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THE PROBABILISTIC MINIMUM VERTEX COVERING PROBLEM

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Le problème de la couverture minimum de sommets probabiliste

Résumé

Une instance du problème de la couverture de sommets probabiliste est une paire $(G = (V, E), \text{Pr})$ obtenue en associant à chaque sommet $v_i \in V$ une probabilité « d'occurrence » p_i . Nous considérons une stratégie de modification M transformant une couverture de sommets C de G en une couverture de sommets C_I pour le sous-graphe de G induit par l'ensemble de sommets $I \subseteq V$. L'objectif, pour la couverture de sommets probabiliste, est de déterminer une couverture de sommets de G minimisant la somme, sur tous les sous-ensembles $I \subseteq V$, des produits : probabilité de I fois C_I . Dans cet article, nous étudions la complexité de résolution optimale du problème de couverture de sommet probabiliste.

Mots-clé : problèmes combinatoires, complexité, couverture de sommets, graphe

The probabilistic minimum vertex covering problem

Abstract

An instance of the *probabilistic vertex-covering problem* is a pair $(G = (V, E), \text{Pr})$ obtained by associating with each vertex $v_i \in V$ an "occurrence" probability p_i . We consider a modification strategy M transforming a vertex cover C for G into a vertex cover C_I for the subgraph of G induced by a vertex-set $I \subseteq V$. The objective for the probabilistic vertex-covering is to determine a vertex cover of G minimizing the sum, over all subsets $I \subseteq V$, of the products: probability of I times C_I . In this paper, we study the complexity of optimally solving probabilistic vertex covering.

Keywords: combinatorial problems, computational complexity, vertex covering, graph

1 Introduction

Given a graph $G = (V, E)$, a vertex cover is a set $C \subseteq V$ such that for every $v_i v_j \in E$ either $v_i \in C$, or $v_j \in C$. The *vertex-covering problem* (VC) consists in finding a vertex cover of minimum size. A natural generalization of VC, denoted by WVC in the sequel, is the one where positive weights are associated with the vertices of V . The objective for WVC is to compute a vertex cover for which the sum of the weights of its vertices is the smallest over all vertex covers of G .

Let $\mathcal{G}$ be the set of graphs. Given $G \in \mathcal{G}$, denote by $\mathcal{C}(G)$ the set of vertex covers of G . For any subset $I \subseteq V$ we denote by $G[I]$ the subgraph of G induced by I . The *probabilistic vertex-covering*, denoted by PVC in what follows, is a quintuple $(\mathcal{H}, \mathcal{C}(G), M, E(G_C^M), \min)$ such that:

$\mathcal{H}$ is the set of instances of PVC defined as $\mathcal{H} = \{G = (G, \text{Pr})\}$, where $G \in \mathcal{G}$ and Pr is an n -vector of vertex-probabilities;

$\mathcal{C}(G)$ is the set of vertex covers of G ($\mathcal{C}(G) = \mathcal{C}(G)$);

M is a *modification strategy*, i.e., an algorithm receiving $C \in \mathcal{C}(G)$ and a $G[I]$, $I \subseteq V$ as inputs and modifying C in order to produce a vertex cover for $G[I]$;

$E(G_C^M)$ is the functional, or objective value of C ; consider $G \in \mathcal{H}$, $C \in \mathcal{C}(G)$, $I \subseteq V$, and let $p_i = \text{Pr}[v_i]$, $v_i \in V$, be the probability that vertex v_i is present; moreover, denote by C_I^M the solution computed by $M(C, G[I])$ and by $|C_I^M|$ the cardinality of C_I^M ; finally, denote by $\text{Pr}[I] = \prod_{i \in I} p_i \prod_{i \in V \setminus I} (1 - p_i)$ the probability that vertex-subset I is present; then $E(G_C^M) = \sum_{I \subseteq V} \text{Pr}[I] |C_I^M|$.

The complexity of PVC is the complexity of computing $\hat{C} = \text{argmin}_{C \in \mathcal{C}(G)} \{E(G_C^M)\}$. Using the terminology of [4, 5], we will call the solution C (in which the modification strategy is applied) a *priori solution*.

A priori optimization, i.e., searching for optimal a priori solutions of probabilistic combinatorial optimization problems, has been studied in restricted versions of routing and network-design probabilistic minimization problems ([1, 2, 3, 4, 5, 6, 7, 8]), defined on complete graphs. Finally, a priori optimization has been used in [9] to study a probabilistic maximization problem, the longest path.

One can remark that in the above definition of PVC, M is part of the instance and this seems somewhat unusual with respect to standard complexity theory where no algorithm intervenes in the definition of a problem. But M is absolutely not an algorithm for PVC, in the sense that it does not compute $C \in \mathcal{C}(G)$. It simply fits C (no matter how it has been computed) to I .

Moreover, let us note that when changing M one changes the definition of PVC itself. Strictly speaking, the PVC-variants induced by the quintuples $Q1 = (\mathcal{H}, \mathcal{C}, M_1, E(G_C^{M_1}), \min)$ and $Q2 = (\mathcal{H}, \mathcal{C}, M_2, E(G_C^{M_2}), \min)$ are two *distinct* probabilistic combinatorial optimization problems.

We use three modification strategies $M1$, $M2$ and $M3$ and study their corresponding functionals. For $M1$ and $M2$, we produce explicit expressions for their functionals, expressions allowing us to completely characterize the solutions minimizing these functionals. On the contrary, the expression for the functional associated with $M3$ is not quite explicit in order to allow achievement of similar results. Therefore, we give bounds for this functional and study their quality. Let us recall that the three strategies introduce in fact three distinct probabilistic PVC-variants.

