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Le stable maximum pondéré est approximable « aussi bien » que le stable maximum non-pondéré

Résumé

Nous présentons tout d'abord une réduction d'expansion $O(\log n)$ (où n est l'ordre du graphe) préservant le rapport d'approximation entre les versions pondérée et non pondérée d'une classe de problèmes de maximisation consistant à trouver un sous-graphe induit de poids maximum vérifiant une propriété héréditaire. Cette réduction nous permet d'effectuer une première amélioration du meilleur rapport connu pour le problème WIS du stable maximum pondéré. Ensuite, en s'appuyant sur la réduction proposée, nous concevons un algorithme approché polynomial dont le rapport est égal au minimum de $O(\log^k n/n)$ et $O(\log \mu / (k^2 \mu (\log \log \mu)^2))$ où μ désigne le degré moyen du graphe pour toute constante k . Dans les deux cas, l'amélioration du rapport est significative : dans le premier cas, le rapport pour WIS surclasse $O(\log^2 n/n)$, le meilleur rapport connu pour IS (le problème du stable non-pondéré) tandis que, dans le second cas, nous obtenons le premier rapport de l'ordre de $\Omega(1/\Delta)$ pour WIS (où Δ est le degré maximum du graphe ; le meilleur rapport connu en fonction de Δ valait jusqu'ici $3/(\Delta + 2)$). Par la suite, à partir du problème de coloration, nous trouvons un algorithme approché polynomial pour WIS ayant pour rapport le minimum entre $O(n^{-4/5})$ et $O(\log \log \Delta / \Delta)$. Ainsi, à moins que WIS ne soit approximable à $O(n^{-4/5})$ (ce qui est peu probable, l'approximation de IS à rapport $n^{\epsilon-1}$ étant difficile pour tout $\epsilon > 0$), notre algorithme obtient le premier rapport en $\Omega(\log \log \Delta / \Delta)$ pour WIS (pour tout Δ). Notons que l'approximation du problème du stable pondéré ou non pondéré à rapport $\Omega(1/\Delta)$ demeurerait pendant longtemps un problème ouvert. Enfin, nous proposons les premiers rapports non triviaux en $\Omega(1/\Delta)$ pour les problèmes de la clique maximum pondérée et non pondérée, le problème du sous-graphe induit l -colorable de poids maximum et le problème de la somme chromatique.

Mots-clé : problèmes combinatoires, complexité, algorithme polynomial d'approximation, NP-complétude, problème héréditaire, stable, clique, graphe induit l -colorable, somme chromatique.

Maximum-weight independent set is as “well-approximated” as the unweighted one

Abstract

We first devise an approximation-preserving reduction of expansion $O(\log n)$ (where n is the order of the input-graph) between weighted and unweighted versions of a class of problems called weighted hereditary induced-subgraph maximization problems. This allows us to perform a first improvement of the best approximation ratio for the weighted independent set problem (WIS). Then, using the reduction developed, we propose a polynomial time approximation algorithm for WIS achieving as ratio the minimum between $O(\log^k n/n)$ and $O(\log \mu / (k^2 \mu (\log \log \mu)^2))$ where μ denotes the average graph-degree of the input-graph, for every constant k . In any of the two cases, this is an important improvement since in the former one, the ratio for WIS outperforms $O(\log^2 n/n)$, the best-known ratio for (unweighted) independent set problem (IS), while in the latter case, we obtain the first $\Omega(1/\Delta)$ ratio for WIS (where Δ is the maximum graph-degree; the best-known ratio in terms of Δ was $3/(\Delta + 2)$). Next, based upon graph-coloring, we devise a polynomial time approximation algorithm for WIS achieving ratio the minimum between $O(n^{-4/5})$ and $O(\log \log \Delta / \Delta)$. Here also, except for the very unlikely case where WIS can be approximated within $O(n^{-4/5})$ (approximation of IS within $n^{\epsilon-1}$ is hard for every $\epsilon > 0$), our algorithm is the first $\Omega(\log \log \Delta / \Delta)$ -approximation algorithm for WIS (for every Δ). Let us note that approximation of both independent set versions within ratios $\Omega(1/\Delta)$ is a very well-known open problem. Finally, we propose the first non-trivial $\Omega(1/\Delta)$ ratios for maximum-size and maximum-weight clique, for maximum-weight l -colorable induced subgraph and for chromatic sum.

Keywords: combinatorial problems, computational complexity, polynomial-time approximation algorithm, NP-completeness, hereditary problem, independent set, clique, l -colorable induced subgraph, chromatic sum.

1 Introduction

1.1 The framework of the paper and the problems studied

Given a graph $G = (V, E)$ and a set $V' \subseteq V$, a subgraph of G induced by V' is a graph $G' = (V', E')$, where $E' = (V' \times V') \cap E$. We consider **NP**-hard graph-problems Π where the objective is to find a maximum-order induced subgraph G' satisfying a non-trivial hereditary property π . For a graph G , anyone of its vertex-subsets specifies exactly one induced subgraph. Consequently, in what follows we consider that a feasible solution for Π is the vertex-set of G' . Let $\mathcal{G}$ be the class of all the graphs. A graph-property π is a mapping from $\mathcal{G}$ to $\{0, 1\}$, i.e., for a $G \in \mathcal{G}$, $\pi(G) = 1$ iff G satisfies π and $\pi(G) = 0$, otherwise. Property π is hereditary if whenever it is satisfied by a graph it is also satisfied by every one of its induced subgraphs; it is non-trivial if it is true for infinitely many graphs and false for infinitely many ones ([4]). Hereditary induced-subgraph maximization problems have a natural generalization to graphs with positive integral weights associated with their vertices (the weights are assumed to be bounded by 2^n – the order of the input-graph – so that every arithmetical operation on them can be performed in polynomial time). Given a graph G , the objective of a weighted induced subgraph problem is to determine an induced subgraph G^* of G such that G^* satisfies π and, moreover, the sum of the weights of the vertices of G^* is the largest possible among those subgraphs. In what follows, we denote by **WII** the weighted version of Π .

More particularly, we consider in this paper the following induced-subgraph problems.

Maximum independent set. Given a graph $G = (V, E)$ of order n , an independent set is a subset $V' \subseteq V$ such that no two vertices in V' are linked by an edge in E , and the maximum independent set problem (IS) is to find a maximum-size independent set.

Maximum clique. Consider a graph $G = (V, E)$. A clique of G is a subset $V' \subseteq V$ such that every pair of vertices of V' are linked by an edge in E , and the maximum clique problem (KL) is to find a maximum size set V' inducing a clique in G (a maximum-size clique).

Maximum l -colorable subgraph. Given a graph $G = (V, E)$ and a positive constant l , the problem of the *maximum l -colorable induced subgraph* (denoted by **Cl**) is to find a maximum-order induced subgraph $G' = G[V']$ of G ($V' \subseteq V$) such that G' is l -colorable (i.e., there exists a coloring for G' of cardinality at most l).

Property “is an independent set” is hereditary (the subset of an independent set is an independent set). The same holds for property “is a clique” (the vertex-subset of a clique induces also a clique), as well as for property “is ℓ -colorable” (if the vertices of a graph G can be feasibly colored by at most ℓ colors, then every subgraph of G induced by a subset of its vertices can be colored by at most ℓ colors).

We also consider the following minimization problem and its weighted version.

Minimum chromatic sum. Given a graph $G = (V, E)$, an l -coloring is a partition of V into independent sets $C_1, \dots, C_l$. The cost of an l -coloring is the quantity $\sum_{i=1}^l i|C_i|$ (in other words, the cost of coloring a vertex $v \in V$ with color i is i). The *minimum chromatic sum* problem, denoted by **CHS** is to determine a minimum-cost coloring. For the weighted version of **CHS**, denoted by **WCHS**, every vertex $v \in V$ is weighted by a rational weight w_v , the cost of coloring v with color i becomes iw_v , the value of an l -coloring becomes $\sum_{i=1}^l iw(C_i)$, where $w(C_i) = \sum_{v \in C_i} w_v$, and the objective becomes now to determine a coloring of minimum value.

