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TOWARDS A GENERAL FORMAL FRAMEWORK FOR POLYNOMIAL APPROXIMATION

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Vers un cadre de travail général pour la théorie de l'approximation polynomiale

Résumé

Nous proposons une extension du formalisme de l'approximation polynomiale permettant d'envisager de nouvelles classes de résultats. Nous montrons d'une part comment les résultats existants s'intègrent dans ce cadre et d'autre part quels types de résultats ou questions, jusqu'alors difficiles à exprimer, y trouvent une place naturelle. Nous exploitons ce formalisme pour établir des résultats d'approximation pour différents problèmes **NP**-difficiles. Nous nous intéressons d'abord à la classe des problèmes de sous-graphe induit de poids maximum vérifiant une propriété héréditaire. Elle comprend notamment les problèmes de stable, clique ou encore sous-graphe induit ℓ-colorable de valeur maximum. Nous proposons d'abord une réduction en approximation qui transforme un rapport ρ pour un problème non pondéré de la classe (cas où tous les poids sont égaux) en un rapport de la forme $O(\rho/\log n)$ pour sa version pondérée générale. Cette approche nous permet d'améliorer le rapport d'approximation du problème de stable de poids maximum (max_WIS) : le rapport finalement établi est le minimum entre une expression du type $O(\log^k n/n)$ et $O(\log \mu(G)/(k^2\mu(G)(\log\log\mu(G))^2)$, où $\mu(G)$ désigne le degré moyen du graphe instance et où k est une constante quelconque. Chacun des deux termes correspond à une avancée significative par rapport à l'existant. En effet, le premier terme est à comparer à $O(\log^2 n/n)$, le meilleur rapport fonction de n établi pour le cas particulier non pondéré (max_IS) (problème de stable maximum). Le second terme quant à lui correspond au premier rapport du type $\Omega(1/\Delta(G))$ pour max_WIS ($\Delta(G)$ est le degré maximum du graphe instance). Dans un second temps, en nous appuyant sur des résultats pour la coloration des graphes, nous proposons un algorithme polynomial pour max_WIS garantissant comme rapport le minimum entre $O(n^{-4/5})$ et $O(\log\log\Delta(G)/\Delta(G))$. Cette fois encore, hormis le cas où max_WIS peut être approché avec le rapport $O(n^{-4/5})$ (l'approximation de max_IS garantissant le rapport $n^{\epsilon-1}$ est **NP**-difficile pour tout $\epsilon > 0$) notre algorithme donne accès au premier résultat d'approximation impliquant un rapport $\Omega(\log\log\Delta(G)/\Delta(G))$ pour max_WIS (ceci pour tout $\Delta(G)$). Notons que l'approximation de chacune des versions (pondérée et non pondérée) du problème de stable maximum avec un rapport $\Omega(1/\Delta(G))$ est un problème ouvert connu. Nous nous intéressons alors au problème de clique maximum pour lequel aucun résultat d'approximation impliquant un rapport non-trivial fonction de $\Delta(G)$ n'est jusqu'à présent établi. Nous proposons des algorithmes polynomiaux garantissant respectivement les rapports $O(\log^2\Delta(G)/\Delta(G))$ et $O(\log^2\Delta(G)/\Delta(G)\log\log\Delta(G))$ pour les problèmes de clique maximum et de clique de poids maximum. Nous obtenons le même type de résultats pour le problème de sous-graphe induit ℓ-colorable maximum et sa version pondérée. Sortant de la classe des problèmes de sous-graphe induit maximum vérifiant une propriété héréditaire, nous nous intéressons alors à deux problèmes de minimisation : le problème de somme chromatique minimum et le problème de coloration minimum. Pour le premier, nous établissons une réduction entre sa version pondérée et max_WIS qui préserve (à une constante multiplicative près) le rapport d'approximation. Elle permet le transfert des résultats obtenus pour max_WIS au problème de somme chromatique pondérée. Enfin, pour la coloration minimum, nous proposons des algorithmes d'approximation améliorant les résultats existants. Plus précisément, nous obtenons un rapport de la forme $\min\{O(n^{-\epsilon}), O(\log n/\Delta(G)\log\log n)\}$. Dans le cas où le minimum correspond au terme $O(n^{-\epsilon})$, cette expression domine les meilleurs résultats possibles fonction de n (pour tout $\epsilon > 0$, l'approximation garantissant pour toute instance de la coloration le rapport $n^{-\epsilon}$ est **NP**-difficile). Par contre, si le rapport correspond à la quantité $O(\log n/\Delta(G)\log\log n)$, il améliore les meilleurs résultats actuels fonction de $\Delta(G)$ et correspond au premier rapport de la forme $\Omega(1/\Delta(G))$ pour la coloration minimum.

Mots-clé : algorithme d'approximation, rapport d'approximation, problème **NP**-complet, complexité, réduction, problème d'optimization **NP**.

