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AN AGGREGATION/DISAGGREGATION APPROACH TO OBTAIN ROBUST CONCLUSIONS WITH ELECTRE TRI

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Une approche par agrégation/désagrégation pour construire des conclusions robustes avec ELECTRE TRI

RESUME: ELECTRE TRI est une méthode bien connue qui permet d'affecter un ensemble d'actions à des catégories prédéfinies en s'appuyant sur une famille de critères. L'utilisation de cette méthode requiert de définir la valeur de divers paramètres, ce qui est souvent une tâche difficile. Nous nous plaçons dans une situation où les décideurs impliqués dans le problème de décision ont des difficultés à déterminer une valeur précise pour l'ensemble des paramètres. De telles difficultés peuvent provenir de la nature incertaine, imprécise ou mal déterminée des données présentes aussi bien que d'une absence de consensus parmi les décideurs.

Cet article analyse la synergie entre deux approches développées indépendamment qui visent à prendre en compte la difficulté mentionnée ci-dessus. Ces deux approches n'exigent pas des décideurs des valeurs précises pour les paramètres. Pour dépasser ce problème, elles demandent aux décideurs une information qui leur impose une charge cognitive plus faible.

La première approche infère les valeurs des paramètres à partir d'exemples d'affectation fournis par les décideurs. Chaque exemple d'affectation génère des contraintes que les paramètres doivent respecter. La seconde approche considère un ensemble de contraintes sur la valeur des paramètres qui traduit la nature imprécise de l'information que les décideurs peuvent fournir. Cette information permet de déterminer la catégorie la meilleure et la pire à laquelle chaque action peut être affectée considérant les contraintes.

Ce travail propose une nouvelle méthodologie intégrant la phase d'élicitation des préférences avec l'analyse de robustesse en vue d'exploiter les synergies entre ces deux phases. L'approche proposée est interactive dans la mesure où les "output" guident les décideurs en vue d'une éventuelle modification des contraintes et/ou exemples d'affectation.

Mots clés: Approche par Agrégation/Désagrégation, ELECTRE TRI, Critères Multiples.

An aggregation/disaggregation approach to build robust conclusions with ELECTRE TRI

ABSTRACT: ELECTRE TRI is a well-known method to assign a set of alternatives to a set of predefined categories, considering multiple criteria. Using this method requires setting many parameters, which is often a difficult task. We consider the case where the Decision Makers (DMs) in the decision process are not sure of which values should each parameter take, which may result from uncertain, imprecise or inaccurately determined information, as well as from a lack of consensus among them.

This paper discusses the synergy between two approaches developed independently to deal with this difficulty. Both approaches avoid asking for precise values for the parameters. Rather, they proceed to solve the problem in a way that requires from the DMs much less effort.

The first approach infers the value of parameters from assignment examples provided by the DMs. Each assignment example originates mathematical constraints to which the parameter values should satisfy. The second approach considers a set of constraints on the parameter values reflecting the imprecise information that the DMs are able to provide. Then, it computes the best and worst categories for each alternative compatible with constraints.

This paper proposes a new approach integrating the elicitation phase with robustness analysis, to exploit the synergy between them. It is an interactive approach, where the insight obtained during robustness analyses guides the DMs during the elicitation phase.

Keywords: Aggregation/Disaggregation Approach, ELECTRE TRI, Multiple Criteria.

1. Introduction

Many real world decision problems can be formalised using a multiple criteria approach, i.e., by defining a set of criteria evaluating the alternatives' performances. However, such approach requires the Decision Makers (DMs) to provide some preference information in order to build a model that discriminates among the alternatives.

A crucial phase in a modelling process grounded on a Multiple Criteria Decision Aiding (MCDA) model consists in eliciting preference information from the DMs and formalising it through preferential parameters. In the various aggregation procedures, these parameters take the form of weights, aspiration levels, thresholds, ... The values assigned to these parameters express the way the evaluation of the alternatives on the different criteria should be combined in order to build comprehensive preferences.

However, DMs have difficulties in defining precise values for preferential parameters. The reasons for such problems concerning preference elicitation are various. The data considered in the decision problem might be imprecise or uncertain; DMs in the decision process may have a vague understanding of what the parameters represent and their point of view can evolve during the elicitation process; moreover a lack of consensus among DMs can be also a critical issue. A possible approach to deal with these difficulties is to work with information which is often called "imprecise" (see Athanassopoulos and Podinovski, 1997), "incomplete" (see Weber, 1987), "partial" (see Hazen, 1986) or "poor" (see Bana e Costa and Vincke, 1995). We will use the expression "imprecise information", meaning that it does not impose a precise combination of values for the parameters.

Some authors (Mousseau, 1995; Bana e Costa and Vansnick, 1994; Jacquet-Lagrèze and Siskos 1982; Belton and Vickers, 1990) have proposed Preference Elicitation Techniques (PETs) to support the analyst in assigning values to preferential parameters. Usually PETs proceed indirectly through a questioning procedure and "translate" the DMs' answers into values for the preferential parameters (by applying the specific aggregation rule in use).

Such PETs provide the analyst with several acceptable combinations of values for the parameters. One of the last phases of a decision aiding process is robustness analysis (see Roy, 1998). It consists in studying the impact of the values of preferential parameters on the overall preferences in order to determine recommendations to the DMs that are valid for all (or most of the) combinations of acceptable values for the parameters.

It seems obvious that these two stages of a decision aiding process interact. Robustness analysis uses a set of acceptable parameter values as input, while the elicitation of values for preferential parameters should be considered in relation with the impacts of the parameters' values on comprehensive preferences. However, methodologies proposed in the literature do not consider preference elicitation and robustness analysis within an integrated approach.

The purpose of this paper is to show how these two phases of a decision aiding process can be viewed within a single integrated approach. Within such an approach, the preference elicitation process is a sequence of question/answers; the DMs are provided with the solution(s) derived by the model using their answers to the preceding questions. Hence, they can react interactively and control their preference information with regard to the robust output of the model. We will restrict the analysis to the ELECTRE TRI method (see Mousseau *et al.* 1999a,b; Roy and Bouyssou 1993). However, we claim that the basic idea underlying our work can be applied to a large class of methods.

The paper is organised as follows. The next section provides a brief reminder on the well-known ELECTRE TRI method. Section 3 presents two approaches developed independently: the first one consists of inferring values for the ELECTRE TRI method's parameters from assignment examples (see Mousseau and Slowinski, 1998, Mousseau *et al.*, 2000), whereas the second one consists of computing robust assignments consistent with constraints on parameters (see Dias and Clímaco, 1999,2000). Both approaches avoid asking for precise values for the parameters. Rather, they proceed to solve the problem in a way that requires much less cognitive effort from the DMs. Section 4, after introducing the concept of constructive learning

procedures, shows how these two approaches can be combined into an integrated methodology to determine robust assignments interactively, within an aggregation/disaggregation approach. An illustrative example is provided in section 5. The last section draws some conclusions and issues for further research.

2. Brief presentation of the ELECTRE TRI method

This section gives a very brief overview of the ELECTRE TRI method and defines some notations that will be used along the paper. For more details, see Mousseau *et al.* (1999a,b), Roy and Bouyssou (1993).

ELECTRE TRI is a multiple criteria sorting method, i.e., a method that assigns alternatives to pre-defined ordered categories. The assignment of an alternative a results from the comparison of a with the profiles defining the limits of the categories. Let F denote the set of the indices of the criteria $g_1, g_2, \dots, g_n$ ($F = \{1, 2, \dots, n\}$) and B the set of indices of the profiles $b_1, b_2, \dots, b_p$ defining $p+1$ categories ($B = \{1, 2, \dots, p\}$), b_b being the upper limit of category C_b and the lower limit of category C_{b+1} , $b=1, 2, \dots, p$. In what follows, we will assume, without any loss of generality, that preferences increase with the value of each criterion.

ELECTRE TRI builds a fuzzy outranking relation S whose meaning is "*at least as good as*". Preferences restricted to the significance axis of each criterion are defined through pseudo-criteria (see Roy and Vincke, 1984, for details on this double-threshold preference representation). The indifference and preference thresholds, $q_j(b_b)$ and $p_j(b_b)$, respectively, constitute the intra-criterion preferential information. Two types of inter-criteria preference parameters intervene in the construction of S :

- the set of weight-importance coefficients $w = (w_1, w_2, \dots, w_n)$ is used in the concordance test when computing the relative importance of the coalitions of criteria being in favour of the assertion aSb_b (and b_bSa);
- the set of veto thresholds $(v_1(b_b), v_2(b_b), \dots, v_n(b_b))$ is used in the discordance test; $v_j(b_b)$ represents the smallest difference $g_j(b_b) - g_j(a)$ incompatible with the assertion aSb_b (and b_bSa).