Given a problem Π defined on a graph $G = (V, E)$, and its weighted version $\text{W}\Pi$, we denote by $\vec{w} \in \mathbb{N}^{|V|}$ the vector of the weights, by w_v the weight of $v \in V$ and by $w_{\max}(G)$ and $w_{\min}(G)$ the largest and the smallest vertex-weight, respectively. Moreover, we adopt the following notations:

- $w(V')$: the total weight of $V' \subseteq V$, i.e., the quantity $\sum_{v \in V'} w_v$;
- $\beta_w(G)$: the value of an optimal solution for $\text{W}\Pi$;
- $\Delta(G)$: the maximum degree of G ;
- $\mu(G)$: the average degree of G ;
- $\mu_w(G)$: the quantity $\sum_{v \in V} w(\Gamma(v))/w(V)$
- $\Gamma(v)$: the neighborhood of $v \in V$;
- $\chi(G)$: the chromatic number of G (the minimum number of colors with which one can feasibly color the vertices of G);
- $\bar{G}$: the complement of G defined by $\bar{G} = (V, \bar{E})$ with $\bar{E} = \{ij \in V \times V, i \neq j, ij \notin E\}$ (obviously, $\bar{\bar{G}} = G$);
- $G[V']$: the subgraph of G induced by $V' \subseteq V$;
- $n^{(k)}$: the order of the graph $G[V^{(k)}]$, $V^{(k)} \subseteq V$;
- $S^*(G[V^{(k)}])$: an optimal (maximum-size) Π -solution in $G[V^{(k)}]$, $V^{(k)} \subseteq V$.

Especially for IS and WIS, using standard notations, we will denote the size of a maximum independent set by $\alpha(G)$ and the value of a maximum-weight independent set by $\alpha_w(G)$.

Moreover, when no ambiguity can occur, we will use Δ , μ and μ_w instead of $\Delta(G)$, $\mu(G)$ and $\mu_w(G)$

Given a square matrix $B = (m_{ij})_{i,j=1,\dots,n}$, we denote by $\text{Tr}(B)$ and tB the trace and the transpose of B , respectively. Finally, given a vector $\vec{u}$, we denote by $|\vec{u}|$ its Euclidean norm.

1.2 Our main contributions

We are interested in devising polynomial time approximation algorithms (PTAAs) computing solutions, the values of which are as close as possible to the value of an optimal one. The quality of a PTAA is expressed, for every instance G , by the ratio of the size (resp., weight) of the computed solution to the optimal size (resp., weight). Given a PTAA $\mathbf{A}$, this ratio will be denoted by $\rho_{\mathbf{A}}(G)$. For a maximization (resp., minimization) problem Π , $\mathbf{A}$ is said to guarantee an approximation ratio $\rho(G)$ if for every instance G , $\rho_{\mathbf{A}}(G)$ is bounded below (resp., above) by $\rho(G)$ ($\rho(G)$ denotes approximation ratios depending on some parameters of G). It is assumed that $\rho(G) \leq \rho(G')$ (resp., $\rho(G) \geq \rho(G')$), for any subgraph G' of G . Finally, we denote by $\rho_{\Pi}(G)$ the best approximation ratio known for Π .

In what follows, we will denote by $\beta'_w(G)$ the weight of an approximated $\text{W}\Pi$ -solution of G . Once more, this weight will be denoted by $\alpha'_w(G)$ when dealing with WIS.

For IS, which is the most famous among the problems studied here, the strongest inapproximability result is the one in [11] affirming that $0 < \epsilon \leq 1$, IS is not approximable within $n^{\epsilon-1}$ unless $\mathbf{NP} = \mathbf{ZPP}$. Concerning positive approximation results for IS, we give in [5] a PTAA guaranteeing asymptotic ($\Delta(G) \rightarrow \infty$) approximation ratio $\min\{k/\mu(G), k'(\log \log \Delta(G))/\Delta(G)\}$ (where k' depends on k) for every constant k . For WIS, the best-known approximation in terms

of maximum degree is $3/(\Delta(G) + 2)$ ([8]). A particularly interesting question frequently mentioned in the relative literature about WIS-approximation is if it can be as well-approximated as IS. Here we prove that the answer is not so far from being true.

The paper is organized as follows. In section 2, we devise a reduction of expansion $O(\log n)$ from weighted hereditary induced-subgraph problems to unweighted ones. This expansion can be improved when dealing with particular problems; for example, it is $O(\log \log n)$ when dealing with the pair IS-WIS (paragraph 2.2). Based upon this reduction we draw first improvements for the ratio of WIS. In section 3, we propose new improved approximation results for IS. Next, always based upon the reduction of section 2, we further improve approximations for WIS and obtain an approximation ratio for WIS with value greater than the minimum between $O(\log^k n/n)$ and $O(\log \mu(G)/(k^2 \mu(G) \log^2 \log \mu(G)))$. This is an important improvement since if the WIS-ratio obtained is $O(\log^k n/n)$, it outperforms $O(\log^2 n/n)$, the best-known ratio for IS. On the other hand, if the ratio obtained is $O(\log \mu(G)/(k^2 \mu(G) \log^2 \log \mu(G)))$, then we achieve the first $\Omega(1/\Delta(G))$ ratio for WIS. Let us note that this is the first time that non trivial results for WIS are produced by a reduction to IS. In section 4, we propose another PTAA for WIS which further improves results of section 3. Next, we generalize a result of [1] by linking the approximation of the class WIS_k of WIS-instances with weighted independence number greater than $w(G)/k$ to the approximation of a class $\mathcal{G}_l$ of graph-coloring instances including the l -colorable graphs. Combining this result with recent works of [12, 15] about $\mathcal{G}_l$, we obtain the main result of the paper, i.e., an approximation ratio for WIS of value greater than the minimum between $O(n^{-4/5})$ and $O(\log \log \Delta(G)/\Delta(G))$. Consequently, except for the very unlikely case where WIS can be approximated within $O(n^{-4/5})$ (recall that approximation of IS within $n^{\epsilon-1}$ is hard for every $\epsilon > 0$, unless $\text{NP}=\text{ZPP}$ ([11])), our algorithm is the first $\Omega(\log \log \Delta(G)/\Delta(G))$ -approximation algorithm for WIS. In section 5, we devise a new reduction between KL and WKL and, based upon it and using our results on WIS, we deduce the first $\Omega(1/\Delta(G))$ approximation ratio for the maximum-size and maximum-weight clique problems. We note once more that the results of this paper work even for unbounded values of $\Delta(G)$. Finally, in section 6 we improve recent approximation results for WCl and WCHS.

2 Reducing the weighted case to the unweighted one

2.1 Hereditary induced-subgraph problems

Theorem 1. *Consider a hereditary property π , an induced subgraph problem Π stated with respect to π and the weighted version $\text{W}\Pi$ of Π (we suppose that weights are positive). For every fixed $M > 2$, every PTAA for Π achieving ratio $\rho_\Pi(G)$ can be transformed into a PTAA for $\text{W}\Pi$ achieving*

$$\rho_{\text{W}\Pi}(G) \geq \Theta\left(\frac{\rho_\Pi(G)}{\log_M n}\right).$$

Proof. Set, for $i = 1, \dots$,

$$\begin{aligned} V^{(i)} &= \left\{ v_j \in V : \frac{w_{\max}(G)}{M^i} < w_j \leq \frac{w_{\max}(G)}{M^{i-1}} \right\} \\ x &= \sup \left\{ \ell : \beta_w \left(G \left[\bigcup_{1 \leq i \leq \ell} V^{(i)} \right] \right) < \frac{\beta_w(G)}{2} \right\} \\ G_x &= G \left[\bigcup_{1 \leq i \leq x} V^{(i)} \right] \\ G_{x+1} &= G \left[\bigcup_{1 \leq i \leq x+1} V^{(i)} \right] \end{aligned}$$

$$G_d = G \left[V \setminus \bigcup_{1 \leq i \leq x} V^{(i)} \right].$$

Of course, $\beta_w(G_d) \geq \beta_w(G)/2$ and $\beta_w(G_{x+1}) \geq \beta_w(G)/2$.

We first remove vertices v_k such that the graph $(\{v_k\}, \emptyset)$ does not satisfy π . Then, the following lemma holds.