Towards a general formal framework for polynomial approximation

Abstract

In a first time we draw a rough shape of a general formal framework for polynomial approximation theory which encompasses the existing one by allowing the expression of new types of results. We show how this framework incorporates all the existing approximation results and, moreover, how new types of results can be expressed within it. Next, we use the framework introduced to obtain approximation results for a number of **NP**-hard problems. In this second part of the paper, we first deal with a class of problems called weighted hereditary induced-subgraph maximization problems, notable representatives of which are maximum independent set, maximum clique, maximum ℓ-colorable induced subgraph, etc. We devise a polynomial approximation-preserving reduction transforming any approximation ratio ρ for any unweighted problem of the class, into approximation ratio $O(\rho/\log n)$ for its weighted version. This allows us to perform subsequent improvements of the approximation ratio for the weighted independent set problem (max_WIS), in order to finally obtain as ratio the minimum between $O(\log^k n/n)$ and $O(\log\mu(G)/(k^2\mu(G)(\log\log\mu(G))^2)$, where $\mu(G)$ denotes the average graph-degree of the input-graph, for every constant k. In any of the two cases, this is an important improvement since in the former one, the ratio for max_WIS outer-performs $O(\log^2 n/n)$, the best-known ratio for the (unweighted) independent set problem (max_IS), while in the latter case, we obtain the first $\Omega(1/\Delta(G))$ ratio for max_WIS (where $\Delta(G)$ is the maximum graph-degree). Next, based upon graph-coloring, we devise a polynomial time approximation algorithm for max_WIS achieving ratio the minimum between $O(n^{-4/5})$ and $O(\log\log\Delta(G)/\Delta(G))$. Here also, except for the very unlikely case where max_WIS can be approximated within $O(n^{-4/5})$ (approximation of max_IS within $n^{\epsilon-1}$ is hard for every $\epsilon > 0$), our algorithm is the first $\Omega(\log\log\Delta(G)/\Delta(G))$-approximation algorithm for max_WIS (for every $\Delta(G)$). Let us note that approximation of both independent set versions within ratios $\Omega(1/\Delta(G))$ is a very well-known open problem. Then we deal with maximum clique problem for which no non-trivial approximation ratios, functions of $\Delta(G)$ are presently known. We propose here algorithms achieving ratios $O(\log^2\Delta(G)/\Delta(G))$ and $O(\log^2\Delta(G)/\Delta(G)\log\log\Delta(G))$ for maximum-size and maximum-weight clique, respectively. We do the same for the unweighted and weighted versions of maximum ℓ-colorable induced subgraph. We then leave the class of hereditary induced-subgraph maximization problems and deal with two minimization problems, the minimum chromatic sum and the minimum coloring. For the former, we show the existence of a polynomial reduction between its weighted version and for max_WIS preserving (up to multiplicative constant) the approximation ratios for both problems. This reduction allows transfer of the results obtained for max_WIS to minimum-weight chromatic sum. Let us note that this result also is entirely new and non-trivially obtained. Finally, for minimum coloring, we produce approximation ratios improving all the known ones. More precisely, we obtain a ratio of $\min\{O(n^{-\epsilon}), O(\log n/\Delta(G)\log\log n)\}$. In the case where the minimum is realized by the quantity $O(n^{-\epsilon})$, it outer-performs all the ratios, functions of n, while if the minimum is realized by the quantity $O(\log n/\Delta(G)\log\log n)$, it outer-performs the best known ratio, function of $\Delta(G)$ and constitutes the first $\Omega(1/\Delta(G))$ ratio for minimum coloring.

Keywords: approximation algorithm, approximation ratio, **NP**-complete problem, complexity, reduction, **NP** optimization problem.

Part I

Introduction

1 A few words about polynomial approximation

An NP optimization (NPO) problem Π is commonly defined (see, for example, [8]) as a four-tuple $(\mathcal{I}, \mathrm{S}, v_I, \mathrm{opt})$ such that:

- $\mathcal{I}$ is the set of instances of Π and it can be recognized in polynomial time;
- given $I \in \mathcal{I}$, $\mathrm{S}(I)$ denotes the set of feasible solutions of I; for every $S \in \mathrm{S}(I)$, $|S|$ is polynomial in $|I|$: given any I and any S polynomial in $|I|$, one can decide in polynomial time if $S \in \mathrm{S}(I)$;
- given $I \in \mathcal{I}$ and $S \in \mathrm{S}(I)$, $v_I(S)$ denotes the value of S; v_I is polynomially computable and is commonly called objective function;
- $\mathrm{opt} \in \{\max, \min\}$.

The class of NP optimization problems is commonly denoted by **NPO**. The most interesting sub-class of **NPO** is the class of the **NP**-hard optimization problems, known to be unsolvable in polynomial time unless **P**=**NP** and people widely thinks that this fact is very unlikely. This probable intractability of **NP**-hard problems motivates both researchers and practitioners in trying to *approximately solve such problems*, i.e., *in trying to find in polynomial time, not the best solution but one solution which, in some sense, is near the optimal one.*

Given an instance I of a combinatorial optimization problem Π, $\omega(I)$, $\lambda(I)$ and $\beta(I)$ will denote the values of the worst solution of I (in the sense of the objective function), the approximated one (provided by a polynomial time approximation algorithm (PTAA) $\mathtt{A}$ supposed to feasibly solve problem Π), and the optimal one, respectively. There exist mainly two thought processes dealing with polynomial approximation.

1.1 Standard approximation

Here, the quality of an approximation algorithm $\mathtt{A}$ is expressed by the ratio

$$\sigma_{\mathtt{A}}(I) = \min\left\{\frac{\lambda(I)}{\beta(I)}, \frac{\beta(I)}{\lambda(I)}\right\}$$

and the quantity $\sigma_{\mathtt{A}} = \sup\{r : \sigma_{\mathtt{A}}(I) > r, I \in \mathcal{I}\}$ is the *constant approximation ratio* of $\mathtt{A}$ for Π. The ratio induced by the standard approximation will be denoted by σ.

Starting from the basic notion of approximation ratio, one can define the one of *constant asymptotic approximation ratio* expressing the performance of a PTAA when working on "limit-instances". This ratio is defined ([23]) as

$$\sup\{r : \exists M, \sigma_{\mathtt{A}}(I) \geqslant r, I \in \mathcal{I}_M\}$$

where $\mathcal{I}_M = \{I \in \mathcal{I} : \beta(I) \geqslant M\}$ ([23]).