In what follows, given a graph $G = (V, E)$ of order n , we denote by C a vertex cover of G , by τ its cardinality and by $\hat{C}_i$, $i = 1, 2, 3$, the optimal PVC-solutions associated with $M1$, $M2$ and $M3$, respectively. Moreover, by $\Gamma(v_i)$, $i = 1, \dots, n$, we denote the set of neighbors of the

vertex v_i (the degree of v_i). We will set $\delta = \min_{v_i \in V} \{|\Gamma(v_i)|\}$ and $\Delta = \max_{v_i \in V} \{|\Gamma(v_i)|\}$. Given a vertex cover C of G and a subset $V' \subseteq V$, we will set $C[V'] = C \cap V'$ and $\bar{C}[V'] = (V \setminus C) \cap V'$. Finally, when dealing with WVC, we will denote by w_i the weight of $v_i \in V$.

2 The strategies M1, M2 and M3 and a general preliminary result

2.1 Specification of M1, M2 and M3

Given a vertex cover C and a vertex-subset $I \subseteq V$. M1 consists in simply moving vertices of $C \setminus I$ (the absent vertices of C) out of C and in retaining $C[I] = C \cap I$ as vertex cover of $G[I]$.

```
BEGIN (*M1(C,G[I])* )
  C[I] ← C ∩ I;
  OUTPUT C[I];
END. (*M1(C,G[I])* )
```

It is easy to see that $C[I]$ constitutes a vertex cover for $G[I]$. In fact, in the opposite case, there would be at least an edge $v_i v_j$ of $G[I]$ for which neither v_i , nor v_j would belong to $C[I]$. But, since $v_i v_j \in E$ and C is a vertex cover for G , at least one of v_i, v_j , say v_i , belongs to C and, consequently, to $C[I] = C \cap I$. Therefore v_i is part of the vertex cover for $G[I]$, contradicting so the statement that edge $v_i v_j$ is not covered in $G[I]$.

The second strategy studied in this paper is a slight improvement of M1 since one removes isolated vertices from $C[I]$.

```
BEGIN (*M2(C,G[I])* )
  C[I] ← C ∩ I;
  R ← {vi ∈ C[I] : Γ(vi) = ∅};
  OUTPUT C2[I] ← C[I] \ R;
END. (*M2(C,G[I])* )
```

Finally, strategy M3 is a further improvement of M2 since it removes from $C2[I]$ vertices all the neighbors of which belong to $C2[I]$. In the following specification of M3, algorithm SORT sorts the vertices of $C2[I]$ in increasing order with respect to their degrees.

```
BEGIN (*M3(C,G[I])* )
  C2[I] ← M2(C,G[I]);
  C3[I] ← SORT(C2[I]);
  FOR i ← 1 TO |C3[I]| DO
    IF ∀vj ∈ Γ(vi), vj ∈ C3[I] THEN C3[I] ← C3[I] \ {vi} FI
  OD
  OUTPUT C3[I];
END. (*M3(C,G[I])* )
```

It is easy to see that the vertex cover $C3[I]$ computed by strategy M3 is minimal (for the inclusion) for $G[I]$.

2.2 A first expression for the functionals

Consider any strategy that starting from $C[I]$ reduces it by removing some of its vertices (if possible) in order to obtain smaller feasible vertex covers. Clearly, M1, M2 and M3 are such strategies. Then, the following proposition provides a first expression for functionals $E(G_C^{M1})$, $E(G_C^{M2})$ and $E(G_C^{M3})$.

Proposition 1. Consider a vertex cover C of G and strategies $M1$, $M2$ and $M3$. With each vertex $v_i \in V$ associate a probability p_i and a random variable $X_i^{\text{Mk},C}$, $k = 1, 2, 3$, defined, for every $I \subseteq V$, by

$$X_i^{\text{Mk},C} = \begin{cases} 1 & v_i \in C_I^{\text{Mk}} \\ 0 & \text{otherwise} \end{cases}$$

Then, $E(G_C^{\text{Mk}}) = \sum_{v_i \in C} \Pr[X_i^{\text{Mk},C} = 1]$.

Proof. By the definition of $X_i^{\text{Mk},C}$ we have $|C_I^{\text{Mk}}| = \sum_{i=1}^n X_i^{\text{Mk},C}$. So,

$$\begin{aligned} E(G_C^{\text{Mk}}) &= \sum_{I \subseteq V} \Pr[I] |C_I^{\text{Mk}}| = \sum_{I \subseteq V} \Pr[I] \sum_{i=1}^n X_i^{\text{Mk},C} = \sum_{i=1}^n \sum_{I \subseteq V} \Pr[I] X_i^{\text{Mk},C} \\ &= \sum_{i=1}^n E(X_i^{\text{Mk},C}) = \sum_{i=1}^n \Pr[X_i^{\text{Mk},C} = 1] = \sum_{i=1}^n \Pr[X_i^{\text{Mk},C} = 1] (1_{\{v_i \in C\}} + 1_{\{v_i \notin C\}}) \\ &= \sum_{i=1}^n \Pr[X_i^{\text{Mk},C} = 1] 1_{\{v_i \in C\}} + \sum_{i=1}^n \Pr[X_i^{\text{Mk},C} = 1] 1_{\{v_i \notin C\}}. \end{aligned}$$

But, if $v_i \notin C$, then, $\forall I \subset V$ such that $v_i \in I$, $X_i^{\text{Mk},C} = 0$. So, $\Pr[X_i^{\text{Mk},C} = 1] = 0$, $\forall v_i \notin C$. Consequently, $E(G_C^{\text{Mk}}) = \sum_{i=1}^n \Pr[X_i^{\text{Mk},C} = 1] 1_{\{v_i \in C\}} = \sum_{v_i \in C} \Pr[X_i^{\text{Mk},C} = 1]$. ■

3 The complexity of $(\mathcal{H}, \mathbf{C}, M1, E(G_C^{\text{M1}}), \min)$

The following theorem can be deduced from proposition 1.

Theorem 1. $E(G_C^{\text{M1}}) = \sum_{v_i \in C} p_i$ and $\hat{C}_1 = \text{argmin}_{C \in \mathcal{C}(G)} \{\sum_{v_i \in C} p_i\}$. In other words $\hat{C}_1$ is a minimum-weight vertex cover of G where the vertices of V are weighted by their corresponding presence-probabilities. In the case where all vertex-probabilities are identical, $E(G_C^{\text{M1}}) = p\tau$ and $\hat{C}_1$ is a minimum vertex cover of G . Consequently, $(\mathcal{H}, \mathbf{C}, M1, E(G_C^{\text{M1}}), \min)$ is NP-hard.