Lemma 1. *There exists a PTAA for WII achieving approximation ratio $M^x/(2n)$.*

Proof of lemma 1. The algorithm claimed consists of simply taking $v^* \in \operatorname{argmax}_{v_i \in V} \{w_i\}$ as WII-solution. Then, $\beta'_w(G) = w_{\max}(G)$ and

$$\beta_w(G) \leq 2\beta_w(G_d) \leq 2|S^*(G_d)| w_{\max}(G_d) \leq 2|S^*(G_d)| \frac{w_{\max}(G)}{M^x}.$$

Consequently,

$$\frac{\beta'_w(G)}{\beta_w(G)} \geq \frac{M^x}{2|S^*(G_d)|} \geq \frac{M^x}{2n} \implies \rho_{\text{WII}}(G) \geq \frac{M^x}{2n} \quad (1)$$

q.e.d. ■

Remark 1. For every $i \geq 1$, the weight of any Π -solution $S^{(i)}$ of $G[V^{(i)}]$ lies in the interval $[|S^{(i)}|w_{\max}(G)/M^i, |S^{(i)}|w_{\max}(G)/M^{i-1}]$: it is at least $|S^{(i)}|w_{\min}(G[V^{(i)}]) \geq |S^{(i)}|w_{\max}(G)/M^i$ and at most $|S^{(i)}|w_{\max}(G[V^{(i)}]) \leq |S^{(i)}|w_{\max}(G)/M^{i-1}$. ■

Let us now prove the following lemma which is the central part of the proof of the theorem.

Lemma 2. *Assume $x > 0$.*

1. *Let $\beta_w(G)/2 \geq \beta_w(G_x) \geq ((M-2)/2M)\beta_w(G)$ and $p \in \operatorname{argmax}_{1 \leq i \leq x} \{\beta_w(G[V^{(i)}])\}$. If there exists a PTAA for Π guaranteeing approximation ratio $\rho_{\Pi}(G[V^{(p)}])$ in $G[V^{(p)}]$, then one can solve WII in G , in polynomial time, within ratio $((M-2)/2xM^2)\rho_{\Pi}(G)$.*
2. *Let $\beta_w(G_x) \leq ((M-2)/2M)\beta_w(G)$. If there exists a PTAA for Π guaranteeing ratio $\rho_{\Pi}(G[V^{(x+1)}])$ in $G[V^{(x+1)}]$, then one can solve WII in G , in polynomial time, within ratio $(1/M^2)\rho_{\Pi}(G)$.*

Proof of item 1. Obviously,

$$\beta_w(G_x) \leq x\beta_w(G[V^{(p)}]) \stackrel{\text{remark 1}}{\leq} x|S^*(G[V^{(p)}])| \frac{w_{\max}(G)}{M^{p-1}}$$

and, by the hypothesis of the item,

$$\beta_w(G) \leq \frac{2M}{M-2}\beta_w(G_x) \leq \frac{2M}{M-2}x|S^*(G[V^{(p)}])| \frac{w_{\max}(G)}{M^{p-1}}.$$

On the other hand, application of a PTAA guaranteeing approximation ratio $\rho_{\Pi}(G) < 1$ for Π in $G[V^{(p)}]$ constructs a solution $S^{(p)}$ of WII of weight at least

$$|S^{(p)}|w_{\min}(G[V^{(p)}]) \geq |S^{(p)}| \frac{w_{\max}(G)}{M^p}.$$

Note that $S^{(p)}$ is Π -feasible for G . Moreover, starting from this solution, one can greedily augment it in order to finally produce a maximal WII-solution for G . This final solution verifies $\beta'_w(G) \geq |S^{(p)}|w_{\max}(G)/M^p$.

Combination of the above expressions for $\beta'_w(G)$ and $\beta_w(G)$ yields

$$\frac{\beta'_w(G)}{\beta_w(G)} \geq \left(\frac{M-2}{2xM^2} \right) \left(\frac{|S^{(p)}|}{|S^*(G[V^{(p)}])|} \right) \geq \frac{M-2}{2xM^2} \rho_\Pi(G[V^{(p)}]) \geq \frac{M-2}{2xM^2} \rho_\Pi(G)$$

and, consequently,

$$\rho_{W\Pi}(G) \geq \frac{M-2}{2xM^2} \rho_\Pi(G). \quad (2)$$

This concludes the proof of item 1. ■

Proof of item 2. We now suppose that $\beta_w(G_x) \leq ((M-2)/2M)\beta_w(G)$. Note that since x is the largest ℓ for which $\beta_w(G[\cup_{1 \leq i \leq \ell} V^{(i)}]) < \beta_w(G)/2$, set $V^{(x+1)}$ is non-empty.

Let $S^{\text{opt}}(G_{x+1})$ be an optimal WII-solution in G_{x+1} (i.e., $\beta_w(G_{x+1}) = w(S^{\text{opt}}(G_{x+1}))$). Let $S(G_x) = S^{\text{opt}}(G_{x+1}) \cap V(G_x)$ (where $V(G_x)$ denotes the vertex-set of G_x) and $S(G[V^{(x+1)}]) = S^{\text{opt}}(G_{x+1}) \cap V^{(x+1)}$ (in other words, $\{S(G_x), S(G[V^{(x+1)}])\}$ is a partition of $S^{\text{opt}}(G_{x+1})$). Since π is hereditary, sets $S(G_x)$ and $S(G[V^{(x+1)}])$, being subsets of $S^{\text{opt}}(G_{x+1})$, also verify π (and, consequently they are feasible WII-solutions for G_x and $G[V^{(x+1)}]$, respectively). We then have:

$$\begin{aligned} w(S(G_x)) &\leq \beta_w(G_x) \\ w(S(G[V^{(x+1)}])) &\leq \beta_w(G[V^{(x+1)}]) \\ \beta_w(G_{x+1}) &= w(S(G_x)) + w(S(G[V^{(x+1)}])) \leq \beta_w(G_x) + \beta_w(G[V^{(x+1)}]) \\ &\leq \frac{M-2}{2M} \beta_w(G) + \beta_w(G[V^{(x+1)}]) \end{aligned}$$

and also $\beta_w(G_{x+1}) \geq \beta_w(G)/2$. It follows from the above expressions that $\beta_w(G[V^{(x+1)}]) \geq \beta_w(G)/M$ and this together with Remark 1 yield, after some easy algebra,

$$\beta_w(G) \leq |S^*(G[V^{(x+1)}])| \frac{w_{\max}(G)}{M^{x+1}}.$$

As previously, suppose that a PTAA provides, in polynomial time, a solution $S^{(x+1)}$ for Π in $G[V^{(x+1)}]$, the cardinality of which is at least $\rho_\Pi(G[V^{(x+1)}])|S^*(G[V^{(x+1)}])|$. Then, $\beta'_w(G) \geq |S^{(x+1)}|w_{\min}(G[V^{(x+1)}]) \geq |S^{(x+1)}|w_{\max}(G)/M^{x+1}$.

Combination of expressions for $\beta'_w(G)$ and $\beta_w(G)$ yields

$$\frac{\beta'_w(G)}{\beta_w(G)} \geq \left(\frac{1}{M^2} \right) \left(\frac{|S^{(x+1)}|}{|S^*(G[V^{(x+1)}])|} \right) \geq \frac{1}{M^2} \rho_\Pi(G[V^{(x+1)}]) \geq \frac{1}{M^2} \rho_\Pi(G).$$

Therefore,

$$\rho_{W\Pi}(G) \geq \frac{1}{M^2} \rho_\Pi(G) \quad (3)$$

and this concludes the proof of item 2 and of the lemma. ■

Remark 2. For the case where $x = 0$, i.e., $\beta_w(G[V^{(1)}]) \geq \beta_w(G)/2$, arguments similar to the ones of the proof of item 2 in lemma 2 lead to $\rho_{W\Pi}(G) = \beta'_w(G)/\beta_w(G) \geq \rho_\Pi(G)/2M$, better than the one of expression (3). ■

