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Unité de Recherche Associée au CNRS ESA 7024

5/12/99

PQI INTERVAL ORDERS

CAHIER N° 165
novembre 1999

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received: July 1999.

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CONTENTS

PQI Ordres d'intervalle

Résumé

Nous présentons la réponse à un problème ouvert dans la modélisation des préférences à l'aide d'intervalles. Soit un ensemble fini A et trois relations binaires P, Q, I, appelées "préférence stricte", "préférence faible" et "indifférence", respectivement. Nous présentons les conditions nécessaires et suffisantes pour pouvoir associer à chaque élément de A un intervalle de façon à ce que si un intervalle est "complètement à droite" de l'autre on obtient la relation P, si un intervalle est inclus dans l'autre on obtient la relation I et si un intervalle "est à droite" de l'autre, mais leur intersection n'est pas vide on obtient la relation Q (Q modélisant l'hésitation entre P et I). Deux structures de préférences spécifiques sont caractérisées: le PQI ordre d'intervalle et le PQI quasi-ordre. Détecter l'existence d'un PQI quasi-ordre est immédiat. Par contre, la détection d'un PQI ordre d'intervalle est plus difficile parce que le théorème d'existence est une formule du deuxième ordre. Pour cette raison nous présentons un algorithme pour la détection de PQI ordres d'intervalle et nous démontrons qu'il est "backtracking free". Avec ce résultat nous pouvons présenter une implémentation matricielle de l'algorithme et montrer qu'il est polynomial.

Mots cles: *Intervalles, Ordre d'intervalle, Indifférence, Préférence Faible, Préférence Stricte.*

PQI Interval Orders

Abstract

We provide an answer to an open problem concerning the representation of preferences by intervals. Given a finite set A with three relations P, Q, I standing for "strict preference", "weak preference" and "indifference" respectively, necessary and sufficient conditions are provided for representing each element of A by an interval in such a way that P holds when one interval is completely to the right of the other, I holds when one interval is included to the other and Q holds when one interval is to the right of the other, but they do have a non empty intersection (Q modeling the hesitation). Two specific preference structures: PQI semi orders and PQI interval orders, will be considered. While the detection of a PQI semi order is straightforward, the case of the PQI interval order is more difficult as the theorem of existence consists in a second-order formula. To this purpose, the paper also presents an algorithm for detecting a PQI interval order and demonstrates that it is backtracking free. This result leads to a matrix version of the algorithm which can be proved to be polynomial.

Keywords: *Intervals, Interval Orders, Indifference, Weak Preference, Strict Preference.*

1 Introduction

Comparing intervals, instead of discrete values, is a frequently encountered problem in preference modelling and decision aid. This is due to the fact that the comparison of alternatives (outcomes, objects, candidates, ...) generally is realized through their evaluations on numerical scales, while such evaluations are often imprecise or uncertain. A well known preference structure, in this context, is the semi order (see Luce 1956) (for a comprehensive presentation see Pirlot & Vincke 1997) and more generally the interval order (see also Fishburn 1985). An interval order is obtained when one considers that an alternative is preferred to another iff its interval is "completely to the right" of the other (hereafter we assume that the larger an evaluation of an alternative is on a numerical scale the better the alternative is), while any two alternatives, the intervals of which have a non empty intersection, are considered indifferent. Such a model has a strict probabilistic interpretation, since the interval associated to each alternative can be viewed as the extremes of the probability distributions of the evaluations of the alternatives. Under such an interpretation a "sure preference" occurs only if the distributions have an empty intersection. A second implicit assumption in this frame is that if there is no preference of an alternative over the other then they are indifferent.

It is easy however to notice that if, in the previous frame, we want to establish a "sure indifference", it is much more natural to consider that two alternatives are indifferent if their associated intervals (or distributions) are embedded. In such a case we obtain a preference relation which is known to be a partial order of dimension 2, that is a partial order obtained from the intersection of exactly two linear orders; (see Roubens & Vincke 1985).

Practically we observe that we have three situations:

- a "sure indifference": when the intervals associated to two alternatives are embedded;
- a "sure preference": when the interval associated to one alternative is "more to the right" with respect to the interval associated to the other alternative and the two intervals have an empty intersection;
- an "hesitation between indifference and preference" which we denote as weak preference: when the interval associated to one alternative is "more to the right" with respect to the interval associated to the other alternative and the two intervals have a non empty intersection.

The preference structure having three relations P, Q, I defined as such is called hereafter PQI interval order. It fits better in the case we have qualitative uncertainties or imprecision and is consistent with the use of specific relations in order to represent situations of hesitation in preference modeling (see Tsoukiàs & Vincke 1997). The problem is to give the necessary and sufficient conditions for which a preference structure characterized by the presence of the relations P, Q and I may admit a representation by intervals as the one previously discussed, and then to detect if a given PQI preference structure satisfies these conditions. Such a problem was considered open for a long time (see Vincke 1988).

In this paper we present an answer for this problem. Section 2 provides the basic notations and definitions. In section 3 we recall some results concerning conventional interval orders. The main theorical results of PQI interval order are presented, demonstrated and discussed in section 4. Section 5 gives a detecting algorithm in a procedural way by which it is possible to demonstrate that it is "backtracking free". Section 6 presents a "matrix implementation" of the algorithm enabling the demonstration that it is polynomial. Some conclusions are given at the end of the paper. Appendix A contains the demonstrations of the propositions used in the proof of Theorem 5.1 (backtracking free).

2 Notations and definitions

In this paper we consider binary relations defined on a finite set A, that is subsets of $A \times A$ (the quantifiers apply therefore always to such a domain). Further on we will use the following notations for any binary relations S, T. If S is a binary relation on A we denote by $S(x,y)$ the fact that $(x,y) \in S$. $\neg$, $\wedge$ and $\vee$ denote the usual negation, conjunction and disjunction operations.

$S^{-1} = \{(x,y): S(y,x)\}$
$S^c = \neg S = \{(x,y): \neg S(x,y)\}$
$S^d = \neg S^{-1} = \{(x,y): \neg S(y,x)\}$
$S \subset T: \forall x,y \; S(x,y) \rightarrow T(x,y)$
$S.T = \{(x,y): \exists z \; S(x,z) \wedge T(z,y)\}$
$S^2 = \{(x,y): \exists z \; S(x,z) \wedge S(z,y)\}$
$S \cup T = \{(x,y): S(x,y) \vee T(x,y)\}$
$S \cap T = \{(x,y): S(x,y) \wedge T(x,y)\}$

We recall some well known definitions from the literature (our terminology follows Roubens & Vincke 1985).

Definition 2.1 *A relation S on a set A is said to be:*
- reflexive: iff $\forall x \; S(x,x)$

- *irreflexive:* *iff* $\forall x \ \neg S(x,x)$
- *symmetric:* *iff* $\forall x,y \ S(x,y) \rightarrow S^{-1}(x,y)$
- *asymmetric:* *iff* $\forall x,y \ S(x,y) \rightarrow S^{d}(x,y)$
- *complete:* *iff* $\forall x,y,\ x \neq y,\ S(x,y) \vee S^{-1}(x,y)$
- *transitive:* *iff* $\forall x,y,z \ S(x,y) \wedge S(y,z) \rightarrow S(x,z)$
- *negatively transitive:* *iff* $\forall x,y,z \ \neg S(x,y) \wedge \neg S(y,z) \rightarrow \neg S(x,z)$

Definition 2.2 *A binary relation S is:*
- *a partial order iff it is asymmetric and transitive;*
- *a weak order iff it is asymmetric and negatively transitive;*
- *a linear order iff it is irreflexive, complete and transitive;*
- *an equivalence iff it is reflexive, symmetric and transitive.*

In this paper we will consider relations representing strict preference, weak preference and indifference situations. We will denote them P, Q, I respectively. Moreover, such relations are expected to satisfy some "natural" properties of the type announced in the following two definitions.