Proof. By strategy $M1$, if $v_i \in C$, then $v_i \in C_I^{\text{M1}}$, for all subgraphs $G[I]$ such that $v_i \in I$. Consequently, $\Pr[X_i^{\text{M1},C} = 1] = p_i$, $\forall v_i \in C$ and the result follows from proposition 1.

Let us now consider G with its vertices weighted by their corresponding probabilities and denote the so obtained instance of WVC by G_w . The total weight of every vertex cover of G_w is $\sum_{v_i \in C} p_i$ and the optimal weighted vertex cover of G_w is the one for which the sum of the weights of its vertices is the smallest over all vertex covers of G_w . Such a vertex cover minimizes also $E(G_C^{\text{M1}})$ and, consequently constitutes an optimal solution for $(\mathcal{H}, \mathbf{C}, M1, E(G_C^{\text{M1}}), \min)$.

If $p_i = p$, $1 \leq i \leq n$, then $\sum_{v_i \in C} p_i = \sum_{v_i \in C} p = p\tau$ and, consequently, the vertex cover minimizing this last expression is the one minimizing τ , a minimum vertex cover of G .

Expression $\sum_{v_i \in C} p_i$ (the objective value of the problem $(\mathcal{H}, \mathbf{C}, M1, E(G_C^{\text{M1}}), \min)$) can be computed in $O(n)$, therefore, $(\mathcal{H}, \mathbf{C}, M1, E(G_C^{\text{M1}}), \min) \in \text{NP}$. Furthermore, by the isomorphism between this problem and WVC, the former problem is hard for NP and this completes the proof of theorem 1. ■

Let us revisit case $p_i = p$, $1 \leq i \leq n$. Since $C_I^{\text{M1}} = C[I]$ and $0 \leq |C[I]| \leq \tau$, we get: $|C_I^{\text{M1}}| = |C[I]| \sum_{i=1}^{\tau} 1_{\{|C[I]|=i\}}$ and the functional for $M1$ can be written as

$$\begin{aligned} E(G_C^{\text{M1}}) &= \sum_{I \subseteq V} \Pr[I] |C[I]| \sum_{i=1}^{\tau} 1_{\{|C[I]|=i\}} = \sum_{i=1}^{\tau} i \sum_{I \subseteq V} \Pr[I] 1_{\{|C[I]|=i\}} \\ &= \sum_{i=1}^{\tau} i \binom{\tau}{i} p^i (1-p)^{\tau-i} \sum_{j=0}^{n-\tau} \binom{n-\tau}{j} p^j (1-p)^{n-\tau-j} = p\tau \end{aligned}$$

where in the last summation we count all the sub-graphs $G[I]$ such that $|C[I]| = i$, we add their probabilities and we take into account that $\sum_{j=0}^{n-\tau} C_j^{n-\tau} p^j (1-p)^{n-\tau-j} = 1$.

The above can be generalized in order to compute every moment of any order for M1. For instance, $E((C_I^M)^2) = \sum_{I \subseteq V} \Pr[I] (|C_I^M|)^2 = \sum_{i=1}^{\tau} i^2 C_i^{\tau} p^i (1-p)^{\tau-i} = p\tau(p\tau + 1 - p)$ and, consequently,

$$\text{Var}(G_C^{M1}) = E((G_C^{M1})^2) - (E(G_C^{M1}))^2 = \tau p(1-p).$$

So, for M1 and for the case of identical vertex-probabilities, the random variable representing the size τ follows a binomial law with parameters τ and p .

4 The complexity of $(\mathcal{H}, \mathbf{C}, \mathbf{M2}, E(G_C^{M2}), \min)$

We recall that strategy M2 consists in removing the isolated vertices from $C[I]$. Then, the following theorem holds.

Theorem 2.

$$E(G_C^{M2}) = \sum_{v_i \in C} p_i \left(1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right)$$

and $\hat{C}_2$ is a minimum-weight vertex cover of G_w where, for every $v_i \in V$,

$$w_i = p_i \left(1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right).$$

Consequently, $(\mathcal{H}, \mathbf{C}, \mathbf{M2}, E(G_C^{M2}), \min)$ is NP-hard.

Proof. Remark first that $C_I^{M2} = C2[I] = C[I] \setminus (\cup_{\{v_i \in C: (\Gamma(v_i) \cap I) = \emptyset\}} v_i)$.

Starting from proposition 1 we get

$$E(G_C^{M2}) = \sum_{v_i \in C} \Pr[X_i^{M2, C} = 1] = \sum_{v_i \in C} \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \in C2[I]\}} = \sum_{v_i \in C} \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] (1 - 1_{\{v_i \notin C2[I]\}}).$$

But, for any $v_i \in C$, $v_i \notin C2[I] \iff I \cap \Gamma(v_i) = \emptyset$; consequently,

$$E(G_C^{M2}) = \sum_{v_i \in C} \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] - \sum_{v_i \in C} \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{I \cap \Gamma(v_i) = \emptyset\}} = \sum_{v_i \in C} p_i \left(1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right).$$

It is easy to see that $E(G_C^{M2})$ can be computed in $O(n^2)$; consequently $(\mathcal{H}, \mathbf{C}, \mathbf{M2}, E(G_C^{M2}), \min) \in \mathbf{NP}$. With a reasoning completely similar to the one of theorem 1 one can immediately deduce that $\hat{C}_2 = \text{argmin}_{C \in C(G)} \{ \sum_{v_i \in C} p_i (1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j)) \}$, i.e., a minimum-weight vertex cover of G_w where, for $v_i \in V$, $w_i = p_i (1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j))$. ■

5 The complexity of $(\mathcal{H}, \mathbf{C}, \mathbf{M3}, E(G_C^{M3}), \min)$

5.1 Building $E(G_C^{M3})$

Recall that M3 consists in removing from $C[I]$ both the isolated vertices and the vertices all the neighbors of which belong in $C[I]$. Consequently, the solution $C3[I]$ computed by M3 is minimal.