Consider now the following algorithm where we take up the ideas of lemmata 1 and 2 and where, for a graph G' , we denote by $\mathbf{A}(G')$ the solution-set provided by the execution of the Π -PTAA $\mathbf{A}$ on the unweighted version of G' .

```

BEGIN (*WA*)
  fix a constant M > 2;
  partition V in sets  $V^{(i)} \leftarrow \{v_k : w_{\max}/M^i < w_k \leq w_{\max}/M^{i-1}\};$ 
   $S^{(0)} \leftarrow v^* \in \{\operatorname{argmax}_{v_i \in V} \{w_i\}\};$ 
  OUTPUT  $\operatorname{argmax}\{w(S^{(0)}), w(A(G[V^{(i)}])), i = 1, \dots\};$ 
END. (*WA*)

```

Revisit expressions (1), (2) and (3). It is easy to see that

$$\rho_{W\Pi}(G) \geq \rho_{WA}(G) \geq \max \left\{ \frac{M^x}{2n}, \min \left\{ \frac{M-2}{2M^2x} \rho_{\Pi}(G), \frac{1}{M^2} \rho_{\Pi}(G) \right\} \right\}. \quad (4)$$

By expression (1) and by the fact that the approximation ratio of any PTAA for WII must be less than 1 (WII being a maximization problem), $x = O(\log_M n)$. Taking this value for x into account in expression 4, concludes the proof of the theorem which, obviously, works also in the case where weights are exponential in n . ■

2.2 A first improvement for the approximation of the maximum-weight independent set

It is well-known ([16]) that, $\forall k \geq 1$, the general weighted independent set problem polynomially reduces to PWIS(k) (the WIS-subproblem where the weights are bounded by n^k), by a simple scaling and rounding process. This reduction preserves (within a factor of $(1 - \epsilon)$) the ratios for WIS and PWIS, and works also for instance-depending ratios. On the other hand, the following approximation preserving reduction from PWIS to IS *working only for constant ratios* is established in [16].

Definition 1. Given a weighted graph $(G = (V, E), \vec{w})$ an unweighted graph $G_w = (V_w, E_w)$ can be constructed in the following way:

$$\begin{aligned}
V_w &= \{(u, i) : u \in V, i \in \{1, \dots, w_u\}\} \\
E_w &= \{(u, i)(v, j) : i \in \{1, \dots, w_u\}, j \in \{1, \dots, w_v\}, u \neq v, uv \in E\}.
\end{aligned}$$

In other words, every vertex u of V is replaced by an independent set of size w_u in G_w and every edge of E corresponds to a complete bipartite graph in G_w . ■

One can easily show that every independent set S of G of total weight $w(S)$ induces, in G_w , the independent set $\{(s, i) : s \in S, i \in \{1, \dots, w_s\}\}$ of size $w(S)$, and conversely, for every independent set S_w of G_w , the set $S = \{u \in V : \exists i \in \{1, \dots, w_u\}, (u, i) \in S_w\}$ is an independent set of weight $w(S) \geq |S_w|$. Consequently, $\alpha_w(G) = \alpha(G_w)$ and, by applying a $\rho(G)$ -approximation IS-algorithm to G_w , one can derive an approximated WIS-solution of $(G, \vec{w})$ guaranteeing ratio $\rho(G_w)$.

By the above reduction, a ratio $\rho(n, \Delta)$, non-increasing in Δ , for IS transforms to a ratio $\rho(w(V), w_{\max}(G)\Delta)$ for WIS and, except from the case of constant approximation ratios, the reduction above results in WIS-ratios depending on the weights. More precisely, the following result can be easily proved.

Proposition 1. For every $\epsilon > 0$ and every constant $k > 0$, there exists a polynomial reduction from WIS to IS transforming every IS-approximation ratio $\rho(n, \Delta, \mu)$ (where ρ is non increasing with respect to its variables) into an approximation ratio $\rho(n^{1+\epsilon}w(V)/w_{\max}(G), n^{1+\epsilon}\Delta, \mu_w) \geq \rho(n^{2+\epsilon}, n^{1+\epsilon}\Delta, \Delta)$ for WIS.

Unfortunately, the approximation results known for IS do not allow achievement of interesting approximation ratios for WIS using the reduction of proposition 1.

Revisit expression 4 and let Π be IS. Set, for every k , $M = 6$. Then:

- if $x \geq k \log \log n / \log M$, then $\rho_{\text{WIS}}(G) \geq \log^k n / 2n$;
- if $x \leq k \log \log n / \log M$, then $\rho_{\text{WIS}}(G) \geq 0.099 \rho_{\text{IS}}(G) / (k \log \log n)$

and the following theorem holds.

Theorem 2. *For every fixed ℓ , every PTAA for IS achieving ratio $\rho_{\text{IS}}(G)$ can be transformed into a PTAA for WIS achieving*

$$\rho_{\text{WIS}}(G) \geq \min \left\{ \frac{\log^\ell n}{2n}, \frac{0.099 \rho_{\text{IS}}(G)}{\ell \log \log n} \right\}.$$

In terms of n , the best-known approximation ratio for IS is, to our knowledge, $\Theta(\log^2 n / n)$ achieved by the IS-PTAA of [3]. Embedding it in $\rho_{\text{IS}}(G)$ -expression of theorem 2, we obtain the following concluding theorem.

Theorem 3.

$$\rho_{\text{WIS}}(G) \geq \Theta \left(\frac{\log^2 n}{n \log \log n} \right).$$

The above result improves by a factor $O(\log \log n)$ the best-known approximation ratio function of n for WIS ($O(\log^2 n / (n(\log \log n)^2))$), due to [8]).

3 Towards $\Omega(1/\Delta)$ -approximation for IS and WIS

In this section we show how one can use the clique-removal method of [3] to obtain, for every Δ , $\Omega(1/\Delta)$ -approximations for IS and WIS. The authors of [3] repeatedly call a procedure computing either a k -clique (clique of order k), or an independent set of expective size. At most n/k cliques can so be detected, while at each clique-deletion the independence number decreases no more than 1. If the independence number of the initial graph is large enough, a large independent set is necessary detected during one execution of the procedure. In all, the following theorem summarizes the thought process of [3].

Theorem 4. *([3]) There is a constant k and an $O(nm)$ algorithm (denoted by **LARGEIS** in what follows), computing, for every graph G of order n with $\alpha(G) \geq n/k + m$ and every k such that $2 \leq k \leq 2 \log n$, an independent set of size $kO(m^{1/(k-1)})$.*

In [3] k is not supposed to be constant; ratio $\log^2 n / n$ for **LARGEIS** is obtained for $k = O(\log n)$. If we set $k = (\log n / (\ell(n) \log \log n)) + 1$, where ℓ is a mapping such that, $\forall x > 0$, $0 < \ell(x) \leq \log \log x$ and $m = n/k^2$, then an analysis similar to the one for theorem 4 leads to the following.

Theorem 5. *For every function ℓ with $0 < \ell(x) \leq \log \log x$, $\forall x > 0$, there exist constants C and K such that algorithm **LARGEIS** computes, for every graph G of order $n > C$, an independent set S such that if $\alpha(G) \geq \ell(n)n \log \log n / \log n$, then $|S| \geq K \log^{\ell(n)} n$.*

Let now ℓ , C and K be as in theorem 5, denote by **GREEDY** the natural greedy IS-algorithm and by **EXHAUST** an exhaustive-search algorithm for maximum independent set, and consider the following algorithm.

```

BEGIN (*STABLE*)
  IF  $n \leq C$ 
    THEN OUTPUT  $S \leftarrow \text{EXHAUST}(G)$ ;
    ELSE OUTPUT  $S \leftarrow \text{argmax}\{\text{LARGEIS}(G), \text{GREEDY}(G)\}$ ;
  FI
END. (*STABLE*)

```

If $n \leq C$, then **EXHAUST** computes in constant time a maximum independent set for G . So, suppose $n > C$ and consider cases $\alpha(G) \geq \ell(n)n \log \log n / \log n$ and $\alpha(G) < \ell(n)n \log \log n / \log n$.

- If $\alpha(G) \geq \ell(n)n \log \log n / \log n$, then, by theorem 5, **STABLE** guarantees $|S| \geq K \log^{\ell(n)} n$; since $\alpha(G) \leq n$, the approximation ratio obtained is at least $K \log^{\ell(n)} n / n$;
- if $\alpha(G) < \ell(n)n \log \log n / \log n$, then by Turán's theorem ([17]) $|S| \geq n / (\mu + 1)$, and the approximation ratio obtained is $\log n / (\ell(n)(\mu + 1) \log \log n)$.

