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APPROXIMATION OF WEIGHTED HEREDITARY INDUCED-SUBGRAPH MAXIMIZATION PROBLEMS

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Résumé

Le but central de cet article est l'étude de l'approximation polynomiale de problèmes héréditaires de maximisation portant sur la recherche de sous-graphes induits pondérés. Après avoir donné un résultat général d'approximation pour cette classe de problèmes, nous montrons que celui-ci a plusieurs corollaires importants, par exemple, il améliore d'un facteur d'$O(\log\log n)$ le meilleur rapport d'approximation connu pour le stable maximum pondéré.

Mots-clé : problèmes combinatoires, complexité, algorithme polynomial d'approximation, NP-complétude, stable, problème héréditaire.

Approximation of weighted hereditary induced-subgraph maximization problems

Abstract

The main purpose of this paper is to study polynomial approximation of weighted hereditary induced subgraph problems. After giving a general approximation result for this class of problems, we show that the result has many important corollaries since, for instance, it allows improvement of the best-known approximation ratio for weighted independent set by a factor of $O(\log\log n)$.

Keywords: combinatorial problems, computational complexity, polynomial-time approximation algorithm, NP-completeness, independent set, hereditary problem.

1 Preliminaries

We consider **NP**-complete optimization graph-problems Π where the objective is to find a maximum-order induced subgraph[1] satisfying a non-trivial hereditary property π. Let $\mathcal{G}$ be the class of all the graphs; a graph-property π is a mapping from $\mathcal{G}$ to $\{0,1\}$, i.e., for a $G \in \mathcal{G}$, $\pi(G) = 1$ iff G satisfies π and $\pi(G) = 0$, otherwise; π is hereditary if whenever it is satisfied by a graph it is also satisfied by every one of its induced subgraphs; it is non-trivial if it is true for infinitely many graphs and false for infinitely many ones ([2]). The property "π: is a clique", or "π: is an independent set" are typical examples of such non-trivial hereditary properties and the maximum clique-problem or the maximum independent set are in their turn typical examples of hereditary induced subgraph problems. A generalisation of the class of problems just introduced is the one where we consider that positive weights are associated with the vertices of the input-graphs. Given a graph G, the objective of a weighted induced subgraph problem is to determine an induced subgraph G^* of G such that G^* satisfies π and, moreover, the sum of the weights of the vertices of G^* must be the largest possible.

Remark 1. By definition of π (as a mapping from the class of graphs to $\{0,1\}$), it is obvious that π is context-free; in other words, if a graph G satisfies π, then G satisfies π into every graph whose G is an induced subgraph. ∎

Given a polynomial-time approximation algorithm (PTAA) $\mathtt{A}$ that solves a maximization **NP**-complete problem Π (we denote by WΠ the weighted version of Π), the quality of its approximation behaviour is expressed by the ratio ρ of the value (size, or weight, or whatever) of the solution found by the algorithm to the optimal (objective) value of Π; the smallest such ratio over all graphs is the approximation ratio $\rho_{\mathtt{A}}$ of the algorithm; the best-known ratio ρ_Π over all PTAA's solving Π is the approximation ratio for Π. Instance-depending approximation ratios for **NP**-complete graph-problems are commonly expressed by means of (one or more) the three standard graph-parameters, namely the order n of the input graph, the maximum vertex-degree and the average vertex-degree of the input graph. In what follows, we will denote by $\rho_\Pi(n)$ (resp., $\rho_{\mathrm{W}\Pi}(n)$) the best-known n-depending approximation ratio for Π (resp., for WΠ). Note that since we deal with maximization problems ratio ρ is non-increasing in n.

Given a vertex-weighted graph $G = (V, E, \vec{w})$ of order n with weight-vector $\vec{w}$, we denote by w_i the weight of a vertex $v_i \in V$, and by $w_{\max}(G)$ and $w_{\min}(G)$ the largest and the smallest vertex-weights, respectively. We denote by $\beta_w(G)$ and by $\beta'_w(G)$ the weight of a best WΠ-solution and the weight of a maximal[2] Π-solution of G, respectively. For a $V' \subseteq V$, we denote by $G[V']$ the subgraph of G induced by V' and by n' its order. Given a subgraph $G[V^{(k)}]$ of G, we denote by $S^*(G[V^{(k)}])$ a maximum-size Π-solution of $G[V^{(k)}]$; also, given any Π-solution S' of a subgraph G' of G, we denote by $w(S')$ the weight of S'.