Proposition 2.

$$E(G_C^{\text{M3}}) = \sum_{v_i \in C} p_i \left(1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right) - \sum_{v_i \in C} \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \notin C_I^{\text{M3}}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \quad (1)$$

Proof. By proposition 1, $E(G_C^{\text{M3}}) = \sum_{v_i \in C} \Pr[X_i^{\text{M3}, C} = 1]$. Moreover,

$$\Pr[X_i^{\text{M3}, C} = 1] = \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}}.$$

Note that $\Gamma(v_i) = C[\Gamma(v_i)] \cup \bar{C}[\Gamma(v_i)]$ and that if $v_i \in C$, then $\bar{C}[\Gamma(v_i)] \neq \emptyset$ (because C is minimal).

Let now $v_i \in C$ and $\mathcal{I}_i = \{I \subseteq V : v_i \in I\}$. The set of graphs $G[I]$, $I \in \mathcal{I}_i$ can be partitioned in the following four subsets, a graph $G[I]$ belonging to only one of them:

- (i) graphs such that $I \cap \Gamma(v_i) = \emptyset$;
- (ii) graphs such that $I \cap C[\Gamma(v_i)] \neq \emptyset$ and $I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset$;
- (iii) graphs such that $I \cap C[\Gamma(v_i)] \neq \emptyset$ and $I \cap \bar{C}[\Gamma(v_i)] = \emptyset$;
- (iv) graphs such that $I \cap C[\Gamma(v_i)] = \emptyset$ and $I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset$.

Let us analyse strategy M3 in cases (i) to (iv).

In case (i) since v_i is isolated, M3 will remove it from $C[I]$.

In cases (ii) and (iv) v_i will be included in C_I^{M3} (since $I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset$, $\exists v_j \in I \cap \Gamma(v_i)$ such that $v_j \notin C$).

In case (iii) the fact that v_i will be included in, or will be removed from, the solution depends on the order on which it will be treated by M3 and on whether or not all its neighbors are part of $C[I]$.

In all,

- $v_i \notin C_I^{\text{M3}}$ for all graphs verifying case (i) and for a part of graphs verifying case (iii);
- $v_i \in C_I^{\text{M3}}$ for all graphs verifying cases (ii) and (iv) and for the rest of graphs verifying case (iii).

We so have:

$$\begin{aligned} \Pr[X_i^{\text{M3}, C} = 1] &= \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}} = \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}} \left(1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset\}} \right. \\ &\quad \left. + 1_{\{I \cap C[\Gamma(v_i)] = \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset\}} + 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \right) \\ &= \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}} \left(1_{\{I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset\}} + 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \right) \\ &= \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] \neq \emptyset\}} \\ &\quad + \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \\ &= \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{\text{M3}}\}} \left(1 - 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{M3}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \\
& = p_i \left(1 - \prod_{v_j \in \bar{C}[\Gamma(v_i)]} (1 - p_j) \right) + \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \in C_I^{M3}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \\
\Pr[X_i^{M3,C} = 0] & = \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} + \sum_{\substack{I \subseteq V \\ v_i \notin I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} \\
& = \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} \left(1_{\{I \cap \Gamma(v_i) = \emptyset\}} + 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \right) + 1 - p_i \\
& = \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} 1_{\{I \cap \Gamma(v_i) = \emptyset\}} + 1 - p_i \\
& \quad + \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \\
& = p_i \prod_{v_j \in \Gamma(v_i)} (1 - p_j) + 1 - p_i + \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}}
\end{aligned}$$

It is easy to see, after some easy but tedious algebra, that $\Pr[X_i^{M3,C} = 1] + \Pr[X_i^{M3,C} = 0] = 1$. Then, the result of proposition 2 is get using

$$E(G_C^{M3}) = \sum_{v_i \in C} \Pr[X_i^{M3,C} = 1] = \sum_{v_i \in C} \left(1 - \Pr[X_i^{M3,C} = 0] \right). \quad \blacksquare$$

5.2 Bounds for $E(G_C^{M3})$

Let us first note that from theorems 1, 2 and proposition 2, we have $E(G_C^{M3}) \leq E(G_C^{M2}) \leq E(G_C^{M1})$. Since the result of proposition 2 does not allow the achievement of a precise characterization of $\bar{C}_3$, we give in the following theorem bounds for $E(G_C^{M3})$.

Theorem 3.

$$\sum_{v_i \in C} p_i \left(1 - \prod_{v_j \in \bar{C}[\Gamma(v_i)]} (1 - p_j) \right) \leq E(G_C^{M3}) \leq \sum_{v_i \in C} p_i \left(1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right).$$

If $p_i = p$, $\forall v_i \in V$ and $\delta_C = \min_{v_i \in C} \{|\Gamma(v_i)|\}$, then

$$p \left(\tau - \max \left\{ \tau(1-p)^\Delta, (1-p)^{\delta_C} \right\} \right) \geq E(G_C^{M3}) \geq p^2 \tau.$$

The upper bound for $E(G_C^{M3})$ is attained for the bipartite graphs when considering the one of the color classes as a priori solution.

Proof. The upper bound is obvious since it is nothing else than the expression of theorem 2 for $E(G_C^{M2})$.

Let

$$A_i = \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \quad (2)$$

(see the last term of the expression (1) in proposition 2). Then,

$$\begin{aligned}
A_i &\leq \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] 1_{\{I \cap C[\Gamma(v_i)] \neq \emptyset\}} 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \leq \sum_{\substack{I \subseteq V \\ v_i \in I}} \Pr[I] (1 - 1_{\{I \cap C[\Gamma(v_i)] = \emptyset\}}) 1_{\{I \cap \bar{C}[\Gamma(v_i)] = \emptyset\}} \\
&\leq p_i \left(\prod_{v_j \in \bar{C}[\Gamma(v_i)]} (1 - p_j) - \prod_{v_j \in \bar{C}[\Gamma(v_i)]} (1 - p_j) \prod_{v_j \in C[\Gamma(v_i)]} (1 - p_j) \right) \\
&\leq p_i \left(\prod_{v_j \in \bar{C}[\Gamma(v_i)]} (1 - p_j) - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right).
\end{aligned}$$

Combining the expression for A_i above with the expression for $E(G_C^{\mathbf{M3}})$ of proposition 2, we easily get the lower bound claimed.