Since **GREEDY** is of time linear in $|E|$, **STABLE** has the same worst-case time-complexity as **LARGEIS**. So, the following theorem concludes the above discussion.

Theorem 6. *Consider ℓ such that, $\forall x > 0$, $0 < \ell(x) \leq \log \log x$. Then **STABLE** achieves approximation ratio*

$$\rho_{\text{STABLE}}(G) \geq \min \left\{ K \frac{\log^{\ell(n)} n}{n}, \frac{\log n}{\ell(n)(\mu + 1) \log \log n} \right\}.$$

If $\log \log n \geq \ell > 2$ (in particular if ℓ is constant), either $\log^{\ell(n)} n / n$, or $\log n / (\ell(n)(\mu + 1) \log \log n)$ corresponds to a significant improvement with respect to $O((\log^2 n)/n)$ and k/μ , respectively. In terms of graph-degree, it is, to our knowledge, the first $\Omega(1/\mu)$ approximation result for IS. Also,

$$\frac{\log n}{\ell(n)(\mu + 1) \log \log n} \geq \frac{\log \mu}{(\mu + 1) \log^2 \log \mu} > \frac{\log \log \Delta}{\Delta}.$$

Ratio $\log \log \Delta / \Delta$ for IS was, up to now, the only known $\Omega(1/\Delta)$ ratio for IS (presented in [10]). But the algorithm of [10] has the drawback to be polynomial only if Δ is bounded above. Moreover, the complexity of algorithm **STABLE** is $O(nm)$, whereas the $O(k/\mu)$ ratio of [5] is obtained by a PTAA of complexity $O(n^k)$. Of course, our result does not guarantee a ratio always bounded below by $\log n / (\ell(n)(\mu + 1) \log \log n)$, but in the opposite case, the ratio guaranteed is superior to the best-known n -depending ratio for IS (for instance, consider the case $\ell(n) = \log \log n$). In fact, if $K \log^{\ell(n)} n / n \leq \log n / (\ell(n)(\mu + 1) \log \log n)$ and $n \geq C_0$, for a fixed C_0 , then $n/\mu \geq K \ell(n) (\log \log n) \log^{\ell(n)-1} n$ and in this case **GREEDY** guarantees ratio bounded below by $\log^{\ell(n)-1} n / n$. In all, the following corollary can be deduced.

Corollary 1. *Given a graph G and ℓ such that $2 < \ell(n) \leq \log \log n$, at least one of the two following conditions holds:*

1. ***GREEDY** guarantees ratio bounded below by $\log^{\ell(n)-1} n / n$ (improving the best known $\rho(n)$ -ratio if $\ell > 3$);*
2. ***STABLE** guarantees ratio bounded below by $\log n / (\ell(n)(\mu + 1) \log \log n)$ (achieving so ratio $\Omega(1/\mu)$).*

The discussion above draws an interesting remark about the instance-parameters expressing non-constant approximation ratios. Until now, studies about the approximation of IS were limited in expressing ratio using one parameter (either the size or the degree). The results of this section show that considering both parameters, it is possible to reach tighter approximation ratios.

Combination of theorem 6 with theorem 2 allows to obtain important approximation results also for WIS.

Theorem 7. *For every constant k ,*

$$\begin{aligned}\rho_{\text{WIS}}(G) &\geq \min \left\{ \frac{\log^k n}{n}, \frac{0.099 \log n}{k(k+1)(\mu+1) \log^2 \log n} \right\} \\ &\geq \min \left\{ \frac{\log^k n}{n}, \frac{0.099 \log \mu}{k(k+1)(\mu+1) \log^2 \log \mu} \right\}.\end{aligned}$$

In other words, theorem 7 guarantees for WIS the existence of PTAAAs achieving either n -depending ratios much better than the ones known for IS (and comparable with the ones of corollary 1), or the first $\Omega(1/\mu)$ ratios for WIS (recall that the best-known $\rho(\Delta)$ -ratio for WIS – without restrictions on Δ – was, until now, the one of [8], bounded below by $3/(\Delta+2)$).

4 Further improvements

In this section, we propose a polynomial approximation result for maximum-weight independent set problem which is not deduced by reduction from the unweighted case. It improves all the approximation results of section 3, for both weighted and unweighted cases, but the corresponding complexity is higher.

4.1 A weighted version of Turán's theorem

Recall that $\mu_w(G) = \sum_{v \in V} w(\Gamma(v))/w(V)$ (note that $\mu_w(G) \leq \Delta(G)$) and consider the following algorithm for WIS.

```
BEGIN (*WGREEDY*)
  S ← ∅;
  WHILE V ≠ ∅ DO
    v ← argmin_v {w(Γ(v))/w(v)};
    S ← S ∪ {v};
    V ← V \ ({v} ∪ Γ(v));
    update E;
  OD
END. (*WGREEDY*)
```

Then, the following easy theorem holds (an unweighted version of it – see also [9] – is the famous Turán's theorem).

Theorem 8. *For every weighted-graph with maximum degree Δ , algorithm WGREEDY computes an independent set of weight at least $w(V)/(\mu_w(G) + 1) \geq w(V)/(\Delta + 1)$.*

Proof. Consider algorithm WGREEDY and suppose that its WHILE loop is executed t times. For $i \in \{1, \dots, t\}$ let us denote by $G_i = (V_i, E_i)$ the surviving graph, by v_i the vertex selected during the i th execution and by $\Gamma_i(v)$ the neighborhood of v in G_i . Then, $\forall i \in \{1, \dots, t\}$,

$$\sum_{v \in (\{v_i\} \cup \Gamma(v_i))} w(\Gamma(v)) \geq \sum_{v \in (\{v_i\} \cup \Gamma_i(v_i))} w(\Gamma_i(v)) \geq \frac{w(\Gamma_i(v_i))}{w_{v_i}} (w_{v_i} + w(\Gamma_i(v_i)))$$

and, consequently, by adding side-by-side the expressions above, for $i = 1 \dots t$, we get (note that $V = \cup_{i=1 \dots t} \{\{v_i\} \cup \Gamma_i(v_i)\}$),

$$w(V)\mu_w \geq \sum_{i=1}^t \frac{w(\Gamma_i(v_i))}{w_{v_i}} (w_{v_i} + w(\Gamma_i(v_i))).$$

On the other hand,

$$w(V) = \sum_{i=1}^t (w_{v_i} + w(\Gamma_i(v_i)))$$

and by Cauchy-Schwarz inequality we get

$$w(V)(\mu_w + 1) \geq \sum_{i=1}^t \frac{(w_{v_i} + w(\Gamma_i(v_i)))^2}{w_{v_i}} \geq \frac{(w(V))^2}{\sum_{i=1}^t w_{v_i}}$$

which concludes the proof since the weight of the greedy solution is $\sum_{i=1}^t w_{v_i}$. ■

An easy corollary of theorem 8 is that whenever $\alpha_w(G) \leq w(V)/k$, then **WGREEDY** achieves in G ratio $k/(\Delta + 1)$ for WIS. Consequently, in order to devise $O(k/\Delta)$ -approximations for every graph, one can focus him/herself on the approximation of WIS in graphs with large weighted independence number. For an integer function $k(n) \leq n$, we denote by WIS_k the class of graphs with $\alpha_w(G) > w(V)/k$; for reasons of simplicity we assimilate this class with the corresponding WIS-subproblem. The work of [3] and its improvement by [1] show how IS_k (the unweighted version of WIS_k) can be approximated within $O(n^{\epsilon_k - 1})$ where ϵ_k depends only on k (note that using the reduction of [16], one immediately gets a ratio $O(w(V)^{\epsilon_k - 1})$). Here, we perform a further improvement by generalizing the algorithm of [1] to the weighted case.

4.2 Graph coloring and the approximation of WIS_k

In this section, we adapt the method of [1] to the weighted IS-case and relate the approximation of WIS_k to the coloring of a class of graphs (called $\mathcal{G}_{k+1}$ in the sequel) containing the $(k + 1)$ -colorable graphs. We first recall two notions very closely related, the *Lovász θ -function* and the *orthonormal representation* of a graph.