2 The generic result

This section is mainly devoted to the proof of the following theorem.

Theorem 1. *Consider a hereditary property π, an induced subgraph problem Π stated with respect to π and the weighted version* WΠ *of Π (we suppose that weights are positive). Let*

[1] In other words, a feasible solution of Π is a subset of the vertices of the input-graph.

[2] Throughout the paper we use the notation "maximal set" to denote a set which is maximal for the inclusion with respect to a given property π; in other words, a set S of items is maximal with respect to π if (a) S fulfils π and (b) if we add any item in S, then the resulting set does not fulfil π; a set is "maximum" with respect to π if it is the largest between the maximal sets (with respect to π).

$M > 2$ *be an integer and let* $V^{(i)} = \{v_j \in V : w_{\max}(G)/M^i < w_j \leq w_{\max}(G)/M^{i-1}\}$, $i = 1, \ldots$ *Finally, let* $x = \sup\{\ell : \beta_w(G[\cup_{1\leq i\leq \ell} V^{(i)}]) < \beta_w(G)/2, \beta_w(G[\cup_{1\leq i\leq \ell+1} V^{(i)}]) \geq \beta_w(G)/2\}$. *Then,*

$$\rho_{\mathrm{WII}}(n) \geq \max\left\{\frac{M^x}{2n}, \min\left\{\frac{M-2}{2M^2x}\rho_{\Pi}(n), \frac{1}{M^2}\rho_{\Pi}(n)\right\}\right\}.$$

Proof: For the sequel, we set $G_x = G[\cup_{1\leq i\leq x} V^{(i)}]$, $G_{x+1} = G[\cup_{1\leq i\leq x+1} V^{(i)}]$ and $G_d = G[V \setminus \cup_{1\leq i\leq x} V^{(i)}]$ (of course, $\beta_w(G_d) \geq \beta_w(G)/2$).

We first remove vertices v_k such that the graph $(\{v_k\}, \emptyset)$ does not satisfy π. Then, the following lemma holds.

Lemma 1. *There exists a PTAA for* WII *achieving approximation ratio* $M^x/(2n)$.

Proof: The algorithm claimed consists of simply taking $v^* = \mathrm{argmax}_{v_i\in V}\{w_i\}$ as WII-solution and then adding vertices in such a way that the solution finally obtained remains feasible[3] for G. Moreover, note that, since vertices v_k such that the graph $(\{v_k\}, \emptyset)$ does not satisfy π have been removed, the graph $(\{v^*\}, \emptyset)$ is already a feasible solution. Consequently,, $\beta'_w(G) \geq w_{\max}(G)$, $\beta_w(G) \leq 2\beta_w(G_d) \leq 2|S^*(G_d)|w_{\max}(G_d) \leq 2|S^*(G_d)|w_{\max}(G)/M^x$. Consequently,

$$\rho(n) = \frac{\beta'_w(G)}{\beta_w(G)} \geq \frac{M^x}{2|S^*(G_d)|} \geq \frac{M^x}{2n} \tag{1}$$

q.e.d. ∎

Remark 2. For every $i \geq 1$, the weight of any Π-solution $S^{(i)}$ of $G[V^{(i)}]$ lies in the interval $[|S^{(i)}|w_{\max}(G)/M^i, |S^{(i)}|w_{\max}(G)/M^{i-1}]$. It is at least $|S^{(i)}|w_{\min}(G[V^{(i)}]) \geq |S^{(i)}|w_{\max}(G)/M^i$ and at most $|S^{(i)}|w_{\max}(G[V^{(i)}]) \leq |S^{(i)}|w_{\max}(G)/M^{i-1}$. ∎

For the rest of the proof, we distinguish two cases with respect to $\beta_w(G_x)$, namely $\beta_w(G)/2 \geq \beta_w(G_x) \geq ((M-2)/2M)\beta_w(G)$ and $\beta_w(G_x) \leq ((M-2)/2M)\beta_w(G)$, respectively.