As the assignment of alternatives to categories does not result directly from the relation S , an exploitation phase is necessary; it requires the relation S to be "defuzzified" using a so-called λ -cut: the assertion aSb_b is considered to be valid if the credibility index of the fuzzy outranking relation is greater than a "cutting level" λ such that $\lambda \in [0.5, 1]$. This λ -cut determines the preference situation between a and b_b .

Two assignment procedures (optimistic and pessimistic) are available, their role being to analyse the way in which an alternative a compares to the profiles so as to determine the category to which a should be assigned. The result of these two assignment procedures differs when the alternative a is incomparable with at least one profile b_b .

3. Preference elicitation and robustness analysis for ELECTRE TRI

3.1. Main features of the inferring procedure

Mousseau and Slowinski (1998) proposed an inference procedure using the paradigm of aggregation/disaggregation, outlined in Figure 1. Its aim is to find an ELECTRE TRI model as compatible as possible with the assignment examples given by the DMs. The assignment examples concern a subset $\mathcal{A}^* \subset \mathcal{A}$ of alternatives for which the DMs have clear preferences, i.e., alternatives that the DMs can easily assign to a category (or a range of consecutive categories), taking into account their evaluation on all criteria. The compatibility between the ELECTRE TRI model and the assignment examples is understood as an ability of the ELECTRE TRI method using this model to reassign the alternatives of $\mathcal{A}^*$ in the same way as the DM did.

In order to minimise the differences between the assignments made by ELECTRE TRI and the assignments made by the DM, an optimisation procedure is used (Appendix B presents the optimisation model corresponding to the inference of the weights w and the cutting level λ). The DMs can tune up the model in the course of an interactive procedure. They may either revise the assignment examples (i.e., remove and/or add some alternatives from/to A^* , change the assignment of some alternatives of A^*) or define constraints for some model parameters (i.e., ordinal information on the importance of criteria, incomplete definition of some profiles defining the limits between categories, ...) basing on their own intuition.

When the model is not perfectly compatible with the assignment examples, the procedure can detect all "hard cases", i.e., the alternatives for which the assignment computed by the model strongly differs from the DMs' assignment. The DMs could then be asked to reconsider their judgement.

The interaction procedure presented in Figure 1 (grey items refer to the phases in which the intervention of DMs is required) stops when the DM is satisfied with the values proposed for the parameters. These values should be compatible with a set of assignment examples and possibly with additional constraints on the parameters value. The interaction provides the DMs a so-called "constructive learning" context in which they can improve their understanding on how the assignment model is affected by the values of the parameters and find a set of parameters values that are consistent with the assignment examples.

This methodology is of a great help in practical decision situation so as to elicit preferences. However, Mousseau and Slowinski (1998) do not provide much support to the DMs in the management of the interaction. There is a lack of control of the interaction process which is due to the absence of features that would ensure a certain form of "cognitive convergence" of the procedure.

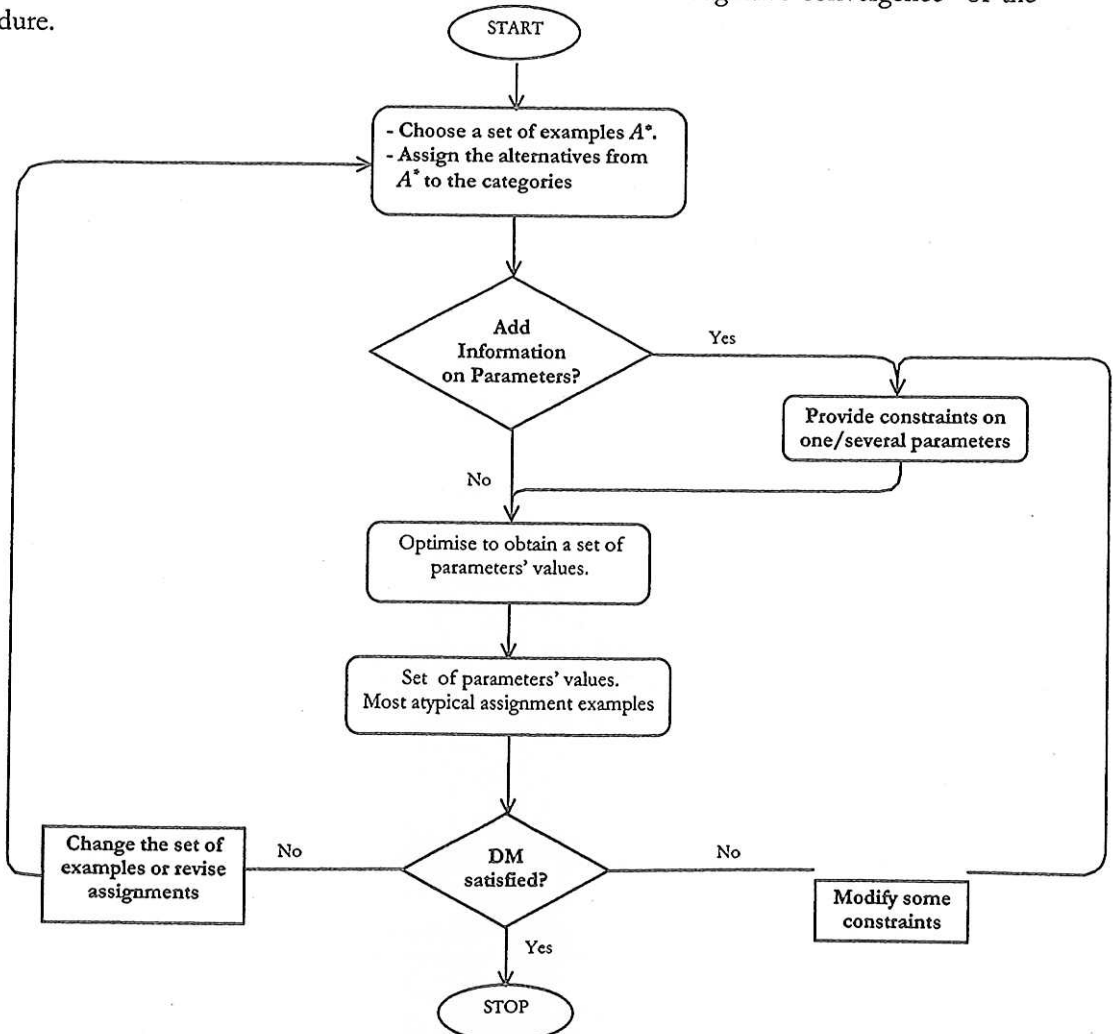


Figure 1: General Scheme of the Inference Procedure

3.2. A procedure to obtain robust conclusions

We consider in this section a situation where several combinations of values for the parameters are acceptable, i.e., a situation with imprecise information. The set of possible combinations are expressed as constraints, rather than as a discrete set of values. The constraints may be explicitly provided by the DMs or inferred from holistic comparisons (as in Mousseau, 1993). Although constraints are not always easy to provide, requiring precise values for the parameters is obviously more demanding.

Let T represent the set of all acceptable combinations of parameter values. One can determine the range of any credibility index subject to T (for details, see Dias and Clímaco, 1999), which may be used to determine the best and worst categories to which an alternative may be assigned, subject to T (for details, see Dias and Clímaco, 2000). This approach determines the best and the worst categories, $B(a_i, T)$ and $W(a_i, T)$, respectively, to which alternative a_i can be assigned by ELECTRE TRI subject to the constraints defining T . If $B(a_i, T)$ coincides with $W(a_i, T)$, then the method is able to assign a_i to a single category, despite the imprecision regarding the input.

Three types of information are considered in Dias and Clímaco's approach:

1. Robust conclusions as regards T : stating that a_i belongs to a category no worse than $W(a_i, T)$ is a robust conclusion, and so is stating that a_i belongs to a category no better than $B(a_i, T)$.
2. Identification of the alternatives that are more affected by the imprecision of data. Indeed, it may be interesting to know that some alternatives have a wide range of categories to which they may be assigned, in contrast with some other alternatives that are precisely assigned to a single category.
3. "Extreme" combinations of parameter values corresponding to best-case and worst-case assignments.