Definition 2.3 *A $\langle P, I \rangle$ preference structure on a set A is a couple of binary relations, defined on A, such that:*
- *I is reflexive and symmetric;*
- *P is asymmetric;*
- *$I \cup P$ is complete;*
- *P and I are mutually exclusive ($P \cap I = \emptyset$).*

Definition 2.4 *A $\langle P, Q, I \rangle$ preference structure on a set A is a triple of binary relations, defined on A, such that:*
- *I is reflexive and symmetric;*
- *P and Q are asymmetric;*
- *$I \cup P \cup Q$ is complete;*
- *P, Q and I are mutually exclusive.*

Finally we introduce an equivalence relation as follows:

Definition 2.5 *The equivalence relation associated to a $\langle P, Q, I \rangle$ preference structure is the binary relation E, defined on the set A, such that, $\forall x, y \in A$:*

$$E(x,y) \quad \textit{iff} \quad \forall z \in A: \begin{cases} P(x,z) \Leftrightarrow P(y,z) \\ Q(x,z) \Leftrightarrow Q(y,z) \\ I(x,z) \Leftrightarrow I(y,z) \\ Q(z,x) \Leftrightarrow Q(z,y) \\ P(z,x) \Leftrightarrow P(z,y) \end{cases}$$

Remark 2.1 *In this paper we consider that two different elements of A are never equivalent for the given $\langle P, Q, I\rangle$ preference structure. This is not restrictive as it suffices to consider the quotient of A by E to satisfy the assumption. Under such an assumption we will use in the numerical representation of the preference relations only strict inequalities without any loss of generality.*

3 Interval Orders

In this section we recall some definitions and theorems concerning conventional interval orders and semi orders.

Definition 3.1 *A $\langle P, I\rangle$ preference structure on a finite set A is a PI interval order iff $\exists\, l, r : A \mapsto \mathcal{R}^+$ such that:*
$\forall\, x :\ r(x) > l(x)$
$\forall\, x, y : P(x, y) \ \Leftrightarrow\ l(x) > r(y)$
$\forall\, x, y : I(x, y) \ \Leftrightarrow\ l(x) < r(y)$ *and* $l(y) < r(x)$

In conventional interval orders when comparing two intervals two situations are considered:
- one interval is completely to the right of the other (strict preference);
- there is a non empty intersection of the intervals (indifference).

Definition 3.2 *A $\langle P, I\rangle$ preference structure on a set A is a PI semi order iff $\exists\, l : A \mapsto \mathcal{R}^+$ and a positive constant k such that:*
$\forall\, x, y : P(x, y) \ \Leftrightarrow\ l(x) > l(y) + k$
$\forall\, x, y : I(x, y) \ \Leftrightarrow\ |l(x) - l(y)| < k$

Such structures have been extensively studied in the literature (see for example Fishburn 1985). We recall here below the two fundamental results which characterize interval orders and semi orders.

Theorem 3.1 *A $\langle P, I\rangle$ preference structure on a finite set A is a PI interval order iff $P.I.P \subset P$.*

Proof. See Fishburn (1985).

Theorem 3.2 *A $\langle P, I\rangle$ preference structure on a finite set A is a PI semi order iff $P.I.P \subset P$ and $I.P.P \subset P$.*

Proof. See Fishburn (1985).

4 $\langle P, Q, I \rangle$ interval orders

As mentioned in the introduction, we are interested in situations where, comparing elements evaluated by intervals, one wants to distinguish three situations: indifference if one interval is included in the other, strict preference if one interval is completely "to the right" of the other and weak preference when one interval is "to the right" of the other, but they have a non empty intersection. Definition 4.1 precisely states this kind of situation, $l(x)$ and $r(x)$ respectively representing the left and right extremities of the interval associated to any element $x \in A$.

Definition 4.1 *A $\langle P, Q, I \rangle$ preference structure on a finite set A is a PQI interval order, iff there exist two real valued functions l and r such that, $\forall x, y \in A, x \neq y$:*
- $r(x) > l(x)$;
- $P(x, y) \Leftrightarrow r(x) > l(x) > r(y) > l(y)$;
- $Q(x, y) \Leftrightarrow r(x) > r(y) > l(x) > l(y)$;
- $I(x, y) \Leftrightarrow r(x) > r(y) > l(y) > l(x)$ or $r(y) > r(x) > l(x) > l(y)$.

The reader will notice that the above definition immediately follows Definition 3.1 since a preference structure characterized a PI interval order can always be seen as PQI interval order also. We give now necessary and sufficient conditions under which such a preference structure exists.

Theorem 4.1 *A $\langle P, Q, I \rangle$ preference structure on a finite set A is a PQI interval order, iff there exists a partial order I_l such that:*
i) $I = I_l \cup I_r \cup I_o$ where $I_o = \{(x, x),\ x \in A\}$ and $I_r = I_l^{-1}$;
ii) $(P \cup Q \cup I_l)P \subset P$;
iii) $P(P \cup Q \cup I_r) \subset P$;
iv) $(P \cup Q \cup I_l)Q \subset P \cup Q \cup I_l$; $RQ \subset Q \cup P$.
v) $Q(P \cup Q \cup I_r) \subset P \cup Q \cup I_r$;

Proof

Necessity.

We first give an outline of necessity demonstration which is the easy part of the theorem. If $\langle P, Q, I \rangle$ is a PQI interval order, then defining
- $I_l(x, y) \Leftrightarrow l(y) < l(x) < r(x) < r(y)$
- $I_r(x, y) \Leftrightarrow l(x) < l(y) < r(y) < r(x)$

we obtain two partial orders satisfying the desired properties. As an example we demonstrate property (v):

$Q(x,y)$ and $(P \cup Q \cup I_r)(y,z)$ imply $r(x) > r(y)$ and $r(y) > r(z)$, hence $r(x) > r(z)$, so that $(P \cup Q \cup I_r)(x,z)$.

Sufficiency.

Conversely let us assume the existence of I_l satisfying the properties of the theorem. Define a set A' isomorphic to A (A' and A being disjoint) and denote by x' the image of $x \in A$ in A'. In the set $A \cup A'$ let us define the relation S as follows: $\forall\, x, y \in A, x \neq y$

- $S(x', x)$
- $S(x,y) \Leftrightarrow (P \cup Q \cup I_l)(x,y)$
- $S(x',y') \Leftrightarrow (P \cup Q \cup I_r)(x,y)$
- $S(x,y') \Leftrightarrow P(x,y)$
- $S(x',y) \Leftrightarrow \neg P(y,x)$

We demonstrate now that S is a linear order (irreflexive, complete and transitive relation) in $A \cup A'$.

Irreflexivity results from irreflexivity of P, Q, I_l and I_r.

To demonstrate completeness of S remark that for $x \neq y$:

$$\begin{aligned}
\neg S(x,y) &\Leftrightarrow \neg(P \cup Q \cup I_l)(x,y) \\
&\Leftrightarrow (P \cup Q \cup I_l)(y,x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_l \cup I_r \cup I_o \\
&\Leftrightarrow S(y,x)
\end{aligned}$$

$$\begin{aligned}
\neg S(x',y') &\Leftrightarrow \neg(P \cup Q \cup I_r)(x,y) \\
&\Leftrightarrow (P \cup Q \cup I_r)(y,x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_l \cup I_r \cup I_o \\
&\Leftrightarrow S(y',x')
\end{aligned}$$

$$\begin{aligned}
\neg S(x,y') &\Leftrightarrow \neg P(x,y) \\
&\Leftrightarrow S(y',x)
\end{aligned}$$

$$\begin{aligned}
\neg S(x',y) &\Leftrightarrow P(y,x) \\
&\Leftrightarrow S(y,x')
\end{aligned}$$

We demonstrate now that S is transitive.