We first consider the case $\beta_w(G)/2 > \beta_w(G_x) \geq ((M-2)/2M)\beta_w(G)$. Then the following lemma holds.

Lemma 2. *Let* $\beta_w(G)/2 \geq \beta_w(G_x) > ((M-2)/2M)\beta_w(G)$, $p = \mathrm{argmax}_{1\leq i\leq x}\{\beta_w(G[V^{(i)}])\}$, $|V^{(p)}| = n^{(p)}$ *and* $S^*(G[V^{(p)}])$ *be the maximum-size* Π*-solution in* $G[V^{(p)}]$. *If there exists a PTAA for* Π *providing a maximal solution* $S^{(p)}$ *such that* $|S^{(p)}| \geq \rho_{\Pi}(n^{(p)})|S^*(G[V^{(p)}])|$, *then one can solve* WII *in* G, *in polynomial time, within ratio* $((M-2)/2xM^2)\rho_{\Pi}(n)$.

Proof: Obviously, $\beta_w(G_x) \leq x\beta_w(G[V^{(p)}])$ and $\beta_w(G_x) \leq x|S^*(G[V^{(p)}])|w_{\max}(G)/M^{p-1}$ (by Remark 2). So, $\beta_w(G) \leq (2M/(M-2))x|S^*(G[V^{(p)}])|(w_{\max}(G)/M^{p-1})$.

On the other hand, application of a PTAA guaranteeing approximation ratio $\rho_{\Pi}(n) < 1$ for Π in $G[V^{(p)}]$ constructs a solution $S^{(p)}$ of WII of weight at least $|S^{(p)}|w_{\min}(G[V^{(p)}]) \geq |S^{(p)}|w_{\max}(G)/M^p$. Note that (by Remark 1) $S^{(p)}$ is Π-feasible for G. Moreover, starting from this solution, one can greedily augment it in order to finally produce a maximal WII-solution for G. This final solution verifies $\beta'_w(G) \geq |S^{(p)}|w_{\max}(G)/M^p$.

Combination of the above expressions for $\beta'_w(G)$ and $\beta_w(G)$ yields

$$\rho(n) = \frac{\beta'_w(G)}{\beta_w(G)} \geq \left(\frac{M-2}{2xM^2}\right)\left(\frac{|S^{(p)}|}{|S^*(G[V^{(p)}])|}\right) \geq \frac{M-2}{2xM^2}\rho_{\Pi}(n^{(p)}) \geq \frac{M-2}{2xM^2}\rho_{\Pi}(n) \tag{2}$$

and this concludes the proof of the case $\beta_w(G)/2 > \beta_w(G_x) \geq ((M-2)/2M)\beta_w(G)$ and of the lemma. ∎

We now suppose that $\beta_w(G_x) \leq ((M-2)/2M)\beta_w(G)$ and prove the following lemma.

[3] In other words, this solution satisfies π.

Lemma 3. *Let $\beta_w(G_x) \leq ((M-2)/2M)\beta_w(G)$ and $S^*(G[V^{(x+1)}])$ be a maximum-size* Π*-solution of $G[V^{(x+1)}]$. If there exists a PTAA for* Π *providing a maximal* Π*-solution $S^{(x+1)}$ of cardinality at least $\rho_\Pi(n^{(x+1)})$-times the cardinality of $S^*(G[V^{(x+1)}])$, then one can solve* WΠ *in G, in polynomial time, within ratio $(1/M^2)\rho_\Pi(n)$.*

Proof: Note that since x is the largest ℓ for which $\beta_w(G[\cup_{1\leq i\leq \ell} V^{(i)}]) < \beta_w(G)/2$, set $V^{(x+1)}$ is not empty.