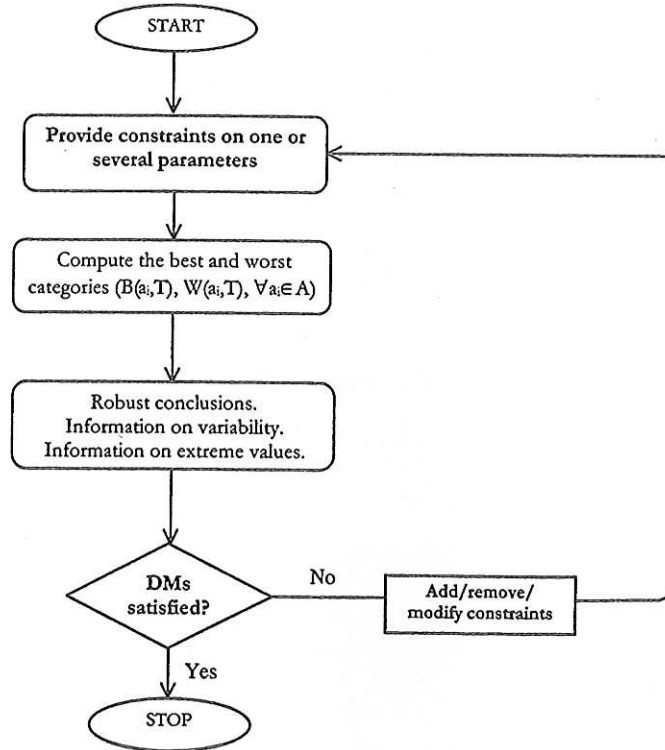


Figure 2: General Scheme of the Robustness Analysis Procedure

We may consider the use of this type of approach to be an interactive learning process, where the results of the analysis may stimulate the DMs to discuss and revise their inputs, as outlined in Figure 2. In general, DMs could begin with little information (starting with a set T not too constrained) and then progressively enrich that information (reducing T) as they form their

convictions. This type of approach then tries to identify conclusions that can be accepted as valid, despite the lack of precision present in the information they provide.

Dias and Clímaco (1999,2000) addressed a general case that involves solving many non-linear maximisation/minimisation problems, which become linear programs when veto thresholds are fixed or only subject to upper and lower bounds (see Appendix C). They did not address explicitly the problem of obtaining those constraints from the DMs. Therefore, the integrated approach presented in the next section can be seen as a complement to the robustness analysis approach, to the same extent that it may be seen as a complement to the inference approach in Section 3.1.

4. A new constructive learning procedure

4.1. The constructive learning process

The process that leads to the definition of a multiple criteria sorting model can be analysed with respect to different perspectives. The role of assignment examples, constraints on preference parameters and the way the model can be validated are very different when they are conceived according to a descriptive or constructive process.

Defining a sorting model can be viewed in a *descriptive learning* perspective, i.e., as a process in which the model reproduces a class of input-output behaviour accounting for a set of learning examples. Machine learning (Michalski, 1983; Quilan, 1986) typically enters this perspective in which the definition of the model should optimise the explanation of the examples. Hence, the validity of the obtained model is grounded on the input data and its ability to reproduce similar classifications. The role of the DMs in the definition of the model is reduced to providing learning examples. Different models that equally explain the data are considered equivalent.

On the other hand, a *constructive learning* perspective regards the definition of the sorting model as a result between the interaction of the assignment examples and the DMs' points of view. The process should result in a model that both explains the examples and accounts for the DMs' perception of the problem. During the construction process, the DMs might modify some assignment examples while increasing their understanding of the problem. The very nature of this constructive learning process is to provide the DMs with an interactive context in which they may check the impact of assignment examples on the model parameters and so find a model that fits both a set of learning examples and their perception of the sorting problem.

Both approaches presented in Sections 3.1 and 3.2 fall into the context of constructive learning. The combined approach proposed in this paper also considered the integration of the DMs in the process of defining the sorting model as a crucial point.

4.2. A combined approach

We can combine the two approaches presented in Sections 3.1 and 3.2 to take advantage of both of them. This combined approach should be used in an interactive manner, where the output of the analyses can help the DMs to revise their inputs (see Figure 3). Although the methodologies proposed by Mousseau and Slowinski (1998) and Dias and Clímaco (1999,2000) apply when all preferential parameters are considered, we restrict ourselves to the case where only the weights $w=(w_1, w_2, \dots, w_n)$ and cutting level λ are variables and when no veto phenomenon occurs. This restriction is mainly due to computational difficulties (the mathematical program becomes non-linear) that arise when all parameters should be inferred

4.2.1. Input information required from the DMs.

Our approach intends to help DMs to fix the parameter values (or to constraint them) and to obtain robust conclusions concerning the alternatives' assignments. To do so, the proposed procedure does not require the DMs to provide precise values for the preferential parameters. At iteration k , the input may consist of:

- explicit preferential information (constraints on the parameter values), defining a domain T_k^w ;
- assignment examples, i.e., a subset of specific alternatives ($A^* \subset A$) that the DMs are able to assign holistically to a category (or a range of categories).

Both types of input information take the form of constraints on the values of preferential parameters as assignment examples are expressed in the model through a set of constraints defining a polyhedron T_k^a (using the ELECTRE TRI assignment rule). $T_k = T_k^w \cap T_k^a$ denotes the set of combination parameters values corresponding to the input information provided by the DMs at the iteration k (either through assignment examples or constraints on the parameter values). At each iteration the input information may be inconsistent or not, i.e., the set T_k may be empty or not.

4.2.2. Output information provided to DMs.

The information provided to the DMs at iteration k is different depending on the consistency of the input information. Technical details concerning the computation of these results are given in section 5.

- The input information is considered as inconsistent if T_k is empty, i.e., there is no combination of parameter values that fully restores the assignment examples and simultaneously conforms to the additional constraints. In this case we can provide the DMs with:
 - a combination of parameter values that least violates the constraints that define T_k ;
 - the constraints in T_k^a that are the most difficult to respect, which yield the alternatives from A^* that are the most difficult to assign;
 - the constraints in T_k^w that are the most difficult to respect;
 - proposals of modification of T_k so that this polyhedron does not become empty.
- The input information is consistent, i.e., T_k is not empty. In this case we can provide the DMs with:
 - a combination of parameter values $t_k^* \in T_k$ that best match the provided information;
 - the "range" of categories $[W(a_i, T_k), B(a_i, T_k)]$ to which each alternative $a_i \in A$ can be assigned to for any combination of parameters values in T_k ; it should be noted that when an alternative a_i is such that $c_j(a_i, b_j) = c_j(a_i, b_{j+1})$, $\forall j$ (b_j and b_{j+1} defining a category C_b in the range $[W(a_i, T_k), B(a_i, T_k)]$), no $t \in T_k$ exist such that $a_i \rightarrow_t C_b$ (see appendix D);
 - the category in $[W(a_i, T_k), B(a_i, T_k)]$ to which ELECTRE TRI assigns each alternative $a_i \in A$ using t_k^* ;
 - the "extreme" combinations $t_k^B(a_i)$ and $t_k^W(a_i)$ leading to the best and worst case assignment, respectively, for each alternative $a_i \in A$;
 - for each category C_b to which any alternative $a_i \in A$ can be assigned, a "typical" set of value t for w and λ leading to this assignment ($a_i \rightarrow_t C_b$);

4.2.3. Interaction process.

The proposed procedure is designed to be used interactively, i.e., the output at a given iteration k is used to guide DMs to discuss and revise the input for the next iteration (T_{k+1}). Figure 3 presents the general scheme of the procedure. The nature of the interaction depends on the consistency of the input information provided at the current iteration.

- a) If $T_k = \emptyset$, the interaction aims at eliminating the inconsistency. The DMs should react by a modification or a deletion of a constraint in T_k . Several elements can support them in such revision of their input information:
- the list of assignment examples that are the most difficult to restore in the ELECTRE TRI model are the ones that should be considered first for an eventual modification or deletion;
 - similarly the constraints of T_k^w that are the most difficult to respect should be considered first;
 - the deletion of specific subsets of constraints can lead to a new polyhedron T_{k+1} which is not empty. Several such sets can be computed and presented to the DMs for them to choose.
- b) If $T_k \neq \emptyset$, the interaction aims at reducing the set of possible combinations of parameter values. Hence, the DMs should react in order to modify a constraint and/or add a new one to T_k . Several elements can support the DMs in such revisions of their input information:
- the combinations of parameters values t_k^* , $t_k^B(a_i)$ and $t_k^W(a_i)$ may suggest additional constraints for the DMs define T_{k+1} ; for instance, if t_k^* is such that $\lambda = 0.5$, the DMs may react by stating that $\lambda \geq 0.75$;
 - the ranges of $[W(a_i, T_k), B(a_i, T_k)]$ for all $a_i \in \mathcal{A}$ provide the DMs with useful information for selecting an alternative to be added to $\mathcal{A}^*$; alternatives with the widest ranges are the most affected by T_k imprecision; hence, giving additional information about their assignment should be highly informative; we conjecture that on average there ought to be less iterations to reach acceptably precise assignments when DMs choose alternatives with larger variability as assignment examples.

This interactive procedure is designed in such a way that the DMs start the first iteration with very little information. Each iteration will provide opportunity to add, delete or modify a specific supplementary constraint. Adding a single piece of information only at each iteration facilitates the control of the information supplied by the DMs.