- $S(x,y)$ and $S(y,z)$ imply $(P \cup Q \cup I_l)(x,y)$ and $(P \cup Q \cup I_l)(y,z)$. From conditions ii) and iv) of the theorem, we know that $(P \cup Q \cup I_l)(x,y)$ and $(P \cup Q)(y,z)$ imply $(P \cup Q \cup I_l)(x,z)$, hence $S(x,z)$. From transitivity of I_l we have that $I_l(x,y)$ and $I_l(y,z)$ imply $I_l(x,z)$, hence $S(x,z)$. Finally, if $(P \cup Q)(x,y)$ and $I_l(y,z)$ then $(P \cup Q \cup I_l)(x,z)$ because, if not, we would have $(P \cup Q \cup I_l)(z,x)$ which with $I_l(y,z)$ would give $(P \cup Q \cup I_l)(y,x)$ (by conditions ii) and iv) and transitivity of I_l), contradiction. So we get $S(x,z)$.

- $S(x,y)$ and $S(y,z')$ imply $(P \cup Q \cup I_l)(x,y)$ and $P(y,z)$, which, by condition ii), give $P(x,z)$, hence $S(x,z')$.

- $S(x,y')$ and $S(y',z)$ imply $P(x,y)$ and $\neg P(z,y)$. If $\neg S(x,z)$, then $(P \cup Q \cup I_l)(z,x)$ which, with $P(x,y)$ and by condition ii) would give $P(z,y)$, a contradiction. Thus $S(x,z)$. This reasoning applies also in the case $y = z$.

- $S(x,y')$ and $S(y',z')$ imply $P(x,y)$ and $(P \cup Q \cup I_r)(y,z)$, which, by condition iii), give $P(x,z)$, hence $S(x,z')$.

- $S(x',y')$ and $S(y',z)$ imply $(P \cup Q \cup I_r)(x,y)$ and $\neg P(z,y)$. If $\neg S(x',z)$, then $P(z,x)$ which, with $(P \cup Q \cup I_r)(x,y)$ and by condition iii) would give $P(z,y)$, a contradiction. Thus $S(x',z)$. This reasoning applies also in the case $y = z$.

- $S(x',y')$ and $S(y',z')$ imply $(P \cup Q \cup I_r)(x,y)$ and $(P \cup Q \cup I_r)(y,z)$. From conditions iii) and v) of the theorem, we know that $(P \cup Q)(x,y)$ and $(P \cup Q \cup I_r)(y,z)$ imply $(P \cup Q \cup I_r)(x,z)$, hence $S(x',z')$. From transitivity of I_r we have that $I_r(x,y)$ and $I_r(y,z)$ imply $I_r(x,z)$, hence $S(x',z')$. Finally, if $I_r(x,y)$ and $(P \cup Q)(y,z)$ then $(P \cup Q \cup I_r)(x,z)$ because, if not, we would have $(P \cup Q \cup I_r)(z,x)$ which with $I_r(x,y)$ would give $(P \cup Q \cup I_r)(z,y)$ (by condition iii) and v) and transitivity of I_r), contradiction. So we get $S(x',z')$.

- $S(x',y)$ and $S(y,z)$ imply $\neg P(y,x)$ and $(P \cup Q \cup I_l)(y,z)$ If $\neg S(x',z)$, then $P(z,x)$ which, with $(P \cup Q \cup I_l)(y,z)$ and by condition ii) would give $P(y,x)$, a contradiction. Thus $S(x',z)$. This reasoning applies also in the case $y = x$.

- $S(x',y)$ and $S(y,z')$ imply $\neg P(y,x)$ and $P(y,z)$. If $\neg S(x',z')$, then $(P \cup Q \cup I_r)(z,x)$ which, with $P(y,z)$ and by condition iii) would give $P(y,x)$, a contradiction. Thus $S(x',z')$. This reasoning applies also in the case $y = x$.

Since S is a linear order on $A \cup A'$, there exists a real valued function u such that, $\forall\, x, y \in A$:
- $S(x, y) \;\Leftrightarrow\; u(x) > u(y)$;
- $S(x', y') \;\Leftrightarrow\; u(x') > u(y')$;
- $S(x, y') \;\Leftrightarrow\; u(x) > u(y')$;
- $S(x', y) \;\Leftrightarrow\; u(x') > u(y)$.

We define $\forall\, x \in A$, $l(x) = u(x)$ and $r(x) = u(x')$ and we obtain:

- $\forall\, x : r(x) > l(x)$, since $S(x', x)$.
- $\forall\, x, y :\; P(x, y) \;\Leftrightarrow\; S(x, y') \;\Leftrightarrow\; l(x) > r(y)$.
- $\forall\, x, y :\; Q(x, y) \;\Leftrightarrow\; S(x, y) \wedge S(x', y') \wedge \neg P(x, y) \;\Leftrightarrow$ $l(x) > l(y)$ and $r(x) > r(y)$ and $r(y) > l(x)$, equivalent to: $r(x) > r(y) > l(x) > l(y)$.
- $\forall\, x, y :\; I(x, y) \;\Leftrightarrow$ $r(x) > r(y) > l(y) > l(x)$ or $r(y) > r(x) > l(x) > l(y)$ since $I(x, y)$ holds in all the remaining cases.

■

We can complete the investigation providing a characterization of PQI semi orders.

Definition 4.2 *A PQI semi order is a PQI interval order such that $\exists\, k > 0$ constant for which $\forall x :\;\; r(x) = l(x) + k$*

In other words, a PQI semi order is a $\langle P, Q, I \rangle$ preference structure for which there exists a real valued function $l : A \mapsto \mathcal{R}$ and a positive constant k such that $\forall\, x, y,\; x \neq y$:
- $P(x, y) \;\Leftrightarrow\; l(x) > l(y) + k$;
- $Q(x, y) \;\Leftrightarrow\; l(y) + k > l(x) > l(y)$;
- $I(x, y) \;\Leftrightarrow\; l(x) = l(y)$ (actually I reduces to I_o).

For such preference structures the following theorem holds.

Theorem 4.2 *A $\langle P, Q, I \rangle$ preference structure is a PQI semi order iff:*
i) I is transitive
ii) $PP \cup PQ \cup QP \subset P$;
iii) $QQ \subset P \cup Q$;

Proof

Necessity is trivial. We give only the sufficiency proof. Since I is an equivalence relation, we consider the relation $P \cup Q$ on the set A/I. Such a relation is clearly a linear order (irreflexivity and completeness result from definition 2.4 and transitivity from conditions ii) and iii) of the theorem). Therefore we can index the elements of A/I by $i = 1, 2 \cdots, n$ in such a way that $\forall\, x_i, x_{i+1} \in A/I$: $(P \cup Q)(x_{i+1}, x_i)$.

Choosing an arbitrary positive value k, we define function l as follows:
$l(x_1) = 0$ and for $i = 2, 3, \cdots n$
$l(x_{i+1}) > l(x_i)$
$l(x_i) > l(x_j) + k \;\; \forall\, j < i$ such that $P(x_i, x_j)$
$l(x_i) < l(x_m) + k \;\; \forall\, m < i$ such that $Q(x_i, x_m)$.

This is always possible because $P(x_i, x_j)$ and $Q(x_i, x_m)$ imply $(P \cup Q)(x_m, x_j)$ (if not, we would have $(P \cup Q)(x_j, x_m)$ which, with $P(x_i, x_j)$ and by condition ii) would give $P(x_i, x_m)$, hence $m > j$ and $l(x_m) > l(x_j)$). By construction the function l satisfies the numerical representation of a PQI semi order.