A particularly interesting case of PTAA (representing an "ideal" approximation behavior), is the one of the *polynomial time approximation schema (PTAS)*. A PTAS is a sequence $\mathtt{A}_\epsilon$ of PTAAs guaranteeing, for every $\epsilon > 0$, approximation ratio $\sigma_{\mathtt{A}_\epsilon} = 1 - \epsilon$ with complexity $O(n^k)$, where k is a constant not depending on n but eventually depending on $1/\epsilon$. A further refinement of PTAS is the one of *fully PTAS (FPTAS)*, i.e., of a PTAS of complexity $O((1/\epsilon)^{k'}O(n^k))$, where k and k' are constants not depending neither on n, nor on ϵ.

1.2 Differential approximation

Here. the quality of an approximation algorithm A is expressed by the ratio

$$\delta_{\mathtt{A}}(I) = \frac{|\omega(I) - \lambda(I)|}{|\omega(I) - \beta(I)|}.$$

The quantity $\delta_{\mathtt{A}} = \inf\{r : \delta_{\mathtt{A}}(I) > r, I \in \mathcal{I}\}$ is the *constant differential-approximation ratio* of A for Π. The ratio adopted by the differential approximation will be denoted by δ. The *constant asymptotic differential-approximation ratio* is defined as ([19])

$$\lim_{k\to\infty} \inf_{\substack{I \\ |S(I)|\geqslant k}} \left\{\frac{\omega(I) - \lambda(I)}{\omega(I) - \beta(I)}\right\}.$$

The differential PTAS and FPTAS are defined analogously to the standard ones. When $\sigma_{\mathtt{A}}$, or $\delta_{\mathtt{A}}$ are equal to 0, we consider instance-depending ratios. i.e., ratios expressed in terms of instance parameters.

In what follows, when we indifferently are referred to either the former or the latter approximation, or when approximation ratios in both theories coincide, we will use ρ instead of σ or δ. Moreover, ***S*-APX** (resp., ***D*-APX**) will denote the class of problems for which the best known standard (resp., differential) PTAAs achieve fixed constant approximation ratios. Analogously, ***S*-** (resp., ***D*-**) **PTAS** or **FPTAS** will denote the classes of problems admitting standard (resp., differential) PTASs or FPTASs, etc. We will also denote by $\sigma_\Pi(I)$ (resp., $\delta_\Pi(I)$), the best standard-approximation (resp., differential approximation) ratio known for Π. When ratios depend on parameters of I, it will be assumed that $\rho(I) \leqslant \rho(I')$ for any sub-instance I' of I.

2 A list of NPO problems

In this paper we speak, in more or less details, about a number of combinatorial problems. Although all these problems are very well-known in complexity theory, in order that the paper is self-contained, we define them in what follows.

2.1 Hereditary induced-subgraph maximization problems

Given a graph $G = (V, E)$ and a set $V' \subseteq V$, the subgraph of G induced by V' is a graph $G' = (V', E')$, where $E' = (V' \times V') \cap E$. Let $\mathcal{G}$ be the class of all the graphs. A graph-property π is a mapping from $\mathcal{G}$ to $\{0, 1\}$, i.e., for a $G \in \mathcal{G}$, $\pi(G) = 1$ iff G satisfies π and $\pi(G) = 0$, otherwise. Property π is hereditary if whenever it is satisfied by a graph it is also satisfied by every one of its induced subgraphs; it is non-trivial if it is true for infinitely many graphs and false for infinitely many ones ([8]). We consider **NP**-hard graph-problems Π where the objective is to find a maximum-order induced subgraph G' satisfying a non-trivial hereditary property π. These problems are called *hereditary induced-subgraph maximization problems.* For a graph G. anyone of its vertex-subsets specifies exactly one induced subgraph. Consequently, in what follows we consider that a feasible solution for Π is the vertex-set of G'. Hereditary induced-subgraph maximization problems have a natural generalization to graphs with positive integral weights associated with their vertices (the weights are assumed to be bounded by 2^n, where n is the order of the input-graph, so that every arithmetical operation on them can be performed in polynomial time). Given a graph G, the objective of a weighted induced subgraph problem is to determine an induced subgraph G^* of G such that G^* satisfies π and, moreover, the sum of the weights of the vertices of G^* is the largest possible among those subgraphs.

Maximum independent set (max_IS).

Given a graph $G = (V, E)$, an *independent set* is a subset $V' \subseteq V$ such that whenever $\{v_i, v_j\} \subseteq V'$, $v_i v_j \notin E$, and the *maximum independent set problem* is to find an independent set of maximum size. In the weighted version of max_IS, denoted by max_WIS, positive weights are associated with the vertices of the input graph and the objective becomes to determine an independent set for which the sum of the weights of its vertices is the largest possible.

Maximum clique (max_KL).

Consider a graph $G = (V, E)$. A *clique* of G is a subset $V' \subseteq V$ such that every pair of vertices of V' are linked by an edge in E, and the *maximum clique problem* (max_KL) is to find a maximum size set V' inducing a clique in G (a maximum-size clique). In the weighted version of max_KL, denoted by max_WKL, positive weights are associated with the vertices of the input graph and the objective becomes to determine a clique for which the sum of the weights of its vertices is the largest possible.

Maximum ℓ-colorable induced subgraph (max_Cℓ).