As the DMs add constraints during the successive iterations, the domain T_k becomes more constrained and the range of categories $[W(a_i, T_k), B(a_i, T_k)]$ (for each $a_i \in \mathcal{A}$) reduces (or is invariant). At a given iteration, it is possible that the domain T_k becomes empty, meaning that the new constraints contradict some previous ones. DMs should then revise T_k . However, such iteration will not occur at the beginning of the interactive process. The interactive procedure stops when DMs are satisfied and the set T_k as well as the assignment of alternatives matches their view of the decision problem. The final results are:

- a set of constraints and assignment examples defining a set T_k of acceptable parameter values;
- an inferred combination of parameter values t_k^* defining a model in a precise manner;
- a precise assignment or range of assignments for each alternative in $\mathcal{A}$ that is robust with respect to T_k .

However, we want to emphasise the fact that this interactive process does not aim only at providing the above mentioned results. During the interactions, the DMs will get insights on their view of the problem and revise and possibly modify their opinions. Such interaction will provide them with an opportunity to learn about their preferences and increase their understanding of the decision problem.

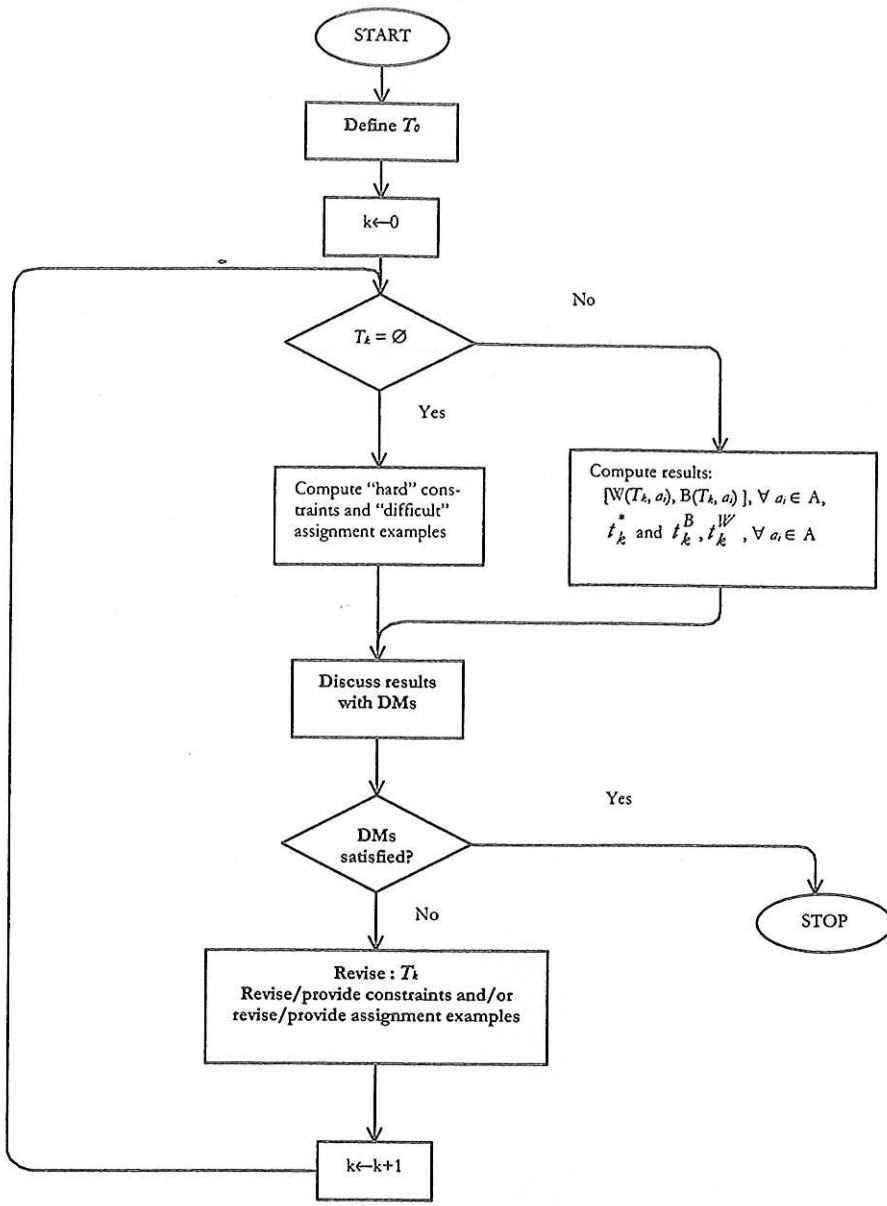


Figure 3: The proposed procedure

5. Computational aspects

At each iteration k , results are computed according to T_k in order to provide the DMs results on the basis of which they can refine the input information T_k and define T_{k+1} . The computations are different depending on the consistency of the input information T_k .

5.1. Computations when T_k is inconsistent

5.1.1. parameter values that least violates the constraints that define T_k

In order to determine the combination of values for w and λ that least violate the constraints defined by T_k , we solve the following linear program (LP1):

Max a

s.t.:

$$a \leq x_i, \forall i \text{ such that } a_i \in A^* \quad (1)$$

$$a \leq y_i, \forall i \text{ such that } a_i \in A^* \quad (2)$$

$$a \leq s_q, \forall q = 1, \dots, n_c \quad (3)$$

$$a \leq s_\lambda^{\min} \quad (4)$$

$$a \leq s_\lambda^{\max} \quad (5)$$

$$\sum_{j=1}^n w_j c_j(a_i, b_{l_i-1}) - x_i = \lambda, \quad \forall i \text{ such that } a_i \in A^* \quad (6)$$

$$\sum_{j=1}^n w_j c_j(a_i, b_{u_i}) + y_i = \lambda + \varepsilon, \quad \forall i \text{ such that } a_i \in A^* \quad (7)$$

$$\sum_{j=1}^n u_{jq} w_j - v_q + s_q = 0, \quad \forall q = 1, \dots, n_c \quad (8)$$

$$\lambda - s_\lambda^{\min} = 0 \quad (9)$$

$$\lambda - s_\lambda^{\max} = 0 \quad (10)$$

$$\varepsilon' \leq w_j \leq 0.5 - \varepsilon', \quad j \in F, \quad \sum_{j=1}^n w_j = 1 \quad (11)$$

$$0.5 \leq \lambda \leq 1 \quad (12)$$

$$x_i, y_i, s_q, s_\lambda^{\min}, s_\lambda^{\max}, a \text{ free} \quad (13)$$

LP1

where :

- l_i (u_i , respectively): the index of the lowest category (highest category, respectively) to which the alternatives $a_i \in A^*$ can be assigned, according to the opinion of the DMs ($l_i = u_i$, means that the DMs are able to assign the alternatives a_i to a single category).
- $\lambda^{\min}$ ($\lambda^{\max}$, respectively) is the lower bound (upper bound, respectively) for the cutting level λ . This information stated by the DMs is such that $0.5 \leq \lambda^{\min} \leq \lambda^{\max} \leq 1$.
- T_k^a : the set of combinations of weights that conform to the constraints implied by the assignment examples at iteration k . In the mathematical program below, T_k^a corresponds to constraints (6) and (7).
- T_k^w : the set of combinations of parameters values that conform to the additional information provided by the DMs at iteration k . T_k^w is defined by n_c constraints (8) of the form $\sum_{j=1}^n u_{jq} w_j \geq v_q, \forall q = 1, \dots, n_c$, and the constraints (9) and (10) defining bounds for λ .
- ε and ε' represents an arbitrary small positive value.
- $x_i, y_i, \forall a_i \in A^*, s_k, k=1, \dots, n_c, s_\lambda^{\min}$ and $s_\lambda^{\max}$: the slack variables that account for the "degree to which" a constraint is fulfilled; if negative, the constraint is violated and T_k is empty.

5.1.2. Alternatives from A^* that are the most difficult to assign

In LP1, these alternatives correspond to the a_i such that $x_i = \alpha$ and/or $y_i = \alpha$ at the optimum.

5.1.3. Constraints in T_k^w that are the most difficult to respect

In LP1, these constraints correspond to the ones for which $s_q = \alpha$ at the optimum.

5.1.4. Proposals of modification of T_k

Proposals of modification of T_k correspond to subsets of constraints that, when removed, lead to a consistent information. Numerous such subsets can exist; in order to identify N of these subsets, it is possible to use the naive enumeration algorithm mentioned below. So as to list all these subsets, more elaborated techniques should be used (see for example Chinneck 1996).

```

i ← 1
num ← 0
while (num < N)
do
    for (each subset Z of i constraint in  $T_k$ )
    do
        if ( $T \setminus Z \neq \emptyset$ ) then print Z
    end do
    num ← num + 1
end do

```

5.2. Computations when T_k is consistent

5.2.1. Parameter values $t_k^* \in T_k$ that best match the provided information

t_k^* is computed solving LP1 (see section 5.1.1).