Let $S^{\rm opt}(G_{x+1})$ be an optimal WΠ-solution in G_{x+1} (i.e., $\beta_w(G_{x+1}) = w(S^{\rm opt}(G_{x+1}))$); let $S(G_x) = S^{\rm opt}(G_{x+1}) \cap V(G_x)$ (where $V(G_x)$ denotes the vertex-set of G_x) and $S(G[V^{(x+1)}]) = S^{\rm opt}(G_{x+1}) \cap V^{(x+1)}$ (in other words, $\{S(G_x), S(G[V^{(x+1)}])\}$ is a partition of $S^{\rm opt}(G_{x+1})$). Since π is hereditary, sets $S(G_x)$ and $S(G[V^{(x+1)}])$, being subsets of $S^{\rm opt}(G_{x+1})$, also verify π (and, consequently they are feasible WΠ-solutions for G_x and $G[V^{(x+1)}]$, respectively). We then have:

$$
\begin{array}{rcl}
w(S(G_x)) & \leq & \beta_w(G_x) \\
w(S(G[V^{(x+1)}])) & \leq & \beta_w(G[V^{(x+1)}]) \\
\beta_w(G_{x+1}) & = & w(S(G_x)) + w(S(G[V^{(x+1)}])) \leq \beta_w(G_x) + \beta_w(G[V^{(x+1)}]) \\
& \leq & \frac{M-2}{2M}\beta_w(G) + \beta_w(G[V^{(x+1)}]) \\
\beta_w(G_{x+1}) & \geq & \frac{\beta_w(G)}{2}.
\end{array}
$$

It follows from the above expressions that $\beta_w(G[V^{(x+1)}]) \geq \beta_w(G)/M$ and this together with Remark 2 yield, after some easy algebra, $\beta_w(G) \leq |S^*(G[V^{(x+1)}])| w_{\max}(G)/M^{x-1}$.

As previously, suppose that a PTAA provides, in polynomial time, a maximal solution $S^{(x+1)}$ for Π in $G[V^{(x+1)}]$, the cardinality of which is at least $\rho_\Pi(n^{(x+1)})|S^*(G[V^{(x+1)}])|$. Then, one can greedily augment this solution in order to produce a maximal Π-solution for G of total weight $\beta'_w(G) \geq |S^{(x+1)}| w_{\max}(G)/M^{x+1}$.

Combination of expressions for $\beta'_w(G)$ and $\beta_w(G)$ yields

$$
\rho(n) = \frac{\beta'_w(G)}{\beta_w(G)} \geq \left(\frac{1}{M^2}\right)\left(\frac{|S^{(x+1)}|}{|S^*(G[V^{(x+1)}])|}\right) \geq \frac{1}{M^2}\rho_\Pi(n^{(x+1)}) \geq \frac{1}{M^2}\rho_\Pi(n) \qquad (3)
$$

and this concludes the proof of the case $\beta_w(G_x) \leq ((M-2)/2M)\beta_w(G)$ and of the lemma. ∎

Remark 3. For the case where $x = 0$, i.e., $\beta_w(G[V^{(1)}]) \geq \beta_w(G)/2$, arguments similar to the ones of the proof of Lemma 3 lead to an approximation ratio $\rho_{\rm W\Pi}(n) = \beta'_w(G)/\beta_w(G) \geq \rho_\Pi(n)/2M$, better than the one of expression (3). ∎

Consider now the following algorithm where we take up the ideas of Lemmata 1, 2 and 3 and where, for a graph G', we denote by $\mathtt{A(G')}$ the solution-set provided by the execution of the Π-PTAA A on the unweighted version of a graph G'.

```
BEGIN (*WA*)
      fix a constant M > 2;
      partition V in sets V^(i) ← {v_k : w_max/M^i < w_k ≤ w_max/M^(i-1)};
      S^(0) ← {argmax_{v_i ∈ V}{w_i}};
      OUTPUT argmax{w(S^(0)), w(A(G[V^(i)])), i = 1,...};
END. (*WA*)
```

Revisit expressions (1), (2) and (3). It is easy to see that, since M is fixed, worst cases for the respective ratios are represented by expressions (1) and (2); so,

$$
\rho_{\rm W\Pi}(n) \geq \rho_{\rm WA}(n) \geq \max\left\{\frac{M^x}{2n}, \min\left\{\frac{M-2}{2M^2x}\rho_\Pi(n), \frac{1}{M^2}\rho_\Pi(n)\right\}\right\} \qquad (4)
$$

and this concludes the proof of Theorem 1 which, obviously, works also in the case where weights are exponential in n. ∎