Given a graph $G = (V, E)$ and a constant $\ell < \Delta(G)$ (the maximum graph-degree), max_Cℓ consists in finding a maximum-order subgraph G' of G such that G' is ℓ-colorable.

Property "is an independent set" is hereditary (a subset of an independent set is an independent set). The same holds for property "is a clique" (a vertex-subset of a clique induces also a clique), as well as for property "is ℓ-colorable" (if the vertices of a graph G can be feasibly colored by at most ℓ colors, then every subgraph of G induced by a subset of its vertices can be colored by at most ℓ colors).

2.2 Some other NPO problems

Minimum vertex-covering (min_VC).

Given a graph $G = (V, E)$, a *vertex cover* is a subset $V' \subseteq V$ such that, $\forall uv \in E$, either $u \in V'$, or $v \in V'$, and the *minimum vertex-covering problem* is to determine a minimum-size vertex cover.

Minimum set-covering (min_SC).

Given a collection $\mathcal{S}$ of subsets of a finite set C, a *set cover* is a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ such that $\cup_{S_i \in \mathcal{S}'} S_i = C$, and the *minimum set-covering problem* consists in finding a cover of minimum size. We denote by min_3-SC the restriction of min_SC on 3-element sets.

Minimum coloring (min_C).

Consider a graph $G = (V, E)$ of order n. We wish to color V with as few colors as possible so that no two adjacent vertices receive the same color. The *chromatic number* of a graph is the smallest number of colors which can feasibly color its vertices. A graph G is called *k-colorable* if its vertices can be legally colored by k colors, in other words if its chromatic number is at most k; it will be called *k-chromatic* if k is its chromatic number.

Minimum chromatic sum (min_CHS).

Given a graph $G = (V, E)$, an l-coloring is a partition of V into independent sets $C_1, \ldots, C_l$. The cost of an l-coloring is the quantity $\sum_{i=1}^{l} i|C_i|$ (in other words, the cost of coloring a vertex $v \in V$ with color i is i). The *minimum chromatic sum* problem, denoted by CHS is to determine a minimum-cost coloring. For the weighted version of CHS, denoted by WCHS, every vertex $v \in V$ is weighted by a rational weight w_v, the cost of coloring v with color i

becomes iw_v, the value of an l-coloring becomes $\sum_{i=1}^{l} iw(C_i)$, where $w(C_i) = \sum_{v\in C_i} w_v$, and the objective becomes now to determine a coloring of minimum value.

Bin packing (BP).
We are given a finite set $L = \{x_1, \dots, x_n\}$ of n rational numbers and an unbounded number of bins, each bin having capacity 1. We wish to arrange all these numbers in the least possible bins in such a way that the sum of the numbers in each bin does not violate its capacity.

Minimum and maximum traveling salesman problem (min_TSP and max_TSP).
Given a complete graph on n vertices, denoted by K_n, with positive distances on its edges, min_TSP (resp., max_TSP) consists in minimizing (resp., maximizing) the cost of a Hamiltonian cycle[1], the cost of such a cycle being the sum of the distances of its edges. An interesting sub-case of TSP is the one in which edge-distances are only 1 or 2 (TSP12).

3 Notations

Given a problem Π defined on a graph $G = (V, E)$, and its weighted version WΠ, we denote by $\vec{w} \in \mathbb{N}^{|V|}$ the vector of the weights, by w_v the weight of $v \in V$, and by $w_{\max}(G)$ and $w_{\min}(G)$ the largest and the smallest vertex-weights, respectively. Moreover, we adopt the following notations:

n:	the order of G, i.e., $n = \|V\|$;
$\Gamma(v)$:	the neighborhood of $v \in V$;
$\delta_S(v)$:	the quantity $\|\Gamma(v) \cap S\|$, $S \subset V$, $v \in V \setminus S$;
$w(V')$:	the total weight of $V' \subseteq V$, i.e., the quantity $\sum_{v\in V'} w_v$;
$\beta_w(G)$:	the value of an optimal solution for WΠ;
$\beta'_w(G)$:	the weight of an approximated WΠ-solution of G
$\vec{d}$:	the vector $(d_1, \dots, d_n)$, where d_i denotes the degree of vertex $v_i \in V$;
$\Delta(G)$:	the maximum degree of G, i.e., $\Delta(G) = \max_i\{d_i\}$;
$\mu(G)$:	the average degree of G, i.e., $\mu(G) = (\sum_i d_i)/n$;
$\mu_w(G)$:	the quantity $\sum_{v\in V} w(\Gamma(v))/w(V)$
$\chi(G)$:	the chromatic number of G (the minimum number of colors with which one can feasibly color the vertices of G);
$\bar{G}$:	the complement of G defined by $\bar{G} = (V, \bar{E})$ with $\bar{E} = \{ij \in V \times V, i \neq j, ij \notin E\}$ (obviously, $\bar{\bar{G}} = G$);
$G[V']$:	the subgraph of G induced by $V' \subseteq V$;
n':	the order of the graph $G[V']$, $V' \subseteq V$, i.e., $\|V'\| = n'$;
$S^*(G[V'])$:	an optimal (maximum-size) Π-solution in $G[V']$, $V' \subseteq V$.

[1] An ordering $(v_1, v_2, \dots, v_n)$ of the vertices of K_n such that $v_n v_1 \in E(K_n)$ and, for $1 \leqslant i < n$, $v_i v_{i+1} \in E(K_n)$.

Especially for IS and WIS, using standard notations, we will denote the size of a maximum independent set by $\alpha(G)$, the value of a maximum-weight independent set by $\alpha_w(G)$ and the value of an approximated max_WIS-solution by $\alpha'_w(G)$. Moreover, when no ambiguity can occur, we will use Δ, μ, μ_w, $w_{\max}$ and $w_{\min}$ instead of $\Delta(G)$, $\mu(G)$, $\mu_w(G)$, $w_{\max}(G)$ and $w_{\min}(G)$.