Definition 2. ([14]) Consider a graph $G = (V, E)$, $V = \{1, \dots, n\}$:

the Lovász θ -function $\theta(G)$ of G is the maximum value of $\sum_{i,j=1}^n b_{ij}$ where $B = (b_{ij})_{i,j=1 \dots n}$ ranges over all positive semidefinite symmetric matrices with trace 1 and such that $b_{ij} = 0$ for every pair (i, j) , $i \neq j, ij \in E$;

the orthonormal representation of G is a system of n unit vectors $(\vec{u}_i)_{i=1 \dots n}$ in a n -dimensional Euclidean space, such that for every (i, j) such that $ij \notin E$, $\vec{u}_i$ and $\vec{u}_j$ are orthogonal. ■

Proposition 2. ([14]) Given a graph G , the following holds:

- given an orthonormal representation $(\vec{u}_i)_{i=1 \dots n}$ of G , and a unit vector $\vec{d}$, $\exists i \in \{1, \dots, n\}$ such that $\theta(G) \leq 1/(\vec{d} \cdot \vec{u}_i)^2$;
- $\alpha(G) \leq \theta(G) \leq \chi(\bar{G})$.

For an integer function $\ell = \ell(n) \leq n$, set $\mathcal{G}_\ell = \{G : \theta(\bar{G}) \leq \ell\}$; by proposition 2, every ℓ -colorable graph belongs to $\mathcal{G}_\ell$. In [12] it is shown that every graph in $\mathcal{G}_\ell$ with order n and maximum degree Δ can be colored with $O(\min\{\Delta^{1-2/\ell} \log^{3/2} n, n^{1-3/(\ell+1)} \log^{1/2} n\})$ colors in randomized polynomial time ([12]). The algorithm of [12] has been derandomized later by [15].

Theorem 9. *Let $k = k(n) \leq n$ be an integer function and $f_k(x, y)$ be a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, non-decreasing with respect to both x and y . If there exists a $O(T(n))$ algorithm A computing, for every graph $G \in \mathcal{G}_{k+1}$, $|V| \leq n$, a $f_k(n, \Delta)$ -coloring, then there exists a $O(\max\{n^3, T(n)\})$ PTAA for WIS_k guaranteeing ratio $1/(k(k+1)f_k(n, \Delta))$.*

Proof. Let $(G = (V, E), \vec{w})$ be an instance of WIS_k of order n and G_w be as in definition 1 of section 2. Since the weights are supposed to be integral and $\alpha_w(G) > w(V)/k$, we have

$$\alpha_w(G) = \alpha(G_w) \geq \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)} + \frac{1}{k}.$$

In [7] it is shown that:

- $\theta(G_w)$ is equal to the maximum value of $\sum_{i,j=1}^n \sqrt{w_i w_j} b_{ij}$, where b_{ij} are as in definition 2;
- one can compute, in $O(n^3)$ by the ellipsoid method, a positive semidefinite symmetric matrix $B = (b_{ij})_{i,j=1\dots n}$ satisfying

$$\begin{cases} \sum_{i=1}^n \sum_{j=1}^n \sqrt{w_i w_j} b_{ij} \geq \theta(G_w) - \frac{1}{2k} > 0 \\ b_{ij} = 0 & ij \in E \end{cases} \quad (5)$$

Given that B is symmetric and positive semidefinite, there exist n vectors $\hat{u}_i, i \in \{1, \dots, n\}$ in $\mathbb{R}^n$ (seen as Euclidean space) such that, $\forall (i, j) \in \{1, \dots, n\}^2$, $\hat{u}_i \cdot \hat{u}_j = b_{ij}$; in particular, we have $b_{ii} \geq 0$. For our purpose we just need to compute n vectors $\vec{u}_i$ satisfying the following expression (6) for $\epsilon = 1/(2knw(V))$:

$$\begin{cases} \vec{u}_i \cdot \vec{u}_j = b_{ij} & i \neq j \\ |\vec{u}_i|^2 = b_{ii} + \epsilon > 0 \\ \sum_{i=1}^n |\vec{u}_i|^2 = \text{Tr}(B) + n\epsilon = 1 + n\epsilon \\ \left| \sum_{i=1}^n \sqrt{w_i} \vec{u}_i \right|^2 \geq \sum_{i=1}^n \sum_{j=1}^n \sqrt{w_i w_j} b_{ij} \end{cases} \quad (6)$$

Such vectors can be seen as non-zero approximations of $\hat{u}_i, i = 1 \dots n$. The system of vectors $(\vec{u}_i)_{i=1\dots n}$ can be computed ([6]) by applying Cholesky's decomposition to the symmetric positive definite matrix $B + \epsilon I$ (where I is the identity matrix); one gets (in $O(n^3)$) an n -dimensional triangular matrix U such that $B = {}^t U U - \epsilon I$. Let then $(\vec{u}_i)_{i=1\dots n}$ be the columns of U ; they clearly satisfy expression (6).

Set, for $i \in \{1, \dots, n\}$,

$$\begin{aligned} \vec{d} &= \frac{\sum_{i=1}^n \sqrt{w_i} \vec{u}_i}{\left| \sum_{i=1}^n \sqrt{w_i} \vec{u}_i \right|} \\ \vec{z}_i &= \frac{\vec{u}_i}{|\vec{u}_i|} \end{aligned}$$

and note that $\vec{z}_i$, constitutes an orthonormal representation of $\bar{G}$. Without loss of generality, we can assume that $(\vec{d} \cdot \vec{z}_1)^2 \geq (\vec{d} \cdot \vec{z}_2)^2 \geq \dots (\vec{d} \cdot \vec{z}_n)^2$.

Lemma 3. *Let*

$$\begin{aligned} j &= \max \left\{ i \in \{1, \dots, n\} : \sum_{l=1}^{i-1} w_l \leq \frac{w(V)}{k(k+1)} \right\} \\ K &= G[\{v_i : i = 1, \dots, j\}]. \end{aligned}$$

Then, $(\vec{d} \cdot \vec{z}_j)^2 \geq 1/(k+1)$ which implies $K \in \mathcal{G}_{k+1}$.

Proof of lemma 3. By Cauchy-Schwarz inequality and expressions (5) and (6), we get:

$$\begin{aligned} (1+n\epsilon) \sum_{i=1}^n w_i (\vec{d} \cdot \vec{z}_i)^2 &= \left(\sum_{i=1}^n |\vec{u}_i|^2 \right) \sum_{i=1}^n w_i (\vec{d} \cdot \vec{z}_i)^2 \geq \left(\sum_{i=1}^n \vec{d} \cdot \sqrt{w_i} \vec{u}_i \right)^2 = \left| \sum_{i=1}^n \sqrt{w_i} \vec{u}_i \right|^2 \\ &\geq \sum_{i=1}^n \sum_{j=1}^n \sqrt{w_i w_j} b_{ij} \geq \theta(G_w) - \frac{1}{2k} \geq \alpha(G_w) - \frac{1}{2k} \\ &\geq \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)} + \frac{1}{2k}. \end{aligned}$$

On the other hand, since for $i \in \{1, \dots, n\}$, $\vec{d} \cdot \vec{z}_i \leq 1$, (recall that $\vec{d}$ and $\vec{z}_i$, $i = 1 \dots n$ are unit vectors),

$$(1+n\epsilon) \sum_{i=1}^n w_i (\vec{d} \cdot \vec{z}_i)^2 \leq \sum_{i=1}^n w_i (\vec{d} \cdot \vec{z}_i)^2 + \frac{1}{2k}.$$

Consequently,

$$\sum_{i=1}^n w_i (\vec{d} \cdot \vec{z}_i)^2 \geq \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)}. \quad (7)$$

Recall that $(\vec{d} \cdot \vec{z}_1)^2 \geq (\vec{d} \cdot \vec{z}_2)^2 \geq \dots (\vec{d} \cdot \vec{z}_n)^2$. We have $(\vec{d} \cdot \vec{z}_j)^2 \geq 1/(k+1) > 0$ since, in the opposite case,

$$\sum_{i=1}^n w_i (\vec{d} \cdot \vec{z}_i)^2 \leq \sum_{i=1}^{j-1} w_i + \sum_{i=j}^n w_i (\vec{d} \cdot \vec{z}_j)^2 < \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)}$$

which contradicts expression (7).