Let us now suppose that $p_i = p$, $\forall v_i \in V$. Then the expression for $E(G_C^{\mathbf{M3}})$ becomes

$$E(G_C^{\mathbf{M3}}) = p\tau - p \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)|} - \sum_{v_i \in C} A_i$$

with $0 \leq A_i \leq p((1 - p)^{|\bar{C}[\Gamma(v_i)]|} - (1 - p)^{|\Gamma(v_i)|})$. Since $A_i \geq 0$ we have

$$E(G_C^{\mathbf{M3}}) \leq p\tau - p \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)|}.$$

Using, $\forall i, |\Gamma(v_i)| \leq \Delta$, we get

$$E(G_C^{\mathbf{M3}}) \leq p(\tau - \tau(1 - p)^\Delta). \quad (3)$$

Remark that

$$\begin{aligned}
p \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)|} &= p(1 - p)^{\delta_C} \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)| - \delta_C} \\
\sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)| - \delta_C} &\geq 1
\end{aligned}$$

because there exists at least a vertex $v_{i_0} \in C$ with $|\Gamma(v_{i_0})| = \delta_C$. Consequently,

$$E(G_C^{\mathbf{M3}}) \leq p(\tau - (1 - p)^{\delta_C}). \quad (4)$$

Combining expressions (3) and (4) one immediately obtains the upper bound claimed.

For identical vertex-probabilities, the lower bound for $E(G_C^{\mathbf{M3}})$ becomes $p\tau(1 - (1 - p)^{|\bar{C}[\Gamma(v_i)]|})$. Since C is minimal, $\bar{C}[\Gamma(v_i)] \neq \emptyset$, i.e., $|\bar{C}[\Gamma(v_i)]| \geq 1$. Using the latter inequality in the former one we get $E(G_C^{\mathbf{M3}}) \geq p\tau(1 - (1 - p)) = p^2\tau$, proving so the lower bound claimed.

Consider now a bipartite graph $B = (V_1, V_2, E)$ and one of its color classes, say color class V_1 , as a priori solution. Remark that, $\forall v_i \in V_1$, $\Gamma(v_i) = \bar{V}_1[\Gamma(v_i)] \in V_2$. Consequently, the upper and lower bounds for $E(B_{V_1}^{\mathbf{M3}})$ coincide. ■

5.3 On the quality of the bounds obtained

We prove in this section that the bounds of $E(G_C^{\mathbf{M3}})$ obtained in theorem 3 are quite tight.

Let $G = (V, E)$ be a graph consisting of a clique K_ℓ (on ℓ vertices) and of an independent set S on σ vertices; moreover consider that any vertex of K_ℓ is linked to any vertex of S . Let

us suppose that all the vertices of G have the same probability p and consider $C = V(K_\ell)$, the vertex-set of K_ℓ , as a priori solution. Finally set $n = \ell + \sigma$.

For $v_i \in C$ we have $C[\Gamma(v_i)] = V(K_\ell) \setminus \{v_i\}$ and $\bar{C}[\Gamma(v_i)] = S$. Then, expression (1) becomes

$$E(G_C^{M3}) = \sum_{v_i \in C} p - \sum_{v_i \in C} p \prod_{v_j \in \Gamma(v_i)} (1-p) - \sum_{v_i \in C} \sum_{I \subseteq V} \Pr[I] 1_{\{v_i \notin C_I^{M3}\}} 1_{\{I \cap (V(K_\ell) \setminus \{v_i\}) \neq \emptyset\}} 1_{\{I \cap S = \emptyset\}} \quad (5)$$

Let $V(K_\ell) = \{v_1, v_2, v_3, \dots, v_\ell\}$ (the degrees of the vertices of $V(K_\ell)$ are all equal) and let $S = \{v_{\ell+1}, v_{\ell+2}, v_{\ell+3}, \dots, v_n\}$. In step i , strategy M3 tests if vertex v_i can be removed.

Revisit now the third term of expression (5) (the double sum). This term deals with instances $G[I]$ such that $I \subseteq V(K_\ell) \setminus \{v_i\}$, in other words, instances where all present vertices are elements of $V(K_\ell)$, i.e., they are also part of C . Since $v_i \in V(K_\ell)$, $\Gamma(v_i) \cap I \subseteq C \cap I$ and vertex $v_i \in C$ can be removed only if, it is not isolated and for $j < i$, $v_j \notin I$ (because, in the opposite case, v_j would be removed at step $j < i$, in other words, before v_i and, since $v_i v_j \in E(K_\ell)$, v_i should not be removed). Consequently, $1_{\{v_i \notin C_I^{M3}\}} = 1_{\{I \cap \{v_1, v_2, v_3, \dots, v_{i-1}\} = \emptyset\}}$ and for a fixed index i , expression (2) becomes

$$\begin{aligned} A_i &= \sum_{I \subseteq V} \Pr[I] 1_{\{I \cap \{v_1, v_2, v_3, \dots, v_{i-1}\} = \emptyset\}} 1_{\{I \cap V(K_\ell) \setminus \{v_i\} \neq \emptyset\}} 1_{\{I \cap S = \emptyset\}} \\ &= \sum_{I \subseteq V} \Pr[I] 1_{\left\{ \begin{array}{l} I \cap v_i = v_i \\ I \cap \{v_1, v_2, v_3, \dots, v_{i-1}\} = \emptyset \\ I \cap \{v_{i+1}, v_{i+2}, v_{i+3}, \dots, v_\ell\} \neq \emptyset \end{array} \right\}} 1_{\{I \cap S = \emptyset\}} \\ &= \prod_{j=1}^{i-1} (1-p)p \left(1 - \prod_{j=i+1}^{\ell} (1-p) \right) \prod_{j=\ell+1}^n (1-p) \\ &= (1-p)^{i-1} p \left(1 - (1-p)^{\ell-(i+1)+1} \right) (1-p)^{n-(\ell+1)+1} \\ &= (1-p)^{i-1} p \left(1 - (1-p)^{\ell-i} \right) (1-p)^{n-\ell}. \end{aligned}$$