Given a square matrix $B = (m_{ij})_{i,j=1,\dots,n}$, we denote by $\mathrm{Tr}(B)$ and ${}^tB$ the trace and the transpose of B, respectively. Finally, given a vector $\vec{u}$, we denote by $|\vec{u}|$ its Euclidean norm.

4 The scope of the paper and its main contributions

In what follows, we first generalize the notion of the approximation algorithm by introducing the one of the *approximation chain*. This generalization allows us, for example, to express in proper terms algorithms, of time-complexity $O(f(|I|)^k)$ (where f is a polynomial of $|I|$), achieving ratios of the form $\varphi(I) - \varphi'(k) - o(\varphi(I))$, where $\varphi(I)$ denotes an approximation ratio depending on parameters of I and φ' is an integer decreasing function of $k \in I\!N$. This kind of doubly asymptotic approximation ratio means that one can pre-fix a value for k and next the smaller k's value the closer to $\varphi(I) - o(\varphi(I))$ the ratio (and, of course, the heavier the algorithm's complexity). This recalls the notion of the polynomial time approximation schema (without being a such one). On the other hand, if the term $\varphi'(k)$ missed, then $\varphi(I) - o(\varphi(I))$ would be very close to the classical notion of asymptotic approximation ratio.

Next, we propose a natural way for classifying **NPO** problems following their approximability behavior, by introducing the notion of the *approximation level*; it informally associates an approximation result with a family of approximation ratios. Premises of such a notion can be found in the definition of the class **F-APX** in [35]. However, the authors of [35] restrain themselves to problem-classifications with respect to orders of possible approximation ratios. Such a classification does not allow distinction between ratios of the same order, or between ratios whose values are polynomially related. For example, based upon [35], one cannot distinguish approximation classes induced by ratios $O(|I|^k)$, where k is a fixed constant, for different values of k. However, numerous recent approximation results are based upon such distinctions, in particular when dealing with inapproximability results bringing to the fore hardness factors, i.e., values of k for which approximation within better than $|I|^k$ is hard. For example, for max_IS, approximation ratio $1/n$ is guaranteed by any approximation algorithm, while no PTAA can guarantee approximation ratio $(1/n)^{1-\epsilon}$, for any $\epsilon > 0$, where n is the order of the input-graph, unless **NP=ZPP** ([31]). Another weak point of [35] is that their problem-classification is defined with respect to $|I|$. Certainly, this parameter is among the most interesting ones for expressing non-constant approximation ratios, but is not the only one. For many **NPO** problems — max_IS, min_C, min_SC, etc., are some notable examples — approximation results are obtained as functions of different instance-parameters. For max_IS, for example, there exist two popular types of positive approximation results, the ones expressed in terms of n and the ones expressed in terms of graph-degree (maximum or average); moreover no links are known until now between these two types. The latter family of results (ratios, functions of degree) cannot be expressed in terms of the class **F-APX**. But even if one tried to mechanically rewrite definitions of [35] in order to take into account ratios functions of graph-degrees, then, in the case of max_IS, one would have to face another obstacle. The best known ratios exclusively in terms of $\Delta(G)$ for max_IS are: $\kappa/\Delta(G)$, for every fixed constant $\kappa \in I\!N$, for unweighted max_IS ([18]) and $3/(\Delta(G) + 2)$ for the weighted case. Following the classification induced by [35], max_IS and max_WIS are of the same hardness, $O(1/\Delta(G))$, with respect to their approximability. This mathematically is not very fair since the improvement of the term in the numerator by just one unit uses very complicated combinatorial arguments and mathematical techniques. On the other hand, the best approximation ratios - exclusively functions of n - are $O(\log^2 n/n)$ ([7])

for max_IS and $O(\log^2 n/(n \log\log n))$ for max_WIS (see theorem 3 of section 10.1, page 15). Consequently, if one chooses n to express ratios, unweighted and weighted max_IS are not (at the current state of knowledge) of the same hardness regarding their approximability. A second example justifying the real need of using classes of approximation ratios larger than these of constant ratios or of ratios depending on $|I|$ can be taken from [12]. It is proved there that for the minimum-weight maximal independent set, no SPTAA can achieve approximation ratio that does not depend on the vertex-weights. In other words, such a problem (in some sense among the hardest ones regarding its approximation) cannot be included into the classification of [35]. The same holds for min_TSP which cannot be approximated within better than $2^{P(n)}$ where P is a polynomial of n (this follows by a simple remark from the result of [44]).

Next, we introduce a kind of approximation-preserving reduction, called FP-reduction, and show that many of the reductions known (for example the ones of [41, 46]) can be seen as special cases of FP-reductions. An interesting feature of FP-reduction is that it can link, for a given problem, its approximation behaviors in standard and differential approximations.

In part III, we show that many known approximation results can be expressed very naturally in the framework proposed in part II and also that this framework is very suitable for the achievement of new ones.

More precisely, in section 9, we devise an FP-reduction from weighted hereditary induced-subgraph problems to unweighted ones transforming any approximation ratio ρ for the latters into approximation ratio $O(\rho/\log n)$ for the formers.