The proof of Theorem 1 can be seen as a reduction transforming every PTAA A achieving ratio $\rho_{\mathtt{A}(n)}$ for Π into a PTAA for WΠ achieving, at worst, ratio $e = \Omega(\rho_{\mathtt{A}(n)}/x)$. Quantity $1/x$ is usually called expansion of the reduction (see [7]). By expression (1) and by the fact that the approximation ratio of any PTAA for WΠ must be less than 1 (WΠ being a maximization problem), $x = O(\log_M n)$ (in fact, as we will see in the next section, it can be much smaller depending on the form of $\rho_\Pi(n)$). But even such an expansion is a meaningful improvement with respect to expansions induced by reductions between weighted – no-weighted versions of problems which do not admit constant-ratio PTAA's. For example, for the case of IS, if we take into account that no PTAA can guarantee fixed constant ratio for it, reductions of [7] have expansion $1/n^2$.

3 Important corollaries of Theorem 1

3.1 Maximum-weight independent set and maximum-weight clique

Consider now one of the most classical **NP**-complete problems, the *maximum independent set*. Given a graph $G = (V, E)$, an independent set is a subset $V' \subseteq V$ such that whenever $\{v_i, v_j\} \subseteq V'$, $v_i v_j \notin E$, and the *maximum independent set problem (IS)* is to find an independent set of maximum size. As we have already noted in section 1, property "is an independent set" is hereditary (the subset of an independent set is an independent set); moreover, it is non-trivial (not all sets $V' \subseteq V$ are independent). A very well-known and largely studied generalization of IS is its weighted version (denoted by WIS); here, we associate positive weights with the vertices of G and the objective becomes to maximize the sum of the weights of an independent set.

In terms of n, the best known approximation ratio for IS is, to our knowledge, achieved by the IS-PTAA of Boppana and Halldórsson ([1]): $\rho_{\mathrm{IS}}(n) = \Theta(\log^2 n/n)$. Embedding it in expression (2), we get

$$\rho(n) \geq \left(\frac{M-2}{2xM^2}\right)\Theta\left(\frac{\log^2 n}{n}\right) = \Theta\left(\frac{\log^2 n}{nx}\right). \tag{5}$$

For instance, if $x \leq \log\log n$, expression (5) induces $\rho(n) \geq \Theta(\log^2 n/(n\log\log n))$, while, if $x \geq \log\log n$, then expression (1) guarantees $\rho(n) \geq \Theta((\log n)^{\log M}/n)$; on the other hand, expression (3) always guarantees $\rho(n) \geq \Theta(\log^2 n/n)$. So, by expression (4), the approximation ratio guaranteed by algorithm WA (supposing $M > 8$) called when used to solve WIS is, at worst,

$$\rho_{\mathtt{WA}}(n) \geq \Theta\left(\frac{\log^2 n}{n\log\log n}\right). \tag{6}$$

Expression (6) and the discussion above introduce the following corollary.

Corollary 1. $\rho_{\mathrm{WIS}}(n) = \Omega(\log^2 n/(n\log\log n))$.

The above result improves by a factor $O(\log\log n)$ the best-known approximation ratio (function of n) for WIS ($\Theta(\log^2 n/(n(\log\log n)^2))$, due to Halldórsson ([5])).

Finally, let us note that the same approximation result holds for another famous **NP**-complete problem, the maximum weighted clique (see [2] for the statement of this problem).

3.2 Maximum-weight ℓ-colorable induced subgraph

Given a graph $G = (V, E)$ and a positive constant ℓ, the problem of the *maximum ℓ-colorable induced subgraph* (denoted by Cℓ) is to find a maximum-order induced subgraph $G' = G[V']$ of G ($V' \subseteq V$) such that G' is ℓ-colorable (i.e., there exists a coloring for G' of cardinality at most ℓ). In the weighted version of Cℓ, denoted by WCℓ, positive weights are associated with the vertices of G and the objective becomes to find the maximum-weight ℓ-colorable induced subgraph. Once more, property "is ℓ-colorable" is hereditary (if the vertices of a graph G can be feasibly colored by at most ℓ colors, then every subgraph of G induced by a subset of its vertices can be colored by at most ℓ colors) and non-trivial (given a number ℓ every graph is not ℓ-colorable).