■

5 Detection of a PQI interval order

The problem is the following:
Given a set A and a $\langle P, Q, I \rangle$ preference structure on it, verify whether it is a PQI interval order. The difficulty resides in the fact that the theorem previously announced contains a second order condition which is the existence of the partial order I_l. For this purpose we give two propositions which show the difficulties in detecting such a structure.

Proposition 5.1 *There exist $\langle P, Q, I \rangle$ preference structures which are $P\hat{I}$ interval orders (where $\hat{I} = Q \cup I \cup Q^{-1}$), but are not PQI interval orders.*

Proof Consider the following case.
- $A = \{a, b, c, d, e\}$;
- $P = \{(a, c), (d, e), (a, e)\}$;
- $Q = \{(d, c), (a, b), (b, e)\}$;
- $I = \{(a, d), (c, e), (b, d), (b, c), (d, a), (e, c), (d, b), (c, b)\} \cup I_o$

On the one hand if we consider the relation $\hat{I} = Q \cup I \cup Q^{-1}$ it is easy to observe that the $\langle P, \hat{I} \rangle$ preference structure is a PI interval order ($P\hat{I}P \subset P$ holds). On the other hand if we accept that the given $\langle P, Q, I \rangle$ preference structure is a PQI interval order then we have (by definition 4.1 and theorem 4.1) that:
- $I(a, d)$ has to be $I_l(a, d)$ because of c;
- $I(d, b)$ has to be $I_l(d, b)$ because of e;
therefore by transitivity we should have $I_l(a, b)$, while we have $Q(a, b)$ which is impossible. Therefore we can conclude that for this particular case the PQI interval order representation is impossible.

■

Proposition 5.2 *There exist $\langle P, Q, I \rangle$ preference structures which have more than one PQI interval order representation.*

Proof Consider the following case.
- $A = \{a, b, c\}$;
- $P = \emptyset$;
- $I = \{(a, c), (b, c), (c, a), (c, b)\} \cup I_o$;
- $Q = \{(a, b)\}$

It is easy to observe that both $I_l(a, c), I_l(b, c)$ and $I_l(c, a), I_l(c, b)$ are possible, thus allowing two different PQI interval orders: one in which the interval of c is included in the intervals of both a and b and the other where the intervals of b and a are included in the interval c. Both representations are correct, although incompatible with each other.

■

The basic theorem 4.1, which gives necessary and sufficient conditions to see if a PQI preference structure is a PQI interval order, is unfortunately a formula in a second order logic (a formula where predicates can be variables). Generally the satisfaction of second order formula can be undecidable. Moreover, the theorem does not give a constructive procedure for verifying its satisfaction. In the following we give a second theorem, equivalent to theorem 4.1, which enables to define an algorithm detecting if a PQI preference structure is a PQI interval order.

Theorem 5.1 *A PQI preference structure on a finite set A is a PQI interval order iff there exists a partial order I_l such that:*
i. $I = I_l \cup I_r \cup I_o$ where $I_o = \{(x, x),\ x \in A\}$ and $I_r = I_l^{-1}$;

ii. $P.Q \cup Q.P \cup P.P \subset P$ *and* $Q.Q \subset P \cup Q$;
iii. $(P.Q^{-1} \cap I) \subset I_l$;
iv. $(P^{-1}.Q \cap I) \subset I_l$;
v. $(I.I \cap P) \subset I_l.I_r$;
vi. $(I.I \cap (Q \cup Q^{-1})) \subset (I_l.I_r) \cup (I_r.I_l))$
vii. $I_l.I_l \subset I_l$;

Proof

We will now prove that the conditions $i - vii$ are equivalent to $i - v$ of theorem 4.1.

Necessity - (i-v) of 4.1 ⇒ (i-vii) of 5.1

ii. $P.Q \cup Q.P \cup P.P \subset P$ and $Q.Q \subset P \cup Q$;
$(P \cup Q \cup I_l)P \subset P \Rightarrow P.P \cup Q.P \subset P$
$P(P \cup Q \cup I_r) \subset P \Rightarrow P.Q \subset P$
$(P \cup Q \cup I_l)Q \subset P \cup Q \cup I_l \Rightarrow Q.Q \subset (P \cup Q) \cup I_l$
$Q(P \cup Q \cup I_r) \subset P \cup Q \cup I_r \Rightarrow Q.Q \subset (P \cup Q) \cup I_r$
I_l asymmetric $\Rightarrow I_l \cap I_r = \emptyset$
We have then $Q.Q \subset (P \cup Q) \cup (I_l \cap I_r) = (P \cup Q)$.

iii. $(P.Q^{-1} \cap I) \subset I_l$;
We will prove that $P.Q^{-1} \cap (I_0 \cup I_r) = \emptyset$. Suppose that:
$\exists x, y, z : \ P(x,y) \wedge Q(z,y) \wedge I_0(x,z) \ \ i.e. \ \ x = z$
Impossible since it implies $P(x,y) \wedge Q(x,y)$.
$\exists x, y, z : \ P(x,y) \wedge Q(z,y) \wedge I_r(x,z) \ \ i.e. \ \ I_l(z,x)$
Since $I_l.P \subset P$ (*ii* of 4.1), we have $P(z,y) \wedge Q(z,y)$, impossible.

iv. $(P^{-1}.Q \cap I) \subset I_l$;
The proof is quite similar to that of [iii].
$P^{-1}.Q \cap (I_0 \cup I_r) = \emptyset$. Suppose that:
$\exists x, y, z : \ P(y,x) \wedge Q(y,z) \wedge I_0(x,z) \ \ i.e. \ \ x = z$
Impossible since it implies $P(y,x) \wedge Q(y,x)$.
$\exists x, y, z : \ P(y,x) \wedge Q(y,z) \wedge I_r(x,z)$
Since $P.I_r \subset P$ (*iii* of 4.1), we have $P(y,z) \wedge Q(y,z)$, impossible.

vii. $I_l.I_l \subset I_l$;
Evident as I_l transitive (we also have $I_r.I_r \subset I_r$).

vi. $(I.I \cap (Q \cup Q^{-1})) \subset (I_l.I_r) \cup (I_r.I_l))$;
We have $I.I \subset (I \cup I_l.I_l \cup I_r.I_r) \cup (I_r.I_l \cup I_l.I_r)$

As P, Q, I are mutually exclusive and the above result (vii), we have $(Q \cup Q^{-1}) \cap (I \cup I_l.I_l \cup I_r.I_r) = \emptyset$

v. $(I.I \cap P) \subset I_l.I_r$;
Similarly to vi, we have $(I.I \cap P) \subset (I_l.I_r) \cup (I_r.I_l))$, we still have to prove that $P \cap I_r.I_l = \emptyset$.
$\exists x, y, z: \;\; P(x,z) \wedge I_r(x,y) \wedge I_r(z,y) \Rightarrow P(x,y) \wedge I_r(x,y)$, impossible.

