_2 \\ z_2 = -4.6x_1 + 7.8x_2 \\ -3.656x_1 + 12.902x_2 \leq 27.885 \\ 4.3x_1 + 4.02x_2 \leq 25 \\ x_1 \leq 5.5 \\ x_1, x_2 \geq 0 \end{array} \right. \right. \quad (38)$$

Three efficient solutions are generated, as well as efficient facets,

	z_1	z_2	x_1	x_2	s_1	s_2
$\bar{u}$	33.15	-5.67	5.50	0.34	43.66	0.00
$\bar{v}$	5.40	4.21	0.00	2.16	0.00	13.71
$\bar{w}$	25.14	2.42	2.99	3.01	0.00	0.00

Finally, the decision maker chooses $\bar{w}$ as his best compromise.

This solution is the target point chosen in (Słowiński and Teghem 1990a) for comparison purposes. It allows to "simulate" actions of the DM going in a given direction.

6 Comparison of MOFAC with STRANGE and FLIP

The same example has been treated by both methods STRANGE and FLIP (see Słowiński and Teghem 1990a). Therefore, we may compare our procedure to a typically stochastic approach — namely STRANGE (Teghem et al. 1986) — and to a fuzzy one — FLIP (Słowiński 1986). Both methods are explicitly described in Słowiński and Teghem's (1990b) book and put into competition.

Since we said that any MOLP procedure can be chosen to solve the associate deterministic problem, we will mainly focus on the model differences. Is uncertainty modelled in the same way? What about risk? These are the main viewpoints for the comparison of MOFAC with STRANGE and FLIP.

6.1 Uncertainty Modelling

STRANGE language for input data is based on scenarios; they encodes in a discrete way a set of possible realizations. They are defined globally for each objective as well as for each constraint. In general, the experts estimate extreme cases and a few intermediate ones.

Increasing the number of scenarios leads to a larger number of criteria and, therefore, to a larger number of linear programming problems to be solved in order to compute the pay-off table.

FLIP uses the fuzzy number language. The information about uncertain coefficients is represented separately and in a continuous way. This less schematic representation goes together with a more extensive mathematical treatment.

The number of objectives and constraints are fixed, since the associated deterministic MOLP problem involves one criterion per fuzzy objective and two deterministic constraints per fuzzy one. But, for each objective, a fuzzy goal has to be determined.

MOFAC approach is somewhere in-between those two methods. The imprecise probability distribution on each uncertain coefficient is modelled by a continuous fuzzy number but doesn't require important mathematical effort. The number of criteria and of constraints are equal in the fuzzy problem and in the deterministic one; and no goal is needed.

6.2 Risk and Safety

In the treatment of the risk of violating constraints, two main methodologies can be exhibited (Roubens and Teghem 1991). On the one hand, *Chance Constrained Programming* (CCP) approach considers the risk independently for each uncertain constraint and imposes a bound on this risk; namely, FLIP belongs to this class of procedures. On the other hand, *Stochastic Programming with Recourse* (SPR) consists of a global measure of the constraint violation, through an additional objective aggregating the different slack variables in excess. This class is exemplified by STRANGE. Our approach to safety is more related to FLIP one and to CCP procedures, since each constraint is characterized by one specific satisfaction requirement σ , which can be fixed by the DM. In FLIP, two satisfaction indices (pessimistic and optimistic) are associated to each constraint.

However, a SPR methodology can be easily designed coherently with the probabilistic interpretation of a fuzzy number. For instance, it could be proposed to aggregate in an additional objective the slack variables in excess and defuzzified them by means of the proposed function $\mathfrak{F}$.

In MOFAC, the information about the constraint safety may be enhanced by the graphical analysis of the fuzzy numbers ($\bar{A}\bar{x}$ and B) representing the fuzzy constraint ($\bar{A}\bar{x} \leq B$). It will give some clues about constraints that need to be reinforced.

6.3 Interactive Procedure – Computation

In STRANGE, whose interactive spirit is similar to the STEM method, the DM is asked in the interactive phases to indicate an objective on which he is ready to relax as well as the maximum level of this relaxation. In FLIP as well as in our approach, the DM specifies the safety requirements and supervises the generation of successive efficient solutions.

On the other hand, our approach requires much less computational effort as others, since FLIP leads to multiobjective linear fractional programming problems (if the objective memberships are linear!) and STRANGE involves a higher number of constraints and criteria.

It is worth noting that the shape of the membership functions has no influence in MOFAC. Each coefficient is essentially represented by its lower and upper mean values, due to the imprecise probability interpretation. Should this be considered as too restrictive, the defuzzification of each coefficient could take into account weight functions as presented in (Fortemps and Roubens 1996).

In STRANGE, precisely defined probability about scenarios are considered and a global risk objective to be minimized is introduced; FLIP uses fuzzy numbers as a kind of possibility distributions and defines different safety requirements for each constraint. In MOFAC, the imprecision on probability risk is modelled by fuzzy numbers and each constraint satisfaction is managed independently.

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