We so have,

$$\begin{aligned} \sum_{v_i \in C} A_i &= \sum_{i=1}^{\ell} p \left((1-p)^{n+i-\ell-1} - (1-p)^{n-1} \right) \\ &= p(1-p)^{n-\ell-1} \sum_{i=1}^{\ell} (1-p)^i - p\ell(1-p)^{n-1} \\ &= (1-p)^{n-\ell-1} \left(1-p - (1-p)^{\ell+1} \right) - p\ell(1-p)^{n-1} \\ &= (1-p)^{n-\ell} - (1-p)^n - p\ell(1-p)^{n-1} \end{aligned} \quad (6)$$

Remark that computation of A_i 's in the expression above is polynomial and, consequently, the whole computation for $E(G_C^{M3})$ is also polynomial.

Combining now expressions (5) and (6), we get

$$\begin{aligned} E(G_C^{M3}) &= \sum_{i=1}^{\ell} p - \sum_{i=1}^{\ell} p \prod_{v_j \in \Gamma(v_i)} (1-p) - ((1-p)^{n-\ell} - (1-p)^n - p\ell(1-p)^{n-1}) \\ &= \ell p - \ell p(1-p)^{n-1} - (1-p)^{n-\ell} + (1-p)^n + p\ell(1-p)^{n-1} \\ &= \ell p - (1-p)^{n-\ell} (1 - (1-p)^\ell). \end{aligned}$$

For reasons of facility, set

$$e(\ell, p, n) \stackrel{\text{def}}{=} \ell p - (1-p)^{n-\ell} \left(1 - (1-p)^\ell\right) \quad (7)$$

$$b(\ell, p, n) \stackrel{\text{def}}{=} \ell p \left(1 - (1-p)^{n-\ell}\right) \quad (8)$$

$$B(\ell, p, n) \stackrel{\text{def}}{=} \ell p \left(1 - (1-p)^{n-1}\right) \quad (9)$$

and remark that expressions (8) and (9) are, respectively, the lower and upper bounds of theorem 3 for $E(G_C^{\text{M3}})$.

We now study the difference $(B - e)(\ell, p, n)$ (expressions (7) and (9)). We have,

$$(B - e)(\ell, p, n) = (1-p)^{n-\ell} - (1-p)^n - \ell p (1-p)^{n-1} \leq 1 \quad (10)$$

Starting from expression (10), we prove in appendix that the following hold:

$$0 \leq D(\ell, n) \stackrel{\text{def}}{=} (B - e)(\ell, p, n) \leq \left(\frac{n-\ell}{n-1}\right)^{\frac{n-\ell}{\ell-1}} - \left(\frac{n-\ell}{n-1}\right)^{\frac{n-1}{\ell-1}} \quad (11)$$

Note that the upper bound of expression (11) is quite tight for $D(\ell, n)$. Let us now estimate this bound for several values of ℓ :

- if $\ell = 1$, then $D(\ell, n) = 0$;
- if $1 < \ell \ll n$ (e.g., $\ell = o(n)$), then $\lim_{n \rightarrow \infty} D(\ell, n) = 0$;
- if $\ell = \lambda n$, for a fixed $\lambda < 1$, then

$$\lim_{n \rightarrow \infty} D(\lambda n, n) = e^{\frac{1-\lambda}{\lambda} \ln(1-\lambda)} - e^{\frac{\ln(1-\lambda)}{\lambda}}$$

and this limit's value is a fixed positive constant;

- if, finally, $\ell = n - 1$, then $\lim_{n \rightarrow \infty} D(n - 1, n) = 1$.

Let us now study the difference $(e - b)(\ell, p, n)$. We first have

$$(e - b)(\ell, p, n) = \ell p (1-p)^{n-\ell} - (1-p)^{n-\ell} + (1-p)^n \leq (\ell - 1) \left((1-p)^{n-\ell} - (1-p)^{n-\ell+1} \right) \quad (12)$$

one can refine the bound of expression (12) obtaining (see in the appendix for the proof)

$$\begin{aligned} 0 \leq d(\ell, n) \stackrel{\text{def}}{=} (e - b)(\ell, p, n) &\leq (\ell - 1) \left(\frac{n-\ell}{n-\ell+1} \right)^{n-\ell} \left(1 - \frac{n-\ell}{n-\ell+1} \right) \\ &\leq (\ell - 1) e^{(n-\ell) \ln\left(1 - \frac{1}{n-\ell+1}\right)} \frac{1}{n-\ell+1} \end{aligned} \quad (13)$$

As previously, we estimate this bound for several values of ℓ :

- if $\ell = 1$, then $d(\ell, n) = 0$;
- if $\ell = o(n)$, then $\lim_{n \rightarrow \infty} d(\ell, n) = 0$;
- if $\ell = \lambda n$, for a fixed $\lambda < 1$, then

$$d(\lambda n, n) \leq \frac{\lambda n - 1}{(1-\lambda)n + 1} e^{n(1-\lambda) \ln\left(1 - \frac{1}{(1-\lambda)n+1}\right)} \sim \frac{\lambda}{1-\lambda} e^{-1}$$

and this last value is a fixed constant;