5.2.2. Best and worst categories to which each alternative $a_i \in A$ can be assigned considering T_k

To determine the best possible assignment $B(a_i, T_k)$ for the alternative a_i considering the domain T_k of possible values for w and λ , we perform the tests by maximising the credibility ($b=1, 2, \dots, p$):

$$\text{LP2: } \max_{t \in T_k} \sum_{j=1}^n w_j c_j(a_x, b_h) - \lambda$$

To determine the worst possible assignment $W(a_i, T_k)$ for the alternative a_i , we perform the tests by minimising the credibility ($b=1, 2, \dots, p$):

$$\text{LP3: } \min_{t \in T_k} \sum_{j=1}^n w_j c_j(a_x, b_h) - \lambda$$

5.2.3. "Extreme" combinations $t_k^B(a_i)$ and $t_k^W(a_i)$ leading to the best and worst case assignment

$t_k^B(a_i)$ and $t_k^W(a_i)$ are obtained directly solving LP2 and LP3.

5.2.4. for each category C_b to which any alternative $a_i \in A$ can be assigned, a "typical" set of values $t(a_i, b)$

This result is obtained by the resolution of LP1 (see 5.1.1) to which the constraints corresponding to the assignment of a_i to C_b are added.

5.2.5. For each category C_b to which $a_i \in A$ can not be assigned, the constraints that makes impossible for a_i to be assigned to C_b .

In order to compute the constraints that forbid the assignment of an alternative $a_i \in A$ to a category C_b , we proceed as follows.

1. Add the assignment example $a_i \rightarrow C_b$ to T_k . The information become inconsistent.
2. Compute proposals of modification of T_k that make the information consistent and containing the constraints corresponding to the assignment example $a_i \rightarrow C_b$.

6. An illustrative example

So as to illustrate the proposed methodology, we consider the data from a real world application in the banking sector (see Dimitras *et al.*, 1995). We consider a model that aims at assigning 40 alternatives to 5 categories on the basis of their evaluations on seven criteria. The data are presented in Appendix A.

Using this data we simulate *a posteriori* the modelling process that could have take place using our methodology. Let us consider a hypothetical preference elicitation process and suppose that our fictitious DMs are able to elicit directly the limits of categories b_b and the thresholds $q_j(b_b)$ and $p_j(b_b)$. (their values are given in Appendix A and correspond to the values considered in Dimitras *et al.* 1995) but have difficulties with assigning values for the importance coefficients w_j and cutting level λ .

Let us suppose that the only initial information that the DMs are able to express is that the most important criterion is g_2 , no assignment example is provided by DMs, and $T_0^w = \{w_2 \geq w_1, w_2 \geq w_3, w_2 \geq w_4, w_2 \geq w_5, w_2 \geq w_6, w_2 \geq w_7\}$. Suppose also that $\lambda \in [0.5, 0.99]$. This information is obviously consistent and t_0^* is such that $w_2=0.49, w_j=0.085, \forall j \neq 2$, and $\lambda=0.5$. Ranges $[W(a_i, T_0), B(a_i, T_0)]$, $\forall a_i \in A$, are presented in Table 1 in which the grey cells correspond to possible assignments and the marked cells correspond to the assignments made by ELECTRE TRI pessimistic rule using t_0^* . Alternatives are sorted decreasingly according to the width of their assignment range.

	C_1	C_2	C_3	C_4	C_5		C_1	C_2	C_3	C_4	C_5
a_{28}				X		a_{27}				X	
a_{14}				X		a_{29}				X	
a_{36}				X		a_{30}				X	
a_{31}		X				a_{32}				X	
a_2					X	a_{37}				X	
a_3			X			a_{24}			X		
a_4				X		a_{34}			X		
a_8				X		a_{38}			X		
a_9				X		a_{39}			X		
a_{10}				X		a_{35}			X		
a_{11}				X		a_1				X	
a_{12}				X		a_5				X	
a_{13}				X		a_6				X	
a_{16}				X		a_7				X	
a_{18}				X		a_{15}				X	
a_{19}				X		a_{17}				X	
a_{20}				X		a_{23}				X	
a_{21}				X		a_{26}				X	
a_{22}				X		a_{33}				X	
a_{25}				X		a_0					X

Table 1: Assignment ranges after the first iteration

Analysing these first results, the DMs are surprised with the wide range (C_1 to C_5 , except C_2 as $c_j(a_{28}, b_1) = c_j(a_{28}, b_2)$, $\forall j \in F$, see appendix D) of possible assignments for a_{28} . Considering evaluations of a_{28} on the criteria and their expertise of the problem, the DMs state that a_{28} should be assigned to categories C_1 . Considering this information T_1 , results are computed and the information is still consistent. Computations show that t_1^* is such that $w_2=0.49$, $w_5=0.25$, $w_7=0.22$, $w_j=0.01$ for all other criteria and $\lambda=0.99$. Ranges $[W(a_i, T_1), B(a_i, T_1)]$, $\forall a_i \in A$, are presented in Table 2.

	C_1	C_2	C_3	C_4	C_5		C_1	C_2	C_3	C_4	C_5
a_2					X	a_{18}			X		
a_3				X		a_{19}			X		
a_9			X			a_{20}			X		
a_{10}			X			a_{21}			X		
a_{11}				X		a_{22}				X	
a_{32}			X			a_{23}				X	
a_{14}				X		a_{25}			X		
a_{34}				X		a_{26}			X		
a_{36}		X				a_{27}			X		
a_{38}		X				a_{29}				X	
a_{39}		X				a_{30}				X	
a_{35}		X				a_{33}			X		
a_5				X		a_{37}			X		
a_6				X		a_{24}		X			
a_7					X	a_{31}	X				
a_4			X			a_0					X
a_8				X		a_1				X	
a_{12}			X			a_{15}				X	
a_{13}			X			a_{17}				X	
a_{16}			X			a_{28}	X				

Table 2: Assignment ranges after the second iteration

Considering the result presented in Table 2, the DMs are surprised that the alternative a_1 is assigned to category C_4 . They consider that this alternative should be assigned to category C_5 . However, it is clear that adding the assignment example " $a_1 \rightarrow C_5$ " would make T_2 inconsistent. Though, we determine which constraints contradict such assertion, i.e., the subsets of constraints with minimal cardinality whose deletion from T_2 makes the information consistent. In our case, the deletion of one constraint can make T_2 consistent. This constraint is one among the following:

- $a_1 \rightarrow C_5$
- $a_{28} \rightarrow C_1$
- $w_2 \geq w_5$

Let us suppose that DMs are rather confident in the two assignment examples but feel less confident about the constraint concerning w_2 and w_5 . Therefore, they propose to remove this constraint. Results are computed considering this updated information. T_3 is consistent and t_3^* is such that $w_5=0.49$, $w_2=0.235$, $w_4=w_6=0.1225$, $w_j=0.01$, $j=1,3,7$ and $\lambda=0.6225$. Ranges $[W(a_i, T_3), B(a_i, T_3)]$, $\forall a_i \in A$, are presented in Table 3.

	C_1	C_2	C_3	C_4	C_5
a_{31}			X		
a_{10}					X
a_{12}					X
a_{13}				X	
a_{14}				X	
a_{15}					X
a_{16}					X
a_{17}					X
a_{18}				X	
a_{19}				X	
a_{20}				X	
a_{21}				X	
a_{22}					X
a_{25}				X	
a_{27}				X	
a_{29}					X
a_{30}					X
a_{32}					X
a_{36}				X	
a_{37}					

	C_1	C_2	C_3	C_4	C_5
a_{24}			X		
a_0					X
a_1					X
a_2					X
a_3					X
a_4					X
a_5					X
a_6					X
a_7					X
a_8					X
a_9					X
a_{11}					X
a_{23}				X	
a_{26}				X	
a_{33}				X	
a_{34}				X	
a_{38}				X	
a_{39}				X	
a_{35}			X		
a_{28}	X				

Table 3: Assignment ranges after the third iteration

At this stage, the DMs would like to add a new assignment example by stating that the alternative a_{31} is assigned to category C_2 . The new information T_4 is consistent and t_4^* is such that $w_5=0.48$, $w_2=0.282$, $w_4=0.198$, $w_j=0.01$, $j=1,3,6,7$ and $\lambda=0.604$. Ranges $[W(a_i, T_4), B(a_i, T_4)]$, $\forall a_i \in A$, are presented in Table 4.