As noticed in [1], $(\vec{d} \cdot \vec{z}_j)^2 \geq 1/(k+1) > 0$ implies that the subgraph K of G induced by vertices v_i , $i = 1 \dots j$, satisfies $\theta(\bar{K}) \leq k+1$; this follows from the fact that $\vec{z}_i$, $i \in \{1, \dots, j\}$, is an orthonormal representation of $\bar{K}$ with value (see [14]) less than $1/(\vec{d} \cdot \vec{z}_j)^2 \leq k+1$. Note that this expression holds for the unweighted θ -function of G ; so, $K \in \mathcal{G}_{k+1}$. This completes the proof of lemma 3. ■

Lemma 3 is originally proved in [1] for the case of (unweighted) IS. As one can see from the above proof, extension of it for WIS is non-trivial.

Let us now continue the proof of theorem 9. By lemma 3, algorithm A (claimed in the statement of theorem 9) computes a $f_k(n, \Delta)$ -coloring of K . Then, the maximum-weight color is an independent set of $K = (V_K, E_K)$ (and of G) of total weight at least $w(V_K)/f_k(j, \Delta(K))$. Since $w(V_K) \geq w(V)/(k(k+1))$ (see the definition of j), $j \leq n$ and $\Delta(K) \leq \Delta$, the maximum-weight color is an independent set of G of weight at least $w(V)/(k(k+1)f_k(n, \Delta))$. On the other hand $\alpha_w(G) \leq w(V)$, and consequently the following algorithm WIS_k guarantees ratio $1/(k(k+1)f_k(n, \Delta))$ for the WIS-subproblem WIS_k.

```

BEGIN (*WIS_k*)
(1)  compute the matrix  $B = (b_{ij})_{i,j=1,\dots,n}$  by the ellipsoid method;
(2)   $\epsilon \leftarrow 1/(2knw(V))$ ;
(3)  compute the Cholesky's decomposition of  $B + \epsilon I$ ;
(4)  compute vectors  $\vec{d}$  and  $\vec{z}_i, i \in \{1, \dots, n\}$ ;
(5)  sort vertices in decreasing order with respect to  $(\vec{d} \cdot \vec{z}_i)^2$ ;
(6)  compute  $j$  and  $K$ ;
(7)  IF  $(\vec{d} \cdot \vec{z}_j)^2 \geq 1/(k+1)$ 
(8)    THEN
(9)      call A to compute a coloring  $(C_1, \dots, C_l)$  of  $K$ ;
(10)      $S \leftarrow \operatorname{argmax}\{w(C_i) : i = 1 \dots l\}$ ;
(11)     complete  $S$  to obtain a maximal independent set of  $G$ ;
(12)  ELSE  $S \leftarrow \emptyset$ ;
(13) FI
(14) OUTPUT  $S$ ;
END. (*WIS_k*)

```

To conclude the proof, let us note that lines (1) and (3) are executed in $O(n^3)$, line (5) in $O(n \log n)$ and line (11) in $O(n^2)$. Finally, the time-complexity of line (9) is bounded above by $T(n)$. ■

Of course, unless $\mathbf{P} = \mathbf{NP}$, inclusion of a graph in $\mathbf{WIS}_k$ cannot be polynomially decided. However, algorithm $\mathbf{WIS}_k$ runs on every graph within the same complexity; in fact (by instruction (7)), if $K \in \mathcal{G}_{k+1}$, then algorithm A is called to return a non-empty independent set S ; otherwise, $S = \emptyset$. Consequently, if $S \neq \emptyset$ (in particular if $G \in \mathbf{WIS}_k$), algorithm $\mathbf{WIS}_k$ guarantees the ratio claimed by theorem 9; in the opposite case, the input-graph does not belong to $\mathbf{WIS}_k$.

Using for A, the derandomized version of [12] presented in [15], theorem 9 leads to the following approximation result for $\mathbf{WIS}_k$, for $2 \leq k(n) \leq n$.

Corollary 2.

$$\rho_{\mathbf{WIS}_k} \geq \Theta \left(\frac{1}{k(k+1)\Delta^{1-2/(k+1)} \log^{3/2} n} \right).$$

4.3 The main result

Consider now the following algorithm, the worst-time complexity of which is the same as the one of $\mathbf{WIS}_k$.

```

BEGIN (*WIS*)
  OUTPUT  $S \leftarrow \operatorname{argmax}\{w(\mathbf{WIS}_k(G)), w(\mathbf{WGREEDY}(G))\}$ ;
END. (*WIS*)

```

By theorems 8 and 9, $\exists c'$ such that

$$\rho_{\mathbf{WIS}} \geq \min \left\{ \frac{k}{\Delta + 1}, \frac{c'}{k(k+1)\Delta^{1-2/(k+1)} \log^{3/2} n} \right\}.$$

By an easy but somewhat tedious algebra, one can prove that the righthand side of the above inequality is at least as large as

$$\min \left\{ \frac{k}{\Delta + 1}, \frac{c^k}{(k+1)^{(1+3k)/2} \log^{3(k+1)/4} n} \right\}$$

for a constant c . Let us suppose k constant. Then, the following theorem holds.

Theorem 10. *For any fixed integer $k \geq 2$ and for $t = 3(k+1)/4$*

$$\rho_{\text{WIS}}(G) \geq \min \left\{ \frac{k}{(\Delta+1)}, O(\log^{-t} n) \right\}.$$

Furthermore, in the case where $\min\{k/(\Delta+1), O(\log^{-t} n)\} = O(\log^{-t} n)$, $\Delta+1 \leq O(\log^t n)$ and then algorithm **WGREEDY** already guarantees a wonderful (given the result in [11]) approximation ratio.

Corollary 3. *Consider a graph G and $k \geq 2$. Then, there exists $t > 0$ such that at least one of the two following conditions holds:*

1. *algorithm **WGREEDY** achieves ratio bounded below by $O(\log^{-t} n)$;*
2. *algorithm **WIS** achieves ratio bounded below by $k/(\Delta+1)$.*

On the other hand, let us revisit expression

$$\min \left\{ \frac{k}{\Delta+1}, \frac{c^k}{(k+1)^{(1+3k)/2} \log^{3(k+1)/4} n} \right\}$$

and set $k = \log n / (3 \log \log n)$; then, $\exists n_0 \geq 1, \forall n \geq n_0$, $(1/c^k)(k+1)^{(1+3k)/2} \log^{3(k+1)/4} n \leq n^{4/5}$ and, since instances with $n \leq n_0$ can be solved by exhaustive search in constant time, the following theorem holds and concludes the section.

Theorem 11.

$$\rho_{\text{WIS}}(G) \geq \min \left\{ \frac{\log n}{3(\Delta+1) \log \log n}, O(n^{-4/5}) \right\}.$$

5 Approximating maximum-weight clique

We describe in this section a new reduction, working for both weighted and unweighted cases, between independent set and clique. Let us note that the classical correspondence “independent set in G - clique in $\bar{G}$ ” preserves constant and $\rho(n)$ ratios but it does not work for ratios depending on Δ .

Let $G = (V, E)$ be an instance of KL and let $V = \{1, \dots, n\}$. We consider the n graphs $G_i = G[\{i\} \cup \Gamma(i)]$, $i = 1, \dots, n$ and denote by n_i their respective orders; we also consider the n graphs $\bar{G}_i$. Then, the following proposition holds (recall that in every graph G , the size of a maximum clique is never greater than $\Delta(G)$).