		$p = 0.1$	$p = 0.5$	$p = 0.9$	$p = 1/n$	$p = 1/\sqrt{n}$	$p = 1/\ln n$
$n = 14$	$E(G_{C^*}^{M3})/E(G_{C^*}^{M1})$	0.34	0.82	0.98	0.26	0.62	0.74
	$E(G_{C^*}^{M3})/E(G_{C^*}^{M2})$	0.73	0.88	0.98	0.73	0.79	0.84
	$E(G^{\text{opt}})/E(G_{C^*}^{M3})$	0.97	0.96	0.99	0.98	0.95	0.95
$n = 15$	$E(G_{C^*}^{M3})/E(G_{C^*}^{M1})$	0.34	0.83	0.97	0.25	0.62	0.73
	$E(G_{C^*}^{M3})/E(G_{C^*}^{M2})$	0.73	0.89	0.98	0.72	0.79	0.83
	$E(G^{\text{opt}})/E(G_{C^*}^{M3})$	0.97	0.95	0.99	0.97	0.95	0.95
$n = 16$	$E(G_{C^*}^{M3})/E(G_{C^*}^{M1})$	0.35	0.83	0.98	0.25	0.62	0.73
	$E(G_{C^*}^{M3})/E(G_{C^*}^{M2})$	0.72	0.88	0.98	0.71	0.78	0.83
	$E(G^{\text{opt}})/E(G_{C^*}^{M3})$	0.96	0.95	0.99	0.97	0.94	0.94

Table 1: Relations between functionals.

		$p = 0.1$	$p = 0.5$	$p = 0.9$	$p = 1/n$	$p = 1/\sqrt{n}$	$p = 1/\ln n$
$n = 14$	$B_\ell^3/E(G_{C^*}^{M3})$	0.34	0.62	0.92	0.32	0.45	0.53
	$B_u^3/E(G_{C^*}^{M3})$	1.67	1.18	1.02	1.71	1.42	1.29
$n = 15$	$B_\ell^3/E(G_{C^*}^{M3})$	0.33	0.61	0.92	0.31	0.44	0.52
	$B_u^3/E(G_{C^*}^{M3})$	1.68	1.18	1.02	1.74	1.42	1.29
$n = 16$	$B_\ell^3/E(G_{C^*}^{M3})$	0.32	0.61	0.92	0.29	0.43	0.51
	$B_u^3/E(G_{C^*}^{M3})$	1.69	1.17	1.02	1.77	1.43	1.29

Table 2: On the quality of the bounds for $E(G_{C^*}^{M3})$.

- if, finally, $\ell = n - 1$, then $d(n - 1, n) \sim n/4$.

One can remark that for $\ell = o(n)$, $E(G_C^{M3})$ tends (for $n \rightarrow \infty$) to both $b(\ell, n, p)$ and $B(\ell, n, p)$. This is due to the fact that $o(n)/\lim_{n \rightarrow \infty}(B - b)(\ell, n, p) \sim 0$.

We see from the above discussion that for the graph and the a priori solution considered and for $n \rightarrow \infty$, the distance of $E(G_C^{M3})$ from the bounds given in theorem 3 can take an infinity of values being either arbitrarily close to, or arbitrarily far from them. Consequently, it seems very unlikely that the bounds computed could be substantially improved.

6 Some simulation results

Limited computational experiments have been conducted in order to study the realistic performances of the three strategies proposed. We have randomly generated graphs of orders $n \in \{14, 15, 16\}$, forty graphs per order, in the following way: for each graph, a number $p_E \in [0, 1]$ is randomly drawn; next, for each edge e , a randomly chosen probability p_e is associated with, and, if $p_e \leq p_E$, then e is retained as an edge in the graph under construction. For all the generated instances, the minimum vertex-degree varies between 0 (graph non-connected) and 14, while the maximum one varies between 1 and 15. The ratio $|E|/n$ varied between 0.06 and 7.3 for all the graphs generated. The a priori solution considered was the minimum vertex cover of each graph, denoted by C^* . We have supposed identical vertex-probabilities and have tested six probability-values: 0.1, 0.5, 0.9, $1/\ln n$, $1/\sqrt{n}$ and $1/n$.

We have computed quantities $E(G_{C^*}^{M1})$, $E(G_{C^*}^{M2})$, $E(G_{C^*}^{M3})$ and $E(G^{\text{opt}})$. The last quantity is the mathematical expectation of a thought process consisting in finding the optimal vertex cover in any subinstance of G . This implies computation, by branch-and-bound, of the optimal solution

		$p = 0.1$	$p = 0.5$	$p = 0.9$	$p = 1/n$	$p = 1/\sqrt{n}$	$p = 1/\ln n$
$n = 14$	$\sigma_{C^*}^{M1}/E(G_{C^*}^{M1})$	1.08	0.36	0.12	1.30	0.60	0.46
	$\sigma_{C^*}^{M3}/E(G_{C^*}^{M3})$	2.26	0.46	0.13	3.08	0.90	0.63
	$\sigma^{\text{opt}}/E(G^{\text{opt}})$	2.23	0.46	0.14	3.05	0.89	0.63
$n = 15$	$\sigma_{C^*}^{M1}/E(G_{C^*}^{M1})$	1.03	0.34	0.11	1.29	0.58	0.45
	$\sigma_{C^*}^{M3}/E(G_{C^*}^{M3})$	2.13	0.44	0.13	3.08	0.88	0.61
	$\sigma^{\text{opt}}/E(G^{\text{opt}})$	2.10	0.44	0.13	3.05	0.87	0.61
$n = 16$	$\sigma_{C^*}^{M1}/E(G_{C^*}^{M1})$	0.99	0.33	0.11	1.28	0.57	0.44
	$\sigma_{C^*}^{M3}/E(G_{C^*}^{M3})$	2.03	0.42	0.12	3.13	0.87	0.61
	$\sigma^{\text{opt}}/E(G^{\text{opt}})$	2.01	0.42	0.13	3.09	0.86	0.60

Table 3: Standard deviations for M1, M3 and reoptimization.

for any subgraph of G , i.e., 2^n executions of branch-and-bound. This is the reason for having restricted our studies in small graphs. In the table 1 we present the mean ratios for the functionals of the different strategies. What we can see from this table is that the ratio $E(G^{\text{opt}})/E(G_{C^*}^{M3})$ varies between 0.94 and 0.99. In other words, C^* seems to be very close to $\hat{C}_3$. In the opposite the computations times for $E(G_{C^*}^{M3})$ are quite large, much more large than the ones for both $E(G_{C^*}^{M1})$ and $E(G_{C^*}^{M2})$, whose the ratios with $E(G_{C^*}^{M3})$ are relatively small. Another remark we can make from the results of table 1 is that the larger the vertex-probabilities, the larger (closer to 1) the ratios.