Sufficiency - (i-vii) of 5.1 $\Rightarrow$ (i-iv) of 4.1

ii. $(P \cup Q \cup I_l).P \subset P$.
$(P.P \cup Q.P) \subset P$ by ii of 5.1
$I_l.P \subset P$. Suppose that:
$\exists x, y, z: \;\; I_l(x,y) \wedge P(y,z) \wedge P(z,x)$.
Impossible since it implies $P(y,x)$ by ii of 5.1
$\exists x, y, z: \;\; I_l(x,y) \wedge P(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x)$ by ii of 5.1
$\exists x, y, z: \;\; I_l(x,y) \wedge P(y,z) \wedge I_l(z,x)$.
Impossible since it implies $I_l(z,y)$ by vii of 5.1
$\exists x, y, z: \;\; I_l(x,y) \wedge P(y,z) \wedge I_l(x,z)$.
Impossible since it implies $I_r(x,y) \wedge I_r(x,z)$ by v of 5.1
$\exists x, y, z: \;\; I_l(x,y) \wedge P(y,z) \wedge Q(x,z)$.
Impossible since it implies $I_l(y,x)$ by iii of 5.1.

iii. $P.(P \cup Q \cup I_l^{-1}) \subset P$.
$(P.P \cup P.Q) \subset P$ by ii of 5.1;
$P.I_l^{-1} \subset P$. Suppose that:
$\exists x, y, z: \;\; P(x,y) \wedge I_l^{-1}(y,z) \wedge P(z,x)$.
Impossible since it implies $P(z,y)$ by ii of 5.1
$\exists x, y, z: \;\; P(x,y) \wedge I_l^{-1}(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x)$ by ii of 5.1
$\exists x, y, z: \;\; P(x,y) \wedge I_l^{-1}(y,z) \wedge I_l(z,x)$.
Impossible since it implies $I_l(x,z) \wedge I_l(y,z)$ by v of 5.1
$\exists x, y, z: \;\; P(x,y) \wedge I_l^{-1}(y,z) \wedge I_l(x,z)$.
Impossible since it implies $I_l(x,y)$ by vii of 5.1
$\exists x, y, z: \;\; P(x,y) \wedge I_l^{-1}(y,z) \wedge Q(x,z)$.
Impossible since it implies $I_l(y,z)$ by iv of 5.1.

iv. $(P \cup Q \cup I_l).Q \subset P \cup Q \cup I_l$.
$(P.Q \cup Q.Q) \subset P$ by ii of 5.1;
$I_l.Q \subset P \cup Q \cup I_l$. Suppose that:

$\exists x, y, z: \ I_l(x,y) \wedge Q(y,z) \wedge P(z,x)$.
Impossible since it implies $P(y,x)$ by *ii* of 5.1
$\exists x, y, z: \ I_l(x,y) \wedge Q(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x) \vee Q(y,x)$ by *ii* of 5.1
$\exists x, y, z: \ I_l(x,y) \wedge Q(y,z) \wedge I_l(z,x)$.
Impossible since it implies $I_l(z,y)$ by *vii* of 5.1.

v. $Q.(P \cup Q \cup I_l^{-1}) \subset P \cup Q \cup I_l^{-1}$.
$Q.P \subset P$ by *ii* of 5.1;
$Q.Q \subset P \cup Q$ by *ii* of 5.1;
$Q.I_l^{-1} \subset P \cup Q \cup I_l^{-1}$. Suppose that:
$\exists x, y, z: \ Q(x,y) \wedge I_l^{-1}(y,z) \wedge P(z,x)$.
Impossible since it implies $P(z,y)$ by *ii* of 5.1
$\exists x, y, z: \ Q(x,y) \wedge I_l^{-1}(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x) \vee Q(y,x)$ by *ii* of 5.1
$\exists x, y, z: \ Q(x,y) \wedge I_l^{-1}(y,z) \wedge I_l(x,z)$.
Impossible since it implies $I_l(x,y)$ by *vii* of 5.1.

■

From this theorem, we have the following algorithm which constructs I_l by converting elements of I either to I_l or I_r. By definition, when $I_l(x,y)$ is established, $I_r(y,x)$ is also established. The algorithm is a direct application of conditions *i* to *vii* of theorem 5.1. Therefore if it succeeds in transforming all elements of I in elements of I_l (or I_r) then the PQI preference structure under investigation is a PQI interval order. If on the other hand it fails then the PQI preference structure under investigation is not a PQI interval order. Failure of the algorithm can occur either because condition *ii* is not satisfied or because during the construction of I_l a contradiction occurs (in the sense that two elements of the set A are linked by two different relations).

Algorithm 5.1

Step 1: *Verify* $P.Q \cup Q.P \cup P.P \subset P$ *and* $Q.Q \subset P \cup Q$;
Step 2: $\forall\, x, y, z \ I(x,y) \wedge P(x,z) \wedge Q(y,z) \rightarrow I_l(x,y)$;
Step 3: $\forall\, x, y, z \ I(x,y) \wedge P(z,x) \wedge Q(z,y) \rightarrow I_l(x,y)$;
Step 4: $\forall\, x, y, z \ I(x,y) \wedge I(y,z) \wedge P(x,z) \rightarrow I_l(x,y) \wedge I_l(z,y)$;
Step 5: $\forall\, x, y, z \ I_l(x,y) \wedge I(y,z) \wedge (Q \cup Q^{-1})(x,z) \rightarrow I_l(z,y)$;
$\forall\, x, y, z \ I(x,y) \wedge I_l(y,z) \wedge (Q \cup Q^{-1})(x,z) \rightarrow I_l(y,x)$;
Step 6: $\forall\, x, y, z \ I_l(x,y) \wedge I_l(y,z) \rightarrow I_l(x,z)$;
Step 7: *If there is one* $I(x,y)$ *not yet established as* I_l *or* I_r*, choose one of them and set it as* $I_l(x,y)$*. Then return to 5. Otherwise stop.*

Steps 1 to 4, are deterministic, in the sense that each I_l established is mandatory. If a contradiction occurs, i.e. a newly established $I_l(x,y)$ has been formerly established as $\Phi(x,y)$, Φ being any among P, Q, P^{-1}, Q^{-1}, the algorithm fails. Steps 5 and 6 however, use already established I_l in order to establish further I_l. The problem arises from Step 7 where I_l is arbitrarily chosen. When the algorithm goes back to Step 5 to continue with establishing I_l, if a contradiction occurs, intuitively, it should backtrack to the last $I_l(x,y)$ established, reverse it to $I_l(y,x)$ and try again. In other terms the algorithm appears to have to explore a "tree structure" defined by the branches created by each arbitrary choice. In such a case the risk is to have to make an exhaustive research of the whole "tree".

In the following we will demonstrate that the algorithm previously presented is "backtracking free". In other words, any contradiction, implies the non-existence of a PQI interval order on A and the algorithm can stop immediately without backtracking.

Theorem 5.2 *The algorithm 5.1 is backtracking free.*

Proof We elaborate the demonstration observing how the setting of $I_l(x,y)$ (steps 5, 6) is propagated and analyzing contradictory situations. The demonstration consists of decomposing the problem in smaller cases and showing for each of them that when a contradiction occurs there is no backtracking necessity and the algorithm fails (the PQI preference structure is not a PQI interval order).

Before reaching step 7 the first time, the process is deterministic, we can therefore construct the graph $G_0 = (A, V_0)$ where A is the usual set of objects on which the PQI preference structure applies and $V_0 = P \cup Q \cup I \cup I_l$ where I consists of (x,y) which are not yet set. G_0 is complete and all its arcs are directed except the ones in I. In the following we denote as a *"triangle"* a set of three elements in A (x,y,z) such that $x\Phi y\Psi z\Theta x$, where Φ, Ψ, Θ are any among $P, P^{-1}, Q, Q^{-1}, I_l, I_l^{-1}, I$.

Proposition 5.3 *In G_0, a triangle with at least an I-arc must be one of the following:*
1 - $I.I.I$
2 - $I.I.I_l$
3 - $I.I.Q$
4 - $I.I_l.I_l^{-1}$
5 - $I.I_l^{-1}.I_l$

6 - $I.P.P^{-1}$
7 - $I.P^{-1}.P$
8 - $I.Q.Q^{-1}$
9 - $I.Q^{-1}.Q$

Proof.
The application of steps 1-6 of the algorithm 5.1 excludes all other possibilities. For example, all triangles $I_l.I.Q$ have changed to $I_l.I_l^{-1}.Q$ by step 5.