Proposition 3. *For every $i \in \{1, \dots, n\}$, the following three facts hold:*

1. $n_i \leq \Delta(G) + 1$ and $\Delta(\bar{G}_i) \leq \Delta(G)$;
2. *cliques (resp., independent sets) of G_i (resp., $\bar{G}_i$) are also cliques (resp., independent sets) of G (resp., $\bar{G}$); moreover, $\exists i^*$ such that the maximum clique (resp., independent set) of G_{i^*} (resp., $\bar{G}_{i^*}$) is exactly the maximum clique (resp., independent set) of G (resp., $\bar{G}$);*
3. *items 1 and 2 hold also for WKL and WIS if we consider weighted cliques and independent sets.*

Let **A** be a PTAA for WIS achieving ratio $\rho(n, \Delta)$; we shall use it to produce a WKL-solution for G . This can be done by the following algorithm.

```

BEGIN (*WKL*)
    OUTPUT  $K \leftarrow \operatorname{argmax}\{w(A(\bar{G}_i)) : i = 1, \dots, n\};$ 
END. (*WKL*)

```

Since K is an independent set of $\bar{G}$, it is a clique (of the same weight) in G . Moreover, by items 1 and 2 of proposition 3, algorithm WKL achieves, in polynomial time, approximation ratio $\rho(\Delta + 1, \Delta)$ for WKL.

The reduction just described is, to our knowledge, the first one preserving $\rho(\Delta)$ ratios between independent set and clique (in both weighted and unweighted cases).

Given the IS result of [3] and the one of theorem 3 we conclude the following.

Theorem 12. *There exist PTAAAs for weighted and unweighted versions of KL such that*

$$\begin{aligned}\rho_{\text{KL}}(G) &\geq \Theta\left(\frac{\log^2 \Delta}{\Delta}\right) \\ \rho_{\text{WKL}}(G) &\geq \Theta\left(\frac{\log^2 \Delta}{\Delta \log \log \Delta}\right).\end{aligned}$$

The results of theorem 12 represent, to our knowledge, the first $\rho(\Delta)$ ratios for the clique-problem.

6 Maximum l -colorable induced subgraph and minimum chromatic sum

6.1 Maximum l -colorable induced subgraph

Let us note that we can assume $l < \Delta$, because, if not, then $G' = G$ (see the definition of Cl in section 1 and recall that there exists the polynomial-time coloring-algorithm of [13] always guaranteeing a Δ -coloring of G , unless G is a $(\Delta + 1)$ -clique).

Consider now, for $l \in \{1, \dots, \Delta - 1\}$, the graph ${}^lG = ({}^lV, {}^lE)$ defined as follows:

$$\begin{aligned}{}^lV &= V \times \{1, \dots, l\} \\ {}^lE &= \{(v, i)(v', j) : [(i = j) \wedge vv' \in E] \vee [(i \neq j) \wedge (v = v')]\}\end{aligned}$$

For WCl the weight of vertex (v, i) equals w_v , $\forall i = 1, \dots, l$.

Clearly,

$$\begin{aligned}|{}^lV| &= ln \\ \Delta({}^lG) &= \Delta(G) + l - 1 \\ \mu({}^lG) &= \mu + l - 1\end{aligned}$$

(in fact, for all $(v_k, i) \in {}^lV$, $v_k \in V$, $i \leq l$, the degree of (v_k, i) equals the degree of v_k plus $l - 1$; the same holds for the average degree $\mu({}^lG)$).

The following holds:

- let us consider an independent set $S \subset {}^lV$ of lG ; the family $S_i = \{v \in V : (v, i) \in S\}$, $i = 1, \dots, l$, is a collection of mutually disjoint independent sets of G ; so the graph $G[\cup_i S_i]$ is l -colorable;
- conversely, for every l -colorable subgraph $G' = (V', E')$ of G and for every l -coloring $(S_1, \dots, S_l)$ of G' , the set $S = \{(v, i) : i \in \{1, \dots, l\}, v \in S_i\}$ is an independent set of lG .

Consequently, every independent set (resp., maximum independent set) of a certain size in lG corresponds to an l -colorable induced (resp., maximum-order) subgraph of the same order in G and vice-versa. Clearly, the same correspondence holds between WIS and WCl if one considers weights instead of sizes.

By the previous discussion and the results of the previous WIS-sections, one can get the following concluding theorem.

Theorem 13. *Consider f such that, $\forall x > 0$, $2 < f(x) \leq \log \log x$. Then (by theorem 6 and corollary 1),*

$$\rho_{Cl}(G) \geq \min \left\{ \Theta \left(\frac{\log^{f(n)-1} n}{n} \right), \frac{\log n}{f(n)(\mu + 1) \log \log n} \right\}.$$

Also (by theorem 11),

$$\rho_{WCl}(G) \geq \min \left\{ \frac{\log n}{3(\Delta + 1) \log \log n}, O \left(n^{-4/5} \right) \right\}.$$

6.2 Minimum chromatic sum

Let us consider the following very common coloring-procedure where by IS we denote an IS-algorithm.

```
BEGIN (*IT-IS*)
  i ← 0;
  WHILE V ≠ ∅ DO
    i ← i + 1;
    Ci ← IS(G);
    V ← V \ Ci;
    remove from E the edges adjacent to vertices of Ci;
END. (*IT-IS*)
```

In other words, algorithm IT-IS iteratively colors the vertices of an independent set by an unused color and removes them from the current graph.

The following results are proved in [2].

Theorem 14. ([2])

1. *If algorithm IS guarantees approximation ratio $\rho(G)$ for IS, then algorithm IT-IS guarantees approximation ratio $4\rho(G)$ for CHS.*
2. *If algorithm IS is an exact (optimal) algorithm for WIS, then algorithm IT-IS guarantees approximation ratio 4 for WCHS.*

On the contrary, no indication for extension of item 2 of theorem 14 in the case where IS is an approximation algorithm exists in the interesting work of [2]. Here we perform an easy extension of their result in order to work also in the case where IS is considered as a PTAA guaranteeing a certain ratio.

Theorem 15. *If algorithm IS guarantees approximation ratio $\rho(G)$ for WIS, then IT-IS guarantees approximation ratio $4\rho(G)$ for CWHS.*

Proof. Given a vertex-weighted graph $(G, \vec{w})$, let us denote by D the LCM of the denominators of the components of $\vec{w}$ and set $\vec{\tilde{w}} = D \cdot \vec{w}$. Obviously, components of $\vec{\tilde{w}}$ are integral. Consider now the graph $G_{\vec{\tilde{w}}}$ (definition 1) and any subgraph $H = (V_H, E_H)$ of G . Then:

- application of algorithm **IS** in H determines an independent set S_H corresponding to an independent set $\tilde{S}_H$ of $H_{\tilde{w}}$ of cardinality $w(S_H)$;
- if $w(S_H)/\alpha_w(H) \geq \rho(H)$, then $|\tilde{S}_H|/\alpha(H_{\tilde{w}}) \geq \rho(H)$;
- if H' and H'' are the subgraphs of H and $H_{\tilde{w}}$, respectively, obtained by removing vertex-sets S_H and $\tilde{S}_H$, then $H'' = H'_{\tilde{w}}$.

Consequently, supposing that an algorithm **WIS** solves **WIS** in G within ratio $\rho_{\text{WIS}}(G) = \rho(G)$, application of algorithm **WIS** in G can be equivalently seen as an application of an algorithm solving (*unweighted*) **IS** in $G_{\tilde{w}}$ within ratio $\rho(G)$ (and not $\rho(G_{\tilde{w}})$).

Every coloring $\tilde{C} = (\tilde{C}_1, \dots, \tilde{C}_l)$ of $G_{\tilde{w}}$ corresponds to a coloring $C = (C_1, \dots, C_l)$ of G such that $\tilde{w}(C_i) = |\tilde{C}_i|$, $i = 1, \dots, l$. Consequently, the value of the WCHS-solution C of G is $\sum_{i=1}^l i\tilde{w}(C_i)$ which is exactly the value of the CHS-solution $\tilde{C}$ in $G_{\tilde{w}}$. Of course the same holds if we divide all the above quantities by D .

In all, if we consider a **WIS-PTAA** **WIS** guaranteeing approximation ratio $\rho(G)$, its application in G can be simulated to the application of an **IS-PTAA** **IS** in $G_{\tilde{w}}$ guaranteeing the same ratio $\rho(G)$. Extending this argument, application of **IT_WIS** in G can be simulated to the application of **IT_IS** in $G_{\tilde{w}}$. By item 1 of theorem 14, this latter algorithm guarantees for CHS approximation ratio $4\rho(G)$. Consequently, this ratio is also the ratio of **IT_WIS** when used to solve WCHS in G , q.e.d. ■

Immediate consequence of theorem 15 is that the results for **WIS** produced above apply also for WCHS.

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