Such a ratio can be improved when dealing with particular problems. For example, it becomes $O(\rho/\log\log n)$ when dealing with the pair max_IS–max_WIS (section 10.1). Based upon this reduction we draw first improvements for the ratio of max_WIS. In section 10.3, we propose new improved approximation results for max_IS. Next, always based upon the reduction of section 9, we further improve approximations for max_WIS and obtain an approximation ratio for max_WIS with value greater than the minimum between $O(\log^k n/n)$ and $O(\log\mu(G)/(\mu(G)\log^2\log\mu(G))$. This is an important improvement since if the max_WIS-ratio obtained is $O(\log^k n/n)$, it outer-performs $O(\log^2 n/n)$, the best-known ratio for max_IS. On the other hand, if the ratio obtained is $O(\log\mu(G)/(\mu(G)\log^2\log\mu(G)))$, then we achieve the first $\Omega(1/\mu(G))$ ratio for max_WIS. Let us note that this is the first time that non-trivial results for max_WIS are produced by a reduction to max_IS. In section 10.4, we propose another PTAA for max_WIS which further improves (although increasing the time-complexity) the results of section 10.3. We generalize a result of [3], by linking, devising another FP-reduction, the approximation of the class max_WIS$_k$ of max_WIS-instances with weighted independence number greater than $w(G)/k$ to the approximation of a class $\mathcal{G}_\ell$ of graph-coloring instances including the ℓ-colorable graphs. Combining this result with recent works of [34, 38] about $\mathcal{G}_\ell$, we obtain an approximation ratio for max_WIS of value greater than the minimum between $O(n^{-4/5})$ and $O(\log\log\Delta(G)/\Delta(G))$. Consequently, except for the very unlikely case where max_WIS can be approximated within $O(n^{-4/5})$ (recall that approximation of IS within $n^{\epsilon-1}$ is hard for every $\epsilon > 0$, unless **NP=ZPP** ([31])), our algorithm is the first PTAA achieving ratio $\Omega(\log\log\Delta(G)/\Delta(G))$ for max_WIS.

In section 11, we devise a new FP-reduction between max_KL and max_WKL. Based upon this reduction and using our results on max_WIS, we deduce the first $\Omega(1/\Delta(G))$ approximation ratio for the maximum-size and maximum-weight clique problems. We note that the results of this section work even for unbounded values of $\Delta(G)$.

In section 12, using another FP-reduction from max_WIS, we obtain improvements of recent approximation results for min_WCℓ, while in section 13 we devise an FP-reduction guaranteeing that min_WCHS is equi-approximable (up to a multiplicative constant) with max_WIS.

In section 14, we deal with min_C and prove that it can be approximately solved within standard-approximation ratio $\min\{O(n^{-\epsilon}), O(\log n/(\Delta(G)\log\log n))\}$. In the unlikely case where the minimum is realized by the quantity $O(n^{-\epsilon})$ (in [21] it is proved that for any positive ϵ, min_C cannot be solved within standard-approximation ratio $n^{\epsilon-1}$, unless NP=ZPP), it outer-performs all the ratios, functions of n, while if the minimum is realized by the quantity $O(\log n/(\Delta(G)\log\log n))$, it outer-performs the best known ratio, function of $\Delta(G)$, and constitutes the first $\Omega(1/\Delta(G))$ ratio for minimum coloring.

Finally, in section 15 we study FP-reductions between standard and differential approximations. We first give sufficient conditions for transferring results between the two thought processes. Next, we mention results dealing with such FP-reductions for a number of **NPO** problems.

Part II
Roughing out a new approximation framework

5 Approximation chains: a generalization of the approximation algorithms

We first define what in the sequel we will call an *approximation chain* which is a generalization of the notion of approximation algorithm.

Definition 1. Approximation chain.
Consider an **NPO** problem $\Pi = (\mathcal{I}, \mathrm{S}, v_I, \mathrm{opt})$ and let $\rho : \mathcal{I} \times I\!N \rightarrow [0,1]$, $(I,k) \mapsto \rho(I,k)$, be a mapping increasing in k. An *approximation chain* with ratio ρ for Π is a sequence of algorithms $(\mathtt{A}_k)_{k\in\mathbb{N}}$, indexed by $k \in I\!N$ (by $\mathtt{A}_k$ we denote the kth algorithm of the chain), such that, for all k, $\mathtt{A}_k$ is an approximation algorithm guaranteeing ratio at least $\rho(I,k)$. ▮

Let $T((\mathtt{A}_k)_{k\in\mathbb{N}})$ be the time-complexity of $(\mathtt{A}_k)_{k\in\mathbb{N}}$. If $T((\mathtt{A}_k)_{k\in\mathbb{N}})$ is polynomial in n (the size of I) but exponential in k, then $(\mathtt{A}_k)_{k\in\mathbb{N}}$ is called polynomial time approximation chain (PTAC), while if $T((\mathtt{A}_k)_{k\in\mathbb{N}})$ is polynomial in both n and k, then $(\mathtt{A}_k)_{k\in\mathbb{N}}$ is called fully polynomial time approximation chain (FPTAC). Such chains will be denoted by SPTAC and SFPTAC (resp., DPTAC and DFPTAC) when dealing with standard (resp., differential) approximation. When ratios in two thought processes coincide, we will simply use terms PTAC and FPTAC.

The use of approximation chains suggests to associate approximability results not only with ratios but rather with sequences of ratios. This allows us, for example, to associate an approximation level with the existence of a PTAS which is impossible by means of **F-APX**. In general, the use of the classification of [35] does not allow us to distinguish, even within ***S*-APX**, problems approximated within constant ratios (for constants strictly smaller than 1) from problems approximated within ratio $1-\epsilon$, for every $\epsilon > 0$.

6 Approximation level

The following definition 2 generalizes the notion of classes **F-APX**.