■

Denote as I-path a path where each of its arcs is an I-arc. Consider then the partial graph G^* of G_0, $G^* = (A, V_1)$ where $V_1 = \{(x,y)|x \neq y, \ \exists \ I\text{-path from } x \text{ to } y\}$. The proofs of propositions 5.4 to 5.11 can be found in Appendix A.

Proposition 5.4 *G^* consists of connected components which:*
i. are complete;
ii. do not contain any P-arc;
iii. are closed under the propagation of the setting of I_l.

We have proved that G^* consists of connected components in which the propagation of the setting of $I_l(x,y)$ is limited. Each component contains only Q or I or I_l arcs, while P arcs exist only among such components. Therefore, we can limit ourselves in analyzing only one connected component, denoted by $G_1 = (A_1, V_1)$.

Let (x^*, y^*) be an I-arc arbitrarily chosen in step 7 to become an I_l-arc. Consider iteration k of the algorithm. Denote as I_l^k the set of I-arcs set in I_l in the current step and as I_l^K the cumulative set of I-arcs set in I_l in all the former iterations of the algorithm. We have that $I_l^K = I_l^k \cup I_l^{K-1}$. Conventionally, in step 5, (x^*, y^*) is added to I_l^k.

Proposition 5.5 *I-arcs set to I_l by transitive closure (step 6) are never used in step 5 when the algorithms iterates.*

Denote as a Q-path a path whose arcs are Q or Q^{-1} ones. In the set A, let us consider now the following equivalence relation: $\Theta(x,y) \Leftrightarrow \exists$ a Q-path from x to y and use X, Y, Z to denote equivalence classes. Therefore we can see graph G_1 as composed by equivalence classes of nodes each of which contains only Q and I arcs. Further on among such equivalence classes only I-arcs do exist.

Proposition 5.6 *In step 5*
i - the propagation of $I_l(x,y) \in X \times Y$ is limited to $X \times Y$.
ii - when $X \neq Y$, the propagation of I_l covers the whole set $X \times Y$.
iii - If $(x^*, y^*) \in X \times X$ then $I_l^k \subset X \times X$
iv - If $(x^*, y^*) \in X \times Y$, $X \neq Y$ then $I_l^k = X \times Y$.
v - Whatever (x,y) is chosen to be set in I_l in Step 5 the result is the same.
vi - If $I_l(y^*, x^*)$ is chosen instead of $I_l(x^*, y^*)$ then all the settings in this step will be reversed.

Proposition 5.6 states that, during the k-th iteration of the algorithm, Step 5 sets to I_l either some I-arcs included in one equivalence class (of relation Θ) or all I-arcs among two equivalence classes.

Consider now Step 6. In each application of step 6, setting $I_l(x,z)$ from $I_l(x,y)$ and $I_l(y,z)$, implies that at least one arc, let's say (x,y), has to be set during, either this step, or the two last steps 5,7. In a formal notation we have:

Proposition 5.7 *In Step 6:*
i - If $(x,y) \in X \times X$ then $z \in X$.
ii - If $(x^*, y^*) \in X \times X$ is set in Step 5 then $I_l^k \subset X \times X$.
iii - If it exists $I_l^k(x,z) \in X \times Z$, $X \neq Z$ then $X \times Z \subset I_l^k$.
iv - If $(x^*, y^*) \in X \times Y$, $X \neq Y$ is set in Step 5, only arcs connecting different classes are set in Step 6 (in other terms if $I_l(x,z) \in X \times Z$ is set in Step 6 then $Z \neq X \wedge Z \neq Y$).

These results show that if we choose an arc (x^*, y^*) to set in I_l, if it is inside one equivalent class it does not propagate I_l outside this class, while if it connects two different classes, it does not propagate I_l into any class. Furthermore, as the algorithm has passed through steps 5, 6 before the establishment of G_1 at least once, all the arcs between two classes X, Y are of the same type (either I-arcs or I_l-arcs). Therefore, the problem can be further decomposed into two sub-problems:
a) - Outside all the equivalent classes, we consider the same problem with G_1 replaced by $G_2 = (A_2, V_2)$ where A_2 is the quotient set A^Θ and $V_2 = \{(X,Y) | X, Y \in A_2 \wedge \exists (x,y) \in X \times Y$ such that I or I_l holds$\}$ according to the type of the arcs connecting X, Y.
b) - Inside each equivalent class, we consider the same problem with G_1 replaced by $G_3 = (A_3, V_3)$.

The sub-problem a) is trivial, as the graph G_2 contains only I or I_l arcs, furthermore, the part of G_2 covered by I_l-arcs is already I_l transitively closed since the algorithm has already gone through Step 6. The problem is reduced to the construction of a linear order. Therefore, we have to deal only with the sub-problem (b).

We have to demonstrate now that the algorithm is backtracking free on G_3 where the arcs are Q, I_l, I and there is a Q-path connecting any two different nodes. We consider now the possible situations where a contradiction may occur.

Proposition 5.8 *In step 5*
i - $I_l^k(x,y) \wedge I_l^k(y,z) \Rightarrow I_l^k(x,z)$ *i.e. if* (x,y) *and* (y,z) *are set in this step, then so is* (x,z).
ii - $I_l^k(x,y) \wedge I_l^{K-1}(y,z) \wedge I_l^k(z,t) \Rightarrow I_l^k(x,t)$.
iii - $I_l^{K-1}(x,y) \wedge I_l^{K-1}(y,z) \Rightarrow I_l^{K-1}(x,z)$.

N.B. We may emphasize that, while in Step 5, $I_l^k(x,y) \wedge I_l^{K-1}(y,z)$ does not necessarily imply $I_l^k(x,z)$.

Proposition 5.9 *In step 5, an I_l-circuit occurs only with a contradiction.*

Proposition 5.10 *If the first contradiction occurs at step 6, then there must be an I_l circuit at the end of step 5 (an I_l^{K-1} circuit).*

Proposition 5.11 *If the first contradiction occurs at step 5, then the problem has no solution.*

From Proposition 5.10 if a contradiction occurs in Step 6 there is an I_l circuit at Step 5. From Proposition 5.9 if such a circuit exists in Step 5 it has to exist also a contradiction in Step 5. And from Proposition 5.11 if a contradiction occurs at Step 5, the problem has no solution and it is not necessary to make any backtracking. And this concludes our demonstration. ■

6 Matrix version of the algorithm

From the previous discussion it is easy to see that the critical part of the PQI graph to analyze is the G_3 graph, so we may study complexity with respect to this subgraph. In the following we give a way to implement the algorithm and discuss its complexity. Let P, Q, I, L be $n \times n$ matrixes representing relations P, Q, I, I_l respectively, where:
$x_{ij} = 1 \Leftrightarrow X(i,j)$, otherwise $x_{ij} = 0$, X being one among P, Q, I, I_l.

Theorem 6.1 *Algorithm 5.1 is in polynomial time ($O(n^5)$)*

Proof The algorithm presented in the previous section can be represented in the following way (including some small variations discussed immediately after):

Algorithm 6.1
Step 1: $p_{ij}+p_{jk} \leq 1+p_{ik}$, $p_{ij}+q_{jk} \leq 1+p_{ik}$, $q_{ij}+q_{jk} \leq 1+p_{ik}+q_{ik}$ $\forall\, i,j,k$;
Step 2: $i_{ij} = p_{ik} = q_{jk} = 1 \Rightarrow l_{ij} = 1 \;\forall\, i,j,k$;
Step 3: $i_{ij} = p_{ki} = q_{kj} = 1 \Rightarrow l_{ij} = 1 \;\forall\, i,j,k$;
Step 4: $p_{ij} = i_{ik} = i_{kj} = 1 \Rightarrow l_{ik} = l_{kj} = 1 \;\forall\, i,j,k$;
Step 5: $q_{ij} + q_{ji} = i_{ik} = i_{kj} = 1 \Rightarrow l_{ik} = l_{kj} \;\forall\, i,j,k$;
Step 6: $l_{ij} = l_{jk} = 1 \Rightarrow l_{ik} = 1 \;\forall\, i,j,k$;
Step 7: For $I(x,y)$ not yet established as I_l or I_r, choose arbitrarily $I_l(x,y)$. If the I_l established belongs to an equivalence class established in Step 5, put all the elements of the class equal to 1. Return to 6 (instead of 5).