	C_1	C_2	C_3	C_4	C_5
a_{10}					X
a_{12}					X
a_{13}					X
a_{14}					X
a_{15}					X
a_{16}					X
a_{17}					X
a_{19}					X
a_{20}					X
a_{21}					X
a_{22}					X
a_{25}					X
a_{27}					X
a_{29}					X
a_{30}					X
a_{32}					X
a_{36}					X
a_{37}					X
a_{24}				X	
a_0					X

	C_1	C_2	C_3	C_4	C_5
a_1					X
a_2					X
a_3					X
a_4					X
a_5					X
a_6					X
a_7					X
a_8					X
a_9					X
a_{11}					X
a_{18}				X	
a_{23}				X	
a_{26}				X	
a_{33}				X	
a_{34}				X	
a_{38}				X	
a_{39}				X	
a_{35}			X		
a_{31}		X			
a_{28}	X				

Table 4: Assignment ranges after the fourth iteration

The DMs want now to add a supplementary constraint concerning the weights $w_3 \geq w_4$. Adding this constraint makes the information T_5 inconsistent. It is however possible to determine a set of parameters that least violates the information contained in T_5 : $w_5=0.333$, $w_2=w_3=0.160$, $w_4=w_6=0.166$, $w_1=w_7=0.01$ and $\lambda=0.667$. In this case, we determine which constraints contradict such assertion, i.e., the subsets of constraints with minimal cardinality whose deletion from T_5 makes the information consistent. In our case, the deletion of one constraint can make T_5 consistent. This constraint is one among the following:

- $a_1 \rightarrow C_5$
- $a_{31} \rightarrow C_2$
- $w_3 \geq w_4$

Let us suppose that the DMs decide to revise their views concerning alternative a_1 in such a way that $a_1 \rightarrow (C_4 \text{ or } C_5)$. Results are computed considering this updated information. T_6 is consistent and t_6^* is such that $w_5=0.406$, $w_2=0.396$, $w_3=0.158$, $w_j=0.01$, $j=1,4,6,7$ and $\lambda=0.743$. Ranges $[W(a_i, T_6), B(a_i, T_6)]$, $\forall a_i \in A$, are presented in Table 5.

	C ₁	C ₂	C ₃	C ₄	C ₅		C ₁	C ₂	C ₃	C ₄	C ₅
a ₃					X	a ₁₃				X	
a ₈				X		a ₁₈				X	
a ₉					X	a ₁₉				X	
a ₁₀				X		a ₂₀				X	
a ₁₁					X	a ₂₁				X	
a ₁₂				X		a ₂₂				X	
a ₁₆				X		a ₂₃				X	
a ₂₉				X		a ₂₅				X	
a ₃₂					X	a ₂₆				X	
a ₁₄				X		a ₂₇				X	
a ₃₄				X		a ₃₀				X	
a ₃₆				X		a ₃₃				X	
a ₃₈				X		a ₃₇				X	
a ₃₉				X		a ₂₄			X		
a ₃₅			X			a ₀					X
a ₅				X		a ₁				X	
a ₆					X	a ₁₅				X	
a ₇					X	a ₁₇				X	
a ₂				X		a ₃₁		X			
a ₄				X		a ₂₈	X				

Table 5: Assignment ranges after the sixth iteration

The DMs would like now to add a new assignment example $a_{16} \rightarrow C_3$. The new information T_7 is consistent and t_7^* is such that $w_5=0.264$, $w_2=0.348$, $w_3=0.094$, $w_1=0.264$, $w_j=0.01$, $j=1,4,6$ and $\lambda=0.819$. Ranges $[W(a_i, T_7), B(a_i, T_7)]$, $\forall a_i \in A$, are presented in Table 6.

Pursuing their analysis, the DMs would like to add a new assignment example $a_{14} \rightarrow C_2$. The new information T_8 is still consistent and t_8^* is such that $w_5=0.225$, $w_2=0.295$, $w_3=0.225$, $w_1=0.225$, $w_j=0.01$, $j=1,4,6$ and $\lambda=0.846$. Ranges $[W(a_i, T_8), B(a_i, T_8)]$, $\forall a_i \in A$, are presented in Table 7.

	C ₁	C ₂	C ₃	C ₄	C ₅		C ₁	C ₂	C ₃	C ₄	C ₅
a ₇				X	X	a ₈			X		
a ₂			X	X		a ₉			X		
a ₃			X	X		a ₁₀			X		
a ₁₁			X	X		a ₁₂			X		
a ₂₂			X	X		a ₁₃			X		
a ₂₃			X	X		a ₁₆			X		
a ₂₉			X	X		a ₁₈			X		
a ₃₀			X	X		a ₁₉			X		
a ₃₃			X	X		a ₂₀			X		
a ₁₄			X			a ₂₁			X		
a ₂₄			X			a ₂₅			X		
a ₃₄			X			a ₂₆			X		
a ₃₅		X				a ₂₇			X		
a ₀					X	a ₃₂			X		
a ₁				X		a ₃₇			X		
a ₅				X		a ₃₁		X			
a ₆				X		a ₃₆		X			
a ₁₅				X		a ₃₈		X			
a ₁₇				X		a ₃₉		X			
a ₄			X			a ₂₈	X				

Table 6: Assignment ranges after the seventh iteration

	C ₁	C ₂	C ₃	C ₄	C ₅		C ₁	C ₂	C ₃	C ₄	C ₅
a ₂₂			X	X		a ₁₃			X		
a ₂₉			X	X		a ₁₆			X		
a ₃₀			X	X		a ₁₈			X		
a ₂₄		X				a ₁₉			X		
a ₃₅		X				a ₂₀			X		
a ₀					X	a ₂₁			X		
a ₁				X		a ₂₃			X		
a ₅				X		a ₂₅			X		
a ₆				X		a ₂₆			X		
a ₇				X		a ₂₇			X		
a ₁₅				X		a ₃₂			X		
a ₁₇				X		a ₃₃			X		
a ₂			X			a ₃₇			X		
a ₃			X			a ₁₄		X			
a ₄			X			a ₃₁		X			
a ₈			X			a ₃₄		X			
a ₉			X			a ₃₆		X			
a ₁₀			X			a ₃₈		X			
a ₁₁			X			a ₃₉		X			
a ₁₂			X			a ₂₈	X				

Table 7: Assignment ranges after the eighth iteration

Finally, the DMs add a new assignment example $a_{35} \rightarrow C_1$. The new information T_9 is consistent and t_9^* is such that $w_2=0.242$, $w_j=0.185$, $j=1,3,5,7$, $w_j=0.01$, $j=4,6$ and $\lambda=0.873$. Ranges $[W(a_i, T_9), B(a_i, T_9)]$, $\forall a_i \in A$, are presented in Table 8. The interaction process stops here, as the DMs are satisfied with this assignment table and the corresponding values of preferential parameters.

	C ₁	C ₂	C ₃	C ₄	C ₅		C ₁	C ₂	C ₃	C ₄	C ₅
a ₂₂			X	X		a ₁₆			X		
a ₂₉			X	X		a ₁₈			X		
a ₃₀			X	X		a ₁₉			X		
a ₂₄		X				a ₂₀			X		
a ₀					X	a ₂₁			X		
a ₁				X		a ₂₃			X		
a ₅				X		a ₂₅			X		
a ₆				X		a ₂₆			X		
a ₇				X		a ₂₇			X		
a ₁₅				X		a ₃₂			X		
a ₁₇				X		a ₃₃			X		
a ₂			X			a ₃₇			X		
a ₃			X			a ₁₄		X			
a ₄			X			a ₃₁		X			
a ₈			X			a ₃₄		X			
a ₉			X			a ₃₆		X			
a ₁₀			X			a ₃₈		X			
a ₁₁			X			a ₃₉		X			
a ₁₂			X			a ₂₈	X				
a ₁₃			X			a ₃₅	X				

Table 8: Assignment ranges after the last iteration

At the end of this interactive elicitation process, the DMs come out with a set of weights and a value for the cutting level that they can apply in their assignment model. Moreover, these values for w and λ are justified by the information they have provided in the process, i.e., the assignment examples and constraints on the parameters.

This example of fictitious elicitation process shows how the interaction is made possible through the proposed procedure. We would like to emphasise that the procedure is designed in such a way that DMs can gain insights on their own preferences during the process. In our example, the DMs began the interaction with the idea that g_2 was at least as important as all other criteria. During the interaction, the DMs revised their point of view by stating that the inequality $w_2 \geq w_5$ does not hold. Similarly, their opinion concerning the assignment of alternative a_1 changed during the process.

Another important issue illustrated through this example deals with the "amount of information" provided at each iteration. In our example, only a single piece of information is added at each iteration. In this way, the DMs keep a good control of the interaction. This feature is particularly important for the management of inconsistencies, i.e., in order for the DMs to identify and understand the reasons of inconsistencies.

7. Conclusion

A classical view of a multiple criteria decision aiding process is frequently the following: ① definition of the criteria and alternatives, ② choice of an aggregation model and preference elicitation, ③ use of the aggregation model, ④ Robustness analysis and recommendations to the DMs.

Preference elicitation and robustness analysis are usually considered separately. In this paper, we show how these two phases of a decision aiding process can be viewed within a single integrated approach. We present a new approach to elicit an ELECTRE TRI model in a way that integrates the preference elicitation phase and the construction of robust conclusions: in our interactive procedure, the information provided by the DMs for preference elicitation purposes is considered as compared to its implications on the robust conclusions that can be derived.

Although this work is dealing with the ELECTRE TRI method only, we claim that the basic ideas underlying our work can be applied to a large class of sorting methods such as additive utility sorting methods (UTADIS), Rough sets based classifiers, etc.

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Appendix A: Data set

This example was adapted from Dimitras *et al.* (1995).