We also have considered the bounds B_ℓ^3 and B_u^3 for $E(G_{C^*}^{M3})$ in the last expression of theorem 3 and computed their ratios with $E(G_{C^*}^{M3})$. The results are presented in table 2. The mean value of ratio $B_\ell^3/E(G_{C^*}^{M3})$ is 0.52, while for $B_u^3/E(G_{C^*}^{M3})$ this value is 1.39. So, although these values are fairly close to 1, it seems to us that the bounds considered have to be improved. Also one can note that once more the larger the vertex-probabilities, the closer to 1 the ratios.

Finally, for strategies M1, M3 we have computed their standard deviations defined, for $k = 1, 3$, as $\sigma_{C^*}^{Mk} = [\sum_{I \subseteq V} \Pr[I] (|C_I^{Mk}|)^2 - (\sum_{I \subseteq V} \Pr[I] |C_I^{Mk}|)^2]^{1/2}$ and compared the expectations with the corresponding deviations. We also have done so for σ^{opt} and $E(G^{\text{opt}})$. The relative results are presented in table 3. As one can see entries of this table show important dispersions of the functionals around the corresponding mathematical expectations.

7 Conclusions

We have first given a formal definition for probabilistic vertex cover and next we have analyzed three modification strategies and studied the complexities for the corresponding probabilistic vertex covering problems. Moreover we have shown that two of the three variants studied have natural interpretations as weighted vertex cover versions.

The framework of our paper differs for the one [1, 2, 3, 4, 5, 6, 7, 8] where besides the fact that all the problems considered (even minimum spanning tree and shortest path) are defined in complete graphs, a single modification strategy consisting in dropping absent vertices out of the a priori solutions (as our M1) is studied.

Dealing with our results, further work has to be made for strategy M3 in order to (a) completely characterize $E(G_{C^*}^{M3})$ and (b) determine $\hat{C}_3$ and the complexity of its computation. We think that if one can answer point (a), then one will be able with very little more work to answer also point (b).

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A Proof of expression (11)

Consider expression (10) and set $x = 1 - p$. Then, expression (10) becomes

$$\begin{aligned}(B - e)(\ell, x, n) &= x^{n-\ell} - x^n - \ell(1-x)x^{n-1} = x^{n-\ell} + (\ell-1)x^n - \ell x^{n-1} \\ &= x^{n-\ell}(1 + (\ell-1)x^\ell - \ell x^{\ell-1}) \leq x^{n-\ell} - x^{n-1} \leq 1\end{aligned}\quad (\text{A.1})$$

Set $f(x) = 1 + (\ell-1)x^\ell - \ell x^{\ell-1}$. Then, $f'(x) = \ell(\ell-1)(x^{\ell-1} - x^{\ell-2}) < 0$. Consequently, $f(x)$ is decreasing in $x \in [0, 1]$, therefore $f(x) \geq f(1) = 0$ and the lower bound of expression (11) is proved.

Revisit expression (A.1) and set $g(x) = x^{n-\ell} - x^{n-1}$ in $[0, 1]$. First remark that $g(0) = g(1) = 0$. Moreover, $g'(x) = (n-\ell)x^{n-\ell-1} - (n-1)x^{n-2}$ and

$$g'(x) = 0 \iff x_0 = \left(\frac{n-\ell}{n-1}\right)^{\frac{1}{\ell-1}}.$$

So, $g(x)$ increases, with respect to x , in $[0, x_0]$ while it is decreasing in x in $(x_0, 1]$. Hence,

$$(B - e)(\ell, x, n) \leq g(x) \leq g(x_0) = \left(\frac{n-\ell}{n-1}\right)^{\frac{n-\ell}{\ell-1}} - \left(\frac{n-\ell}{n-1}\right)^{\frac{n-1}{\ell-1}}$$

and the upper bound of expression (11) is also proved.

B Proof of expression (13)

Revisit expression (12) and, as previously, set $x = 1 - p$. Then, expression (12) becomes

$$\begin{aligned}(e - b)(\ell, x, n) &= \ell(1-x)x^{n-\ell} - x^{n-\ell} + x^n = (\ell-1)x^{n-\ell} - \ell x^{n-\ell+1} + x^n \\ &= x^{n-\ell}(x^\ell - \ell x + (\ell-1)) \leq (\ell-1)(x^{n-\ell} - x^{n-\ell+1})\end{aligned}\quad (\text{B.1})$$

Set $h(x) = x^\ell - \ell x + (\ell-1)$. Then, $h'(x) = \ell(x^{\ell-1} - 1) < 0$, so, $h(x)$ is decreasing in $x \in [0, 1]$ and, consequently, $h(x) \geq h(1) = 0$, proving so the lower bound of expression (13).

Revisit expression (B.1) and set $\varphi(x) = x^{n-\ell} - x^{n-\ell+1}$. Then, $\varphi'(x) = (n-\ell)x^{n-\ell-1} - (n-\ell+1)x^{n-\ell}$ and

$$\varphi'(x) = 0 \iff x_0 = \frac{n-\ell}{n-\ell+1}.$$

So, $\varphi(x)$ increases in $[0, x_0]$ and decreases in $(x_0, 1]$. Hence, in $[0, 1]$,

$$(e - b)(\ell, x, n) \leq (\ell-1) \left(\frac{n-\ell}{n-\ell+1}\right)^{n-\ell} \left(1 - \frac{n-\ell}{n-\ell+1}\right) \leq (\ell-1)e^{(n-\ell)\ln(1-\frac{1}{n-\ell+1})} \frac{1}{n-\ell+1}$$

and the lower bound of expression (13) is also proved.