A critical step in this algorithm is step 5 since it introduces implicitly a recursive establishment of I_l. In order to avoid an infinite recursion and the associated contradictions it is necessary to "fix" I_l as soon as it is generated by step 5 so that only $I(x,y)$ which are not yet established may still be considered in the recursive application of step 5. This is possible partitioning the set of non zero elements of the matrix I into classes which will have the same value of l_{ij} because of step 5. Then as soon as one element of one of these classes turns to 1, the whole class will turn to 1. Under such an adjustment the following positive consequences hold:
- if there is no solution then a contradiction in establishing an I_l will appear before step 6;
- after step 7 you just have to return to step 6.

We can now discuss complexity. Steps 1 to 4 are obviously in $O(n^3)$ as step 6 (transitive closure) is. Step 5 is in $O(n^5)$ as can be seen by the following implementation (remark that in the worst case $n = |G_3|$):

```
function step5:  boolean
      forall i, j, k
           if (Iik*Ikj*(Qij+Qji) == 1)
                if ( not setLabel(i,j,k) )
                     return false
      return true
```

```
function setLabel(i,j,k:  integer)
      if (Lik, Lkj no label)
           set new label to Lik and Lkj
      else if (Lik = L1, Lkj no label)
           set Lkj to L1
      else if (Lik no label, Lkj = L2)
           set Lik to L2
      else if (Lik = L1 et Lkj = -L1)
           return false (conflict)
      else if (Lik = L1 et Lkj = L2)
           unify these two labels
      endif
      return true
```

Furthermore it is easy to see that the decomposition of the PQI graph in G_1 and its connected components, the decomposition in G_2 and G_3 and the construction of the linear order in G_2 are all in polynomial time. Therefore the whole algorithm is in polynomial time.

■

7 Conclusions

The paper has presented an answer for the problem concerning the representation of preferences by intervals by showing necessary as well as sufficient conditions to see if a preference structure is a PI interval order, PQI semi order or PQI interval order. For PQI interval order, it provided also an algorithm to verify whether a PQI preference structure on a finite set A is a PQI interval order. In other words verify if it is possible to associate to each element of A an interval such that if the interval associated to x is completely to the right of the interval associated to y, then x is strictly preferred to y, if one interval is included in the other, then x is indifferent to y and if the interval associated to x is to the right of the interval associated to y, their intersection being not empty, then x is weakly preferred to y. We first demonstrate that the algorithm, although it appears having to explore a tree generated by branches of arbitrary choices, is backtracking free and then we demonstrate that runs in polynomial time. We consider such a result very promising, since it enables an efficient check of the existence of PQI interval orders which are very common in many different cases, including preference modeling and temporal logic.

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Appendix A

Proof of Proposition 5.4

i. If x, y belongs to a connected component then there exists a path $a_0 = x, a_1, ...a_k = y$. $\forall\ i = 0...k-1$, if exists an I-path from a_i to a_{i+1} then there exists an I-path from x to y and therefore $(x, y) \in V_1$.
ii. If a P-arc exists, choose $P(x, y)$ such that the length k of the I-path $a_0 = x, a_1, ...a_k = y$ is minimal. Consider then the arc a_1, a_k (it exists from the completeness of the component), then from proposition 5.3 we have $P(a_1, a_k)$ and therefore we have another P-arc withlength of the I-path $< k$. Impossible.
iii. Immediate from conditions *vi* and *vii* of Theorem 5.1 (steps 5 and 6 of the algorithm).

■

Proof of Proposition 5.5

First consider (x_1, x_2) such that $I_l^k(x_1, x_2)$ in step 6. Therefore it exists $I_l(x_1, x_3) \wedge I_l(x_3, x_2)$. If, for example, (x_1, x_3) was also established in I_l^k (in the current step 6) then it exists $I_l(x_1, x_4) \wedge I_l(x_4, x_3)$ and so on until an I_l^{K-1}-path is obtained. Therefore for all (x, y) such that I_l is established in the current step 6 exists an I_l^{K-1}-path from x to y.
Let now (x, y) to be an arc set to I_l in the last step 6, participating to the setting of arc (x, z) in step 5 through let's say $Q(z, y)$. Let us consider the situation in the last step 6:
$I_l^k(x, y) \Rightarrow \exists\ I_l^{K-1}$-path $t_0 = x, t_1, ...t_k = y$.
Consider the triangle z, t_{k-1}, y where $Q(z, y) \wedge I_l^{K-1}(t_{k-1}, y)$.
If $Q(t_{k-1}, z)$ then $Q(t_{k-1}, z) \wedge Q(z, y) \Rightarrow (P \cup Q)(t_{k-1}, y)$, conflict with $I_l(t_{k-1}, y)$.
If $I(t_{k-1}, z)$ then $I_l^{K-1}(t_{k-1}, y) \wedge Q(y, z) \wedge I(t_{k-1}, z) \Rightarrow I_l^{K-1}(t_{k-1}, z)$ (at least in the last step 5). Therefore it exists an I_l^{K-1}-path from x to z, that is $I_l(x, z)$ must be set at least at the same time as (x, y). We conclude that $Q(z, t_{k-1})$. Repeat this procedure, and we get at last $Q(z, t_1)$, which together with $I_l^{K-1}(x, t_1)$ gives $I_l^{K-1}(x, z)$ i.e. (x, z) must have been set before (x, y).

■

Proof of Proposition 5.6

i - In each application of step 5, consider $(x, y) \in X \times Y$ such that $I_l(x, y)$. Relation I_l will propagate to (x', y) or (x, y'), x', y' arbitrary. From Theorem 5.1 and proposition 5.4 (no P-arcs in G^1) we know that there have to exist

Q-paths from x to x' and from y to y'. Therefore $(x', y') \in X \times Y$.
ii - $(x', y') \in X \times Y$ implies that there exist Q-paths $a_0 = x, a_1, ... a_k = x'$, and $b_0 = y, b_1, ... b_l = y'$. Applying consecutively step 5 on these two paths we obtain the setting in I_l of $(x, y), (a_1, y), ... (x', y)$ and then of $(x', b_1), (x', b_2), ... (x', y')$.
iii and iv - Immediate from propositions (5.5), (5.6.i) and (5.6.ii).
v and vi - Immediate from Theorem 5.1.

■

Proof of Proposition 5.7
i - Otherwise, consider the first setting with $z \in Z \neq X$. It implies that $I_l(y, z) \in X \times Z$, $Z \neq X$ and since (x, z) is the first such setting, $I_l^{K-1}(y, z)$ holds. We have $x, y \in X \wedge z \in Z \wedge I_l^{K-1}(y, z)$ which implies $I_l^{K-1}(x, z)$ as it must be set at least in the last step 5 (proposition (5.6.ii)). Contradiction.
ii - Immediate from (5.6.iii),(5.7.i).
iii - Otherwise it should exist $(x', z') \in X \times Z \setminus I_l^k$. In the next step 5 (x, z), which is set in this step 6, will propagate I_l to (x', z'), which is impossible because of (5.5).
iv - Suppose that $(x^*, y^*) \in X \times Y$, $X \neq Y$ is introduced in step 5. Then all the arcs of $X \times Y$ are set to I_l and only these arcs. The setting in step 6 is the propagation of such arcs. Let $I_l(x, y) \wedge I_l(y, z) \Rightarrow I_l(x, z)$, $x \in X$, $y \in Y$, $z \in Z$ the first setting in step 6 with $I_l^{K-1}(x, y)$, $I_l^{K-1}(y, z)$ and (x, y) set in the last steps 5,7, i.e. $(x, y) \in X \times Y$ and $X \times Y \subset I_l^{K-1}$. If $Z = X$ then (z, y) should have been set at the same time as (x, y), which contradicts $I_l^k(y, z)$. We conclude that $Z \neq X$ (and similarly that $Z \neq Y$).