Definition 2. Approximation level.
Consider an **NPO** problem Π, let $\mathcal{P}_\Pi = \{\rho : \mathcal{I} \times I\!N \rightarrow [0,1], (I,k) \mapsto \rho(I,k)\}$ and let $P \in 2^{\mathcal{P}_\Pi}$. We say that P is *min-invariant* if $\forall(\rho,\rho') \in P \times P$, $\min\{\rho,\rho'\} \in P$. An approximation level is a set $P \in 2^{\mathcal{P}_\Pi}$ min-invariant. A problem Π is approximable on level P if it can be solved by a PTAC achieving ratio in P. ▮

For example:

- if we denote by $\tilde{P}_\Pi$ the approximation level $\{\rho : \mathcal{I} \to [0,1], I \mapsto \rho(I)\}$, i.e., the set of ratios independent on k, then a PTAC with ratio in $\tilde{P}_\Pi$ corresponds to the classical notion of PTAA for Π;
- consider level $\hat{P}_\Pi = \{\rho : (I,k) \to [0,1], (I,k) \mapsto \rho(k)\}$, i.e., the class of approximation ratios independent on I: then, by simple functional analysis arguments, $\exists \eta \in [0,1]$, $\rho(I,k) \to \eta$ when $k \to \infty$; if $\eta = 1$, then our PTAC $(\mathtt{A}_k)_{k\in\mathbb{N}}$ is nothing else than a PTAS for Π;
- if we consider level $P^*_\Pi = \tilde{P}_\Pi \cap \hat{P}_\Pi$, then a PTAC with ratio in P^*_Π is exactly the very well-known constant-ratio PTAA; in other words, $\Pi \in$ **APX**.

Let us note that one can associate with an approximation level (a set of ratios) P the set of problems approximately solved by algorithms guaranteeing ratios in P. For example, P^*_Π (the set of constant — independent on both I and k — ratios) can be seen as the class **APX**. In the sequel, when no confusion arises, we will indifferently use P to denote either an approximation level, or the set of problems with ratios in P. For example, following this convention, **APX** means either the class of problems, or the class of fixed constant ratios admitted by these problems. The same holds also for **PTAS**.

7 Convergence and hardness threshold

We now introduce and discuss the concept of the *convergence* of the ratio of a PTAC, which naturally follows the notion of PTAC.

Definition 3. Convergence (limit with respect to k).
Given a problem Π and $\tilde{\rho} \in \tilde{P}_\Pi$, the approximation ratio of $(\mathtt{A}_k)_{k\in\mathbb{N}}$ converges to $\tilde{\rho}$ if,

$$\forall \epsilon > 0, \exists \kappa, \forall k \geqslant \kappa, \forall I \in \mathcal{I}, \quad \rho(I,k) \geqslant \tilde{\rho}(I)(1-\epsilon). \quad \blacksquare$$

For instance, a PTAS is a PTAC, the ratio of which converges to 1. From the above definition, one can easily see that $\rho(I,k)$, seen as I-depending function-sequence, is uniformly equivalent to $\tilde{\rho}(I)$ when $k \to \infty$. This remark implies the existence of another weaker convergence referring to the classical notion of weak equivalence of function-sequences. Under the same notations as in definition 3, a PTAC $(\mathtt{A}_k)_{k\in\mathbb{N}}$ admits approximation ratio *weakly* converging to $\tilde{\rho}$ if,

$$\forall \epsilon > 0, \forall I \in \mathcal{I}, \exists \kappa, \forall k \geqslant \kappa, \rho(I,k) \geqslant \tilde{\rho}(I)(1-\epsilon).$$

For instance, for a problem Π, an approximation ratio of the form $p(|I|)^{-1/k}$, where p is a polynomial of $|I|$, is an example of weak convergence to 1.

Let us now introduce the second type of ratio-convergence, the convergence with respect to I. This generalizes what in polynomial approximation is commonly called "asymptotic approximation ratio". In general, in either standard or differential approximation, in order to define the asymptotic ratio, a set of instances, called also "interesting" instances, is used. Many criteria are used to express these instances as, for example, the optimal value (see section 1.2), the size of the instance or, for graph-problems, the maximum degree, or even (especially in differential approximation) the number of feasible values of an instance ([19]). Using asymptotic ratio consists of dropping the non-interesting instances out and of studying the approximation behavior of an algorithm only on the interesting ones. Interesting or not-interesting instances in which sense? A very popular general criterion for delimiting interesting from non-interesting instances of a problem is their computational hardness. Given an **NP**-complete problem Π, a sub-class $\mathcal{C}$ of $\mathcal{I}$ is considered as *interesting*, or *hard*, if Π is better approximable (or even optimally solved in polynomial time) in $\mathcal{I} \setminus \mathcal{C}$ than in $\mathcal{C}$. Moreover, it is natural that one is interested in how an algorithm behaves on large-size instances, i.e., on instances the size of which tends to ∞. The notion of instance-hardness together with the one of instance tending to ∞ have produced several more specific criteria about which instances are considered as interesting or not. For example, in [4] a notion of problem's simplicity, called AAP-simplicity in the sequel, is defined as follows: a problem Π is *AAP-simple* if its restriction to the set of instances

$$\mathcal{J}_M = \{I \in \mathcal{I} : |\omega(I) - \beta(I)| \leqslant M, M \text{ any fixed constant}\}$$

is in **P**. Remark that many **NP**-complete problems, for example max_IS, min_SC, max_$C\ell$, min_C, BP, etc., are AAP-simple. On the other hand, in [43] another notion of simplicity, called *PM-simplicity* in what follows is defined: a problem Π is *PM-simple* if its restriction to the set of instances