1. Criteria

- g_1 : (Financial ratio) Earning Before Interest and Taxes/Total Assets [Increasing preferences].
- g_2 : (Financial ratio) Net Income/Net Worth [Increasing preferences].
- g_3 : (Financial ratio) Total Liabilities/Total Assets [Decreasing preferences].
- g_4 : (Financial ratio) Interest Expenses/Sales [Decreasing preferences].
- g_5 : (Financial ratio) General and Administrative Expenses/Sales [Decreasing preferences].
- g_6 : (Qualitative criterion) Managers Work Experience [Increasing preferences].
- g_7 : (Qualitative criterion) Market Niche/Position [Increasing preferences].

2. Alternatives

Altern.	g_1	g_2	g_3	g_4	g_5	g_6	g_7
a0	35.8	67.0	19.7	0.0	0.0	5.0	4.0
a1	16.4	14.5	59.8	7.5	5.2	5.0	3.0
a2	35.8	24.0	64.9	2.1	4.5	5.0	4.0
a3	20.6	61.7	75.7	3.6	8.0	5.0	3.0
a4	11.5	17.1	57.1	4.2	3.7	5.0	2.0
a5	22.4	25.1	49.8	5.0	7.9	5.0	3.0
a6	23.9	34.5	48.9	2.5	8.0	5.0	3.0
a7	29.9	44.0	57.8	1.7	2.5	5.0	4.0
a8	8.7	5.4	27.4	4.5	4.5	5.0	2.0
a9	25.7	29.7	46.8	4.6	3.7	4.0	2.0
a10	21.2	24.6	64.8	3.6	8.0	4.0	2.0
a11	18.3	31.6	69.3	2.8	3.0	4.0	3.0
a12	20.7	19.3	19.7	2.2	4.0	4.0	2.0
a13	9.9	3.5	53.1	8.5	5.3	4.0	2.0
a14	10.4	9.3	80.9	1.4	4.1	4.0	2.0
a15	17.7	19.8	52.8	7.9	6.1	4.0	4.0
a16	14.8	15.9	27.9	5.4	1.8	4.0	2.0
a17	16.0	14.7	53.5	6.8	3.8	4.0	4.0
a18	11.7	10.0	42.1	12.2	4.3	5.0	2.0
a19	11.0	4.2	60.8	6.2	4.8	4.0	2.0
a20	15.5	8.5	56.2	5.5	1.8	4.0	2.0
a21	13.2	9.1	74.1	6.4	5.0	2.0	2.0
a22	9.1	4.1	44.8	3.3	10.4	3.0	4.0
a23	12.9	1.9	65.0	14.0	7.5	4.0	3.0
a24	5.9	-27.7	77.4	16.6	12.7	3.0	2.0
a25	16.9	12.4	60.1	5.6	5.6	3.0	2.0
a26	16.7	13.1	73.5	11.9	4.1	2.0	2.0
a27	14.6	9.7	59.5	6.7	5.6	2.0	2.0
a28	5.1	4.9	28.9	2.5	46.0	2.0	2.0
a29	24.4	22.3	32.8	3.3	5.0	3.0	4.0
a30	29.5	8.6	41.8	5.2	6.4	2.0	3.0
a31	7.3	-64.5	67.5	30.1	8.7	3.0	3.0
a32	23.7	31.9	63.6	12.1	10.2	3.0	2.0

16. Mousseau, V., Slowinski, R. & Zielniewicz, P. (1999a): "ELECTRE TRI 2.0a Methodological Guide and User's Manual", *Document N° 111*, LAMSADE, Université Paris-Dauphine.
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2. Alternatives

Altern.	g_1	g_2	g_3	g_4	g_5	g_6	g_7
a0	35.8	67.0	19.7	0.0	0.0	5.0	4.0
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a12	20.7	19.3	19.7	2.2	4.0	4.0	2.0
a13	9.9	3.5	53.1	8.5	5.3	4.0	2.0
a14	10.4	9.3	80.9	1.4	4.1	4.0	2.0
a15	17.7	19.8	52.8	7.9	6.1	4.0	4.0
a16	14.8	15.9	27.9	5.4	1.8	4.0	2.0
a17	16.0	14.7	53.5	6.8	3.8	4.0	4.0
a18	11.7	10.0	42.1	12.2	4.3	5.0	2.0
a19	11.0	4.2	60.8	6.2	4.8	4.0	2.0
a20	15.5	8.5	56.2	5.5	1.8	4.0	2.0
a21	13.2	9.1	74.1	6.4	5.0	2.0	2.0
a22	9.1	4.1	44.8	3.3	10.4	3.0	4.0
a23	12.9	1.9	65.0	14.0	7.5	4.0	3.0
a24	5.9	-27.7	77.4	16.6	12.7	3.0	2.0
a25	16.9	12.4	60.1	5.6	5.6	3.0	2.0
a26	16.7	13.1	73.5	11.9	4.1	2.0	2.0
a27	14.6	9.7	59.5	6.7	5.6	2.0	2.0
a28	5.1	4.9	28.9	2.5	46.0	2.0	2.0
a29	24.4	22.3	32.8	3.3	5.0	3.0	4.0
a30	29.5	8.6	41.8	5.2	6.4	2.0	3.0
a31	7.3	-64.5	67.5	30.1	8.7	3.0	3.0
a32	23.7	31.9	63.6	12.1	10.2	3.0	2.0

a33	18.9	13.5	74.5	12.0	8.4	3.0	3.0
a34	13.9	3.3	78.7	14.7	10.1	2.0	2.0
a35	-13.3	-31.1	63.0	21.2	29.1	2.0	1.0
a36	6.2	-3.2	46.1	4.8	10.5	2.0	1.0
a37	4.8	-3.3	71.1	8.6	11.6	2.0	2.0
a38	0.1	-9.6	42.5	12.9	12.4	1.0	1.0
a39	13.6	9.1	76.0	17.1	10.3	1.0	1.0

3. Categories

- C_1 : Very High Risk (worst category).
- C_2 : High Risk.
- C_3 : Medium Risk.
- C_4 : Low Risk.
- C_5 : Very Low Risk (best category).

4. Profiles

- b_1 : High Risk/Very High Risk
- b_2 : Medium Risk/High Risk.
- b_3 : Low Risk/Medium Risk.
- b_4 : Very Low Risk/Low Risk.

Prof.	g_1	g_2	g_3	g_4	g_5	g_6	g_7
b_1	-10.0	-60.0	90.0	28.0	40.0	1.0	0.0
b_2	0.0	-40.0	75.0	23.0	32.0	2.0	2.0
b_3	8.0	-20.0	60.0	18.0	22.0	4.0	3.0
b_4	25.0	30.0	35.0	10.0	14.0	5.0	4.0

5. Thresholds

Crit. Thresh.	g_1		g_2		g_3		g_4		g_5		g_6		g_7	
	q_1	p_1	q_2	p_2	q_3	p_3	q_4	p_4	q_5	p_5	q_6	p_6	q_7	p_7
b_1	1.0	2.0	4.0	6.0	1.0	3.0	1.0	2.0	0.0	3.0	0.0	0.0	0.0	0.0
b_2	1.0	2.0	4.0	6.0	1.0	3.0	1.0	2.0	0.0	3.0	0.0	0.0	0.0	0.0
b_3	1.0	2.0	4.0	6.0	1.0	3.0	1.0	2.0	0.0	3.0	0.0	0.0	0.0	0.0
b_4	1.0	2.0	4.0	6.0	1.0	3.0	1.0	2.0	0.0	3.0	0.0	0.0	0.0	0.0

Appendix B: The inference optimisation model

Let us recall the following notations:

- $A^* \subset A$: a set of alternatives that the DMs can assign intuitively to a single category, or to a range of categories, taking into account their evaluation on all the criteria.
- l_i (u_i , respectively): the index of the lowest category (highest category, respectively) to which the alternatives $a_i \in A^*$ can be assigned, according to the opinion of the DMs ($l_i = u_i$ means that the DMs are able to assign the alternatives a_i to a single category).
- T^a : the set of combinations of weights that conform to the constraints implied by the assignment examples. In the mathematical program below, T^a corresponds to constraints (3) and (4).
- T^w : the set of combinations of weights that conform to the additional information (bounds, ranking or more general constraints on weights) provided by the DMs.
- $\lambda^{\min}$ ($\lambda^{\max}$, respectively): the lower bound (upper bound, respectively) for the cutting level λ . This information stated by the DMs is such that $0.5 \leq \lambda^{\min} \leq \lambda^{\max} \leq 1$.
- $T = T^a \cap T^w$: the set of combinations of weights that conform to the constraints implied by the assignment examples and the additional information provided by the DMs, i.e., constraints (3)-(6).
- ε represents an arbitrary small positive value.

$$\max \quad \alpha$$

s. t. :

$$\alpha \leq x_i, \quad \forall i \text{ such that } a_i \in A^* \quad (1)$$

$$\alpha \leq y_i, \quad \forall i \text{ such that } a_i \in A^* \quad (2)$$

$$\sum_{j=1}^n w_j c_j(a_i, b_{l_i-1}) - x_i = \lambda, \quad \forall i \text{ such that } a_i \in A^* \quad (3)$$

$$\sum_{j=1}^n w_j c_j(a_i, b_{u_i}) + y_i = \lambda + \varepsilon, \quad \forall i \text{ such that } a_i \in A^* \quad (4)$$

$$w \in T^w \quad (5)$$

$$\lambda^{\min} \leq \lambda \leq \lambda^{\max}, \text{ with } 0.5 \leq \lambda^{\min} \leq \lambda^{\max} \leq 1 \quad (6)$$

$$0.5 - \varepsilon \geq w_j \geq \varepsilon, \quad j \in F, \sum_{j=1}^n w_j = 1 \quad (7)$$

$$x_i, y_i, \alpha \text{ free}$$

For more details concerning this model, see Mousseau et Slowinski (1998).