The best-known ratios for Cℓ are the ones of [4]: $\Theta((\log n)^2/n)$ (as function of n) and of [3]: $\min\{\kappa/\mu, \Theta((\log\log\Delta)/\Delta)\}$, for every fixed $\kappa \in I\!N^+$, where Δ, μ are the maximum and the average degrees of the graph, respectively (as degree-function). For WCℓ, no approximation result is, to our knowledge, mentioned until now.

Application of Theorem 1, using the algorithm of [4] as approximation algorithm for Cℓ (the algorithm A in algorithm WA of section 2), leads to the following corollary.

Corollary 2. $\rho_{\mathrm{WC}\ell} = \Omega(\log^2 n/(n\log\log n))$.

4 About continuous reductions between weighted and unweighted versions of independent set

In what follows, we use the term "continuous reduction", introduced by Simon ([7]), to denote the approximation-ratio-preserving reductions. Informally, given two maximization problems Π and Π', a reduction $\Pi \propto \Pi'$ is called *continuous with an expansion of e* if each polynomial-time ρ'-approximation algorithm A$'$ for Π' can be converted into a polynomial-time ρ-approximation algorithm A for Π such that $\rho \geq e\rho'$. The corresponding notation is $\Pi \stackrel{e}{\Longrightarrow} \Pi'$. It is very well-known and largely noted until now by many authors that the concept of continuous reductions is strongly based upon the very restrictive hypothesis that the approximation ratio ρ' of A$'$ has to be a fixed constant. Since IS does not admit such algorithms (in 1996 it has been proved that it is hard to approximate IS within $1/n^{1-\epsilon}$, for any $\epsilon > 0$, [6]), continuous reductions proposed until now between IS and WIS fail to produce WIS-ratios as good as the ones known for IS. In fact, the best known ratios for IS are $\Theta(\log^2 n/n)$ (in terms of n, [1]) and $\min\{\kappa/\mu, \Theta((\log\log\Delta)/\Delta)\}$, for every fixed $\kappa \in I\!N^+$, where Δ, μ are the maximum and the average degrees of the graph, respectively ([3]). On the other hand, the best corresponding ratios for WIS are $\Theta(\log^2 n/(n\log\log n))$ (proved in this paper) and $3/(\Delta+2)$ (see [5] for this last ratio), respectively. We think that devising a continuous reduction of constant expansion between IS and WIS, or proving that such a reduction does not exist, is a very interesting open problem, the solution of which is far from being obvious. In what follows, we give some easy cases of such reductions for restrictive WIS-classes recognizable in polynomial time. We denote by $\alpha_w(G)$ and by $\alpha'_w(G)$ the weight of a maximum-weight independent set and the weight of an approximately maximum-weight independent set of G, respectively.

Theorem 2. *Consider a fixed constant ℓ and graphs $G = (V, E)$ of order n where at least one of the following conditions is verified:*

1. *there exist at most ℓ distinct weights,*
2. $w_{\max}(G)/w_{\min}(G) \leq \ell$,
3. *weights are rational and fixed constants,*

4. $w_{\max}(G) \geq nw'$, *where by* w' *we denote the second largest vertex weight.*

For all these graph-families, there exist continuous reductions of constant expansion between IS and WIS. Consequently, in all the above families of graphs, WIS can be approximated in polynomial time within

$$\rho_{\text{WIS}} = \rho_{\text{IS}} \geq \max\left\{\Theta\left(\frac{\log^2 n}{n}\right), \min\left\{\frac{\kappa}{\mu}, \Theta\left(\frac{\log\log\Delta}{\Delta}\right)\right\}\right\}$$

for every fixed $\kappa \in \mathbb{N}^+$.

Proof: We consider the ℓ subgraphs, $G^{(1)}, \ldots, G^{(\ell)}$, of G, induced by the vertices of the same weight. Let $p = \text{argmax}_{1\leq i\leq \ell}\{\alpha_w(G^{(i)})\}$ and w_p be the weight of the vertices of $G^{(p)}$; then, $\alpha_w(G) \leq \ell\alpha_w(G^{(p)}) = \ell w_p|S^*(G^{(p)})|$. On the other hand, one can easily take $\alpha'_w(G) = \max_{1\leq i\leq \ell}\{w_i|S^{(i)}|\}$, where $S^{(i)}$ is the largest among the independent sets provided by the algorithms of [1] and of [3] for IS. It is easy to see that, in this case, $\rho_{\text{WIS}} = \alpha'_w(G)/\alpha_w(G) \geq w_p|S^{(p)}|/(\ell w_p|S^*(G^{(p)})|) \geq \rho_{\text{IS}}/\ell$ and that this ratio verifies the final statement of the theorem, concluding so the proof of item 1.