$$\mathcal{K}_M = \{I \in \mathcal{I} : \beta(I) \leqslant M, M \text{ any fixed constant}\}$$

is in **P**. Under this definition of simplicity, max_IS, min_SC, or, finally max_$C\ell$ are PM-simple, while min_C. or BP are not. Both notions delimit the interesting instances from the non-interesting ones in a fairly intuitive way. For example for an AAP-simple problem, interesting instances are the ones in $\mathcal{I} \setminus \mathcal{J}_M$, while for a PM-simple problem the interesting instances are the ones in $\mathcal{I} \setminus \mathcal{K}_M$. Another instance-interest criterion is the one defined in [19] relying upon the notion of the *radial* problem. Informally, a problem Π (with integer objective values) is radial if, given an instance I of Π and a feasible solution $S \in \mathrm{S}(I)$, one can, in polynomial time, on the one hand deteriorate S as much as one wants (up to finally obtain a worst-value solution) and, on the other hand, one can greedily improve S in order to obtain (always in polynomial time) a sub-optimal solution (eventually the optimal one). This definition generates another boundary between interesting and non-interesting instances, since, as it is proved in [19], denoting by $|v(\mathrm{S}(I))|$ the number of the feasible-solution values of I and setting

$$\mathcal{L}_M = \{I \in \mathcal{I} : |v(\mathrm{S}(I))| \leqslant M, M \text{ any fixed constant}\}$$

then the restriction of a radial problem (with integer objective values) to instances in $\mathcal{J}_M$, or in $\mathcal{L}_M$ is in **P**. Here, in order to unify the different criteria upon which the several definitions of interesting instances are based, we propose the following definition of what we call *hardness threshold.*

Definition 4. Hardness threshold.
Consider $\Pi = (\mathcal{I}, \mathrm{S}, v_I, \mathrm{opt})$, and an approximation level $\bar{P} \subseteq \mathcal{P}_\Pi$ such that, under a complexity theory hypothesis (for example **P**$\neq$**NP**) Π is not approximable within $\bar{P}$. Then, $h : \mathcal{I} \rightarrow \mathbb{N}$ is a *hardness threshold with respect to* $\bar{P}$ if, $\forall M \in \mathbb{N}$, the restriction of Π to the instance-set

$$\{I \in \mathcal{I}, h(I) \leqslant M\}$$

admits a PTAC with ratio in $\bar{P}$. ∎

For example, under the hypothesis **P**$\neq$**NP**:

- $h(I) = n$ is a hardness threshold for *every* **NP**-hard problem, with respect to the approximation level $\bar{P} = \{1\}$ (the exact solution);
- $h(I) = \log(\max(I))$, where $\max(I)$ is the largest number of the instance, is a hardness threshold for the *weakly* **NP**-hard problems ([23]) with respect to $\bar{P} = \{1\}$;
- $h(I) = \beta(I)$ is a hardness threshold for the PM-simple **NP**-hard problems with respect to $\bar{P} = \{1\}$;

- $h(I) = \omega(I) - \beta(I)$ is a hardness threshold for the AAP-simple **NP**-hard problems with respect to $\bar{P} = \{1\}$;
- $h(I) = |v(\mathrm{S}(I))|$ and $h(I) = \omega(I) - \beta(I)$ are hardness thresholds for the radial **NP**-hard problems (with integer objective values) with respect to $\bar{P} = \{1\}$;
- $h(I) = \Delta(G)$ is a hardness threshold for max_IS with respect to $\bar{P} = \mathbf{APX}$;
- $h(I) = |\{x_i : x_i \leqslant 1/3\}|$ is a hardness threshold for BP with respect to $\bar{P} = \{1\}$ ([33]).

Definition 5. Asymptotic approximation ratio (limit with respect to I).
Given a problem Π and a hardness threshold h with respect to an approximation level $\bar{P}$, chain $(\mathtt{A}_k)_{k\in\mathbb{N}}$ has asymptotic ratio $\rho' \in \mathcal{P}_\Pi$ if

$$\forall \epsilon > 0, \forall k \in \mathbb{N}, \exists H, \forall I \in \mathcal{I}, h(I) \geqslant H, \quad \rho(I,k) \geqslant \rho'(I,k)(1-\epsilon). \quad \blacksquare$$

Finally, let us remark that one can combine definitions 3 and 5 to obtain the following definition of asymptotic convergence.

Definition 6. Asymptotic convergence.
Under the hypotheses of definition 5, $(\mathtt{A}_k)_{k\in\mathbb{N}}$ admits approximation ratio asymptotically converging to $\tilde{\rho} \in \bar{P}_\Pi$ if

$$\forall \epsilon > 0, \exists \kappa \in \mathbb{N}, \forall k \geqslant \kappa, \exists H, \forall I \in \mathcal{I}, h(I) \geqslant H, \quad \rho(I,k) \geqslant \tilde{\rho}(I)(1-\epsilon). \quad \blacksquare$$

Let us now show that many results can very naturally (and quite elegantly) be expressed by means of chains convergence.

- *There exists an $O(n^k)$ PTAC for max_IS achieving approximation ratio asymptotically converging to* $6/\Delta(G)$ ([30]).
- *There exists an $O(n|E|)$ FPTAC for max_IS guaranteeing asymptotic approximation ratio* $6/\Delta(G)$ ([18]).
- *There exists a SPTAC for min_3-SC with approximation ratio converging to $\sigma = 5/7$* ([28]).
- *There exists an $O(n \log n)$ DFPTAC with ratio