Appendix C: The robustness analysis optimisation

When no veto phenomenon occurs, to determine the best possible assignment $B(a_x, T)$ for the alternative a_x considering the domain T of possible values for w and λ , we maximise the credibility index $\sigma(a_x, b_b)$, $b=1, 2, \dots, p$:

$$\begin{aligned} \max \quad & \sum_{j=1}^n w_j c_j(a_x, b_b) - \lambda \\ \text{s. t. : } \quad & w \in T \\ & \sum_{j=1}^n w_j = 1, w_j \geq 0, j \in F \\ & \lambda^{\min} \leq \lambda \leq \lambda^{\max} \end{aligned}$$

If b_b' is the highest profile for which the optimum value of the objective function is positive, then $B(a_x, T) = C_{b_b'}$.

To determine the worst possible assignment for the alternative a_x considering the domain T of possible values for w and λ , we perform the tests by minimising the credibility index $\sigma(a_x, b_b)$, $b=1, 2, \dots, p$:

$$\begin{aligned} \min \quad & \sum_{j=1}^n w_j c_j(a_x, b_b) - \lambda \\ \text{s. t. : } \quad & w \in T \\ & \sum_{j=1}^n w_j = 1, w_j \geq 0, j \in F \\ & \lambda^{\min} \leq \lambda \leq \lambda^{\max} \end{aligned}$$

If b_b'' is the lowest profile for which the optimum value of the objective function is positive, then $W(a_x, T) = C_{b_b''}$.

For details, see Dias and Clímaco (1999, 2000).

Appendix D: Nature of the "range" $[W(a_i, T_k), B(a_i, T_k)]$

Let us consider an ELECTRE TRI model in which $g_j(b_h)$, $q_j(b_h)$ and $p_j(b_h)$, $\forall j, \forall h$, are defined such that $g_j(b_h) + p_j(b_h) \leq g_j(b_{h+1}) + p_j(b_{h+1})$ (the direction of preference on each criterion is increasing), i.e., categories are consistent. Let us suppose that no veto phenomenon occurs. T defines a polyhedron of possible values for w and λ .

Let us denote $T(a, C_h) = \{t \in T / a \rightarrow_t C_h\}$. It should be noticed that:

Result 1: $\exists T, \exists a \in A$ such that $T(a, C_{b+2}) \neq \emptyset$, and $T(a, C_b) \neq \emptyset$ with $T(a, C_{b+1}) = \emptyset$

Proof: a simple counter example can be build considering a situation where $c_j(aSb_h) = c_j(aSb_{h+1})$, $\forall j \in F$. Let us consider the following data set in which the scales on the four criteria are such that $g_j(a) \in [0, 20] \forall j \in F, \forall a \in A$. Four categories are defined such that:

$$g_j(b_1) = 5, g_j(b_2) = 10, g_j(b_3) = 15, \forall j \in F,$$

$$p_j(b_h) = q_j(b_h) = 0, v_j(b_h) > 20, \forall j \in F, \forall j = 1, 2, 3$$

We consider a such that $g_1(a) = g_2(a) = 8, g_3(a) = g_4(a) = 16$ and $T = \{w_j \in [0.05, 0.45]; \lambda \in [0.55, 0.55]\}$. It holds $T(a, C_2) = \{t \in T / w_1 + w_2 \geq \lambda\}$, $T(a, C_4) = \{t \in T / w_3 + w_4 \geq \lambda\}$, and $T(a, C_3) = \emptyset$

In other terms, there exist situations in which an alternative $a \in A$ cannot be assigned to an intermediary category between two categories to which it can be assigned. However, it is possible to characterise the situations in which such phenomenon occurs. Result 2 shows that the situations in which it is not possible to assign an alternative a to an intermediate category C_b between $W(a, T_k)$ and $B(a, T_k)$ corresponds to a case in which a equally outranks b_i and b_{i+1} .

Let us denote $h(a, b_i, t) = \sum_j w_j c_j(a, b_i) - \lambda$, where $t = [w_1, w_2, \dots, w_n, \lambda]$ ($i \in B$). $H_i := \{t : h(a, b_i, t) = 0\}$ ($i \in B$); since h is a linear function, H_i is a hyperplane.

Lemma 1: Consider two profiles b_i and b_{i+1} ($b_i \Delta b_{i+1}$), and the positive open orthant where $w_j > 0$ ($j = 1, \dots, n$). Then, either H_i does not intersect H_{i+1} ($H_i \cap H_{i+1} = \emptyset$), or H_i coincides with H_{i+1} ($H_i = H_{i+1}$).

Proof: obvious since the fact that b_i dominates b_{i+1} implies that $h(a, b_i, t) \leq h(a, b_{i+1}, t)$ for $w_j > 0$ ($j = 1, \dots, n$). If H_i were to intersect H_{i+1} in the open orthant while not coinciding with it, we could find t such that $h(a, b_i, t) > h(a, b_{i+1}, t)$, which is impossible.

Let T be a convex set such that $w_j > 0$ ($j \in F$) for all $t \in T$ containing the admissible combinations of parameter values. $T(a, C_i)$ denotes the set of parameter combinations such that ELECTRE TRI pessimistic rule assigns alternative a to C_i , i.e., $T(a, C_i) = \{t \in T : h(a, b_i, t) < 0 \text{ and } h(a, b_{i+1}, t) \geq 0\}$.

Result 2:

Suppose there exists sup and inf , with $sup > inf + 1$, such that $T(a, C_{sup}) \neq \emptyset$ and $T(a, C_{inf}) \neq \emptyset$. For any i such that $sup > i > inf$, we have the following equivalence:

$$T(a, C_i) = \emptyset \Leftrightarrow c_j(a, b_i) = c_j(a, b_{i+1}), \forall j \in F$$

Proof:

$$T(a, C_i) = \emptyset \Leftrightarrow c_j(a, b_i) = c_j(a, b_{i+1}), \forall j \in F$$

If $c_j(a, b_i) = c_j(a, b_{i+1})$, $\forall j \in F$, then $h(a, b_i, t) = h(a, b_{i+1}, t)$. Hence, $aSb_i \Leftrightarrow aSb_{i+1}$ and a cannot be assigned to C_i , i.e., $T(a, C_i) = \emptyset$.

$$T(a, C_i) = \emptyset \Rightarrow c_j(a, b_i) = c_j(a, b_{i+1}), \forall j \in F$$

Let us choose any $t_{sup} \in T(a, C_{sup})$.

We must then have $h(a, b_{sup-1}, t) \geq 0$, which implies that $h(a, b_i, t) \geq 0$ and $h(a, b_{i+1}, t) \geq 0$.

Let us now choose any $t_{inf} \in T(a, C_{inf})$.

We must then have $h(a, b_{inf}, t) < 0$, which implies that $h(a, b_i, t) < 0$ and $h(a, b_{i+1}, t) < 0$.

Since T is convex, the line segment $L = \{\alpha t_{sup} + (1-\alpha) t_{inf}, 1 \geq \alpha \geq 0\}$ is contained in T . Given that $h(a, b_i, t)$ and $h(a, b_{i-1}, t)$ are continuous functions that change their sign when going from t_{inf} to t_{sup} , there must exist $t_1 \in L$ and $t_2 \in L$ such that $h(a, b_i, t_1) = 0$ and $h(a, b_{i-1}, t_2) = 0$. If $t_1 \neq t_2$, then the combination $0.5t_1 + 0.5t_2$ is such that a would be assigned to C_i , which contradicts the hypothesis. Hence, $T(a, C_i) = \emptyset \Rightarrow t_1 = t_2$. However, since $t_1 \in H_i$ and $t_2 \in H_{i-1}$, this means that H_i intersects H_{i-1} at t_1 (t_2 , resp.). Therefore, by Lemma 1, H_i must coincide with H_{i-1} , which implies $c_j(a, b_i) = c_j(a, b_{i-1}), \forall j \in F$.