For item 2, it suffices to remove weights from the vertices of graph and to solve IS in G. Let $S(G)$ be the largest among the independent sets provided by the algorithms of [1] and of [3] for IS and $S^*(G)$ be the optimal one in G. Then, it is easy to see that $\rho_{\text{WIS}} = \alpha'_w(G)/\alpha_w(G) \geq w_{\min}(G)|S(G)|/(w_{\max}(G)|S^*(G)|) \geq \rho_{\text{IS}}/\ell$. Once more the final statement of the theorem is verified and the proof of item 2 is concluded.

The proof of item 3 is standard and already used by many authors (see for example [7] for the proof of the result $\text{WIS}(k) \stackrel{1-\epsilon}{\Longrightarrow} \text{IS}$, where $\text{WIS}(k)$ denotes the version of WIS where vertex-weights are bounded above by n^k, $k \geq 1$; this proof unfortunately works only for fixed-constant ratios). Given an instance G_w of WIS, one can construct an instance G of IS by replacing every vertex v_i of weight w_i by an independent set W_i of size w_i; next, if two vertices v_i and v_j are linked by an edge in G_w, one draws a complete bipartite graph between the two independent sets W_i and W_j of sizes w_i and w_j, respectively (in the case where weights are rationals we simply replace each weight by itself multiplied by the LCM of the weight-denominators). Now, note that if an algorithm delivers a solution for G (instance of IS), one can directly obtain a solution of G_w by simply replacing the independent sets of the former by the corresponding vertices of the latter, this operation leading to a solution for G_w with the same value as the cardinality of the solution for G. Denoting by n_w and Δ_w the order and the maximum vertex-degree of G_w, respectively, and by n and Δ the corresponding parameters of G, we have $n \leq w_{\max}(G)n_w$, $\Delta \leq w_{\max}(G)\Delta_w$. It is easy to see that following the discussion just above and thanks to the fact that the integral weights are fixed constants, applying the algorithms of [1] and [3] in G and retaining the best of the computed solutions, achieves for WIS in G_w the same orders of ratios as for IS in G, concluding so the proof of item 3.

For the proof of item 4, let us denote by G_m the subgraph of G induced by the vertices of weight $w_{\max}(G)$ and by $S(G_m)$ the best of the solutions provided when running the algorithms of [1] and [3] on G_m. Obviously $\alpha'_w(G) \geq \alpha'_w(G_m) \geq w_{\max}(G)|S(G_m)|$. On the other hand, $\alpha_w(G) \leq \alpha_w(G_m) + (n - |S^*(G_m)|)w' \leq w_{\max}(G)|S^*(G_m)| + (n - |S^*(G_m)|)w_{\max}(G)/n$. Divising, member-by-member, the above expressions for $\alpha'_w(G)$ and $\alpha_w(G)$, we get

$$\rho_{\text{WIS}} = \frac{\alpha'_w(G)}{\alpha_w(G)} \geq \frac{|S(G_m)|}{|S^*(G_m)| + \frac{n-|S^*(G_m)|}{n}} \geq \frac{|S(G_m)|}{|S^*(G_m)| + 1}.$$

Let us now consider any fixed $\epsilon \geq 0$. The above expression for ρ_{WIS} is greater than, or equal to, $\rho_{\text{IS}}(1-\epsilon)$ for $|S^*(G_m)| \geq (1-\epsilon)/\epsilon$. On the other hand, if $|S^*(G_m)| \leq (1-\epsilon)/\epsilon$, then $|S^*(G_m)|$

being a fixed constant, it can be computed in polynomial time by exhaustive search. In this case, $\alpha'_w(G)/\alpha_w(G) \geq |S^*(G_m)|/(|S^*(G_m)|+1) \geq 1/2$ (since $|S^*(G_m)| \geq 1$). Consequently, once more, the final statement of the theorem is verified and this concludes the proof of item 4 and of the theorem. ∎

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