■

Proof of Proposition 5.8
i - Let $\mathcal{Q} = Q \cup Q^{-1}$ and $\Psi = I_l \cup I_l^{-1} \cup \mathcal{Q}$. In an equivalence class we have $\forall (x, z)\ \Psi(x, z)$.
If in Step 5 we had $\mathcal{Q}(x, z)$ then $I_l^k(x, y) \wedge \mathcal{Q}(x, z) \wedge I(z, y) \Rightarrow I_l^k(z, y)$, in contradiction with $I_l^k(y, z)$. Therefore we have $\neg \mathcal{Q}(x, z)$ and $\Psi = I_l \cup I_l^{-1}$. The transition from $I_l^k(x, y)$ to $I_l^k(y, z)$ in step 5 passes through 2 Q-paths $x_1 = x, x_2, ... x_n = y$ and $y_1 = y, y_2, ... y_n = z$ where ($x_i = x_{i+1}$ and $y_i \neq y_{i+1}$ and $I(x_i, y_i), I(x_{i+1}, y_{i+1})$) or ($x_i \neq x_{i+1}$ and $y_i = y_{i+1}$ and $I(x_i, y_i), I(x_{i+1}, y_{i+1})$). We consider the two different transitions from (x, y) to (y, z).
1. If $y_2 = y$ then $x \neq x_2$ and therefore $\mathcal{Q}(x, x_2) \wedge I(x_2, y)$. We have then $\mathcal{Q}(x, x_2) \wedge I_l^k(x, y) \wedge I(x_2, y) \Rightarrow I_l^k(x_2, y)$. But $\mathcal{Q}(x_2, z)$ is in contradiction with $I_l^k(x_2, y)$. Therefore $I_l^k(y, z) \Rightarrow \neg \mathcal{Q}(x_2, z)$.

We have $\mathcal{Q}(x,x_2)\wedge I(x_2,z)\wedge\Psi(x,z)\Rightarrow\Psi(x_2,z)$. Therefore the situation is not changed (x_2 plays now the role of x).
2. If $y_2\neq y$ then $x=x_2$. Therefore $\mathcal{Q}(y,y_2)\wedge I(x,y_2)\Rightarrow I_l^k(x,y_2)$. If $\mathcal{Q}(y_2,z)$ holds we have $\mathcal{Q}(y_2,z)\wedge I_l^k(x,y_2)\wedge I(x,z)\Rightarrow I_l^k(x,z)$. Otherwise, $\mathcal{Q}(y,y_2)$ and $I_l^k(y,z)$ give $I_l^k(y_2,z)$ and the situation is not changed (y_2 plays now the role of y).
In order to pass from y to z, it must exist a k such that $y_{k+1}=z$ and $y_k\neq z$, i.e. $\mathcal{Q}(y_k,y_{k+1})\rightarrow\mathcal{Q}(y_k,z)\Rightarrow I_l^k(x,z)$.
ii - Let $\Psi(x,t)$. Since $\mathcal{Q}(y,t)$ or $\mathcal{Q}(x,z)$ is in contradiction with $I_l^k(x,y)$, $I_l^{K-1}(y,z)$ and $I_l^k(z,t)$ we have $I(y,t)$ and $I(x,z)$.
If $\Psi=\mathcal{Q}$ then $I_l^k(x,y)\wedge I(y,t)\wedge\mathcal{Q}(x,t)\Rightarrow I_l^k(t,y)$. But $I_l^k(t,y)\wedge I_l^k(z,t)\Rightarrow I_l^k(z,y)$ (5.8.i) in contradiction with $I_l^{K-1}(y,z)$. Therefore $\Psi=I_l\cup I_{l-1}$. The transition from $I_l^k(x,y)$ to $I_l^k(z,t)$ in step 5 passes through 2 $\mathcal{Q}$-paths $x_1=x,x_2,...x_n=z$ and $y_1=y,y_2,...y_n=z$ where ($x_i=x_{i+1}$ and $y_i\neq y_{i+1}$ and $I(x_i,y_i),I(x_{i+1},y_{i+1})$) or ($x_i\neq x_{i+1}$ and $y_i=y_{i+1}$ and $I(x_i,y_i),I(x_{i+1},y_{i+1})$).
We consider the two different transitions from (x,y) to (z,t).
1. If $y_2=y$ then $x\neq x_2$ therefore $\mathcal{Q}(x,x_2)$.
$\mathcal{Q}(x,x_2)\wedge I_l^k(x,y)\wedge I(x_2,y)\Rightarrow I_l^k(x_2,y)$. If $\mathcal{Q}(x_2,t)$ we have $\mathcal{Q}(x_2,t)\wedge I_l^k(x_2,y)\wedge I(y,t)\Rightarrow I_l^k(t,y)$. But $I_l^k(z,t)\wedge I_l^k(t,y)\Rightarrow I_l^k(z,y)$ (5.8.i), in contradiction with $I_l^{K-1}(y,z)$.
We have then $\neg\mathcal{Q}(x_2,t)\Rightarrow I(x_2,t)$. $\mathcal{Q}(x_2,x)\wedge I(x_2,t)\wedge\Psi(x,t)\Rightarrow\Psi(x_2,t)$ and the situation is not changed (x_2 plays now the role of x).
2. If $y_2\neq y$ then $x_2=x$, therefore $\mathcal{Q}(y,y_2)$.
$\mathcal{Q}(y,y_2)\wedge I_l^k(x,y)\wedge I(x,y_2)\Rightarrow I_l^k(x,y_2)$.
If $\mathcal{Q}(y_2,t)$, we have $\mathcal{Q}(y_2,t)\wedge I_l^k(x,y_2)\wedge\Psi(x,t)\Rightarrow I_l^k(x,t)$.
Otherwise, if $\mathcal{Q}(y_2,z)$ then $\mathcal{Q}(y_2,z)\wedge I_l^k(x,y_2)\wedge I(x,z)\Rightarrow I_l^k(x,z)$.
Therefore, we have $I_l^k(x,z)\wedge I_l^k(z,t)\Rightarrow I_l^k(x,t)$ (5.8.i).
If $I(y_2,z)$ then $\mathcal{Q}(y,y_2)\wedge I_l^{K-1}(y,z)\wedge I(y_2,z)\Rightarrow I_l^{K-1}(y_2,z)$ and the situation is not changed (y_2 plays now the role of y).
So, we have either $I_l(x,t)$ when it exists $\mathcal{Q}(y_i,z)$ or the only way to pass from y to t is through some $y_{k+1}=t$ and $y_k\neq t$ i.e. $\mathcal{Q}(y_k,y_{k+1})\Rightarrow\mathcal{Q}(y_k,t)\Rightarrow I_l(x,t)$.
iii - If $I_l^{K-1}(x,y)$ and $I_l^{K-1}(y,z)$ are set at least in the last step 6 then $I_l^{K-1}(x,z)$ is also set at least in the last step 6.

■

Proof of Proposition 5.9

Let an I_l-circuit with arcs I_l^k or I_l^{K-1}. With (5.8.i), we can replace all I_l^k-paths with I_l^k-arcs. With (5.8.iii), we can replace all I_l^{K-1}-paths with I_l^{K-1}-arcs. We get at last an I_l-circuit with alternative I_l^k-arcs and I_l^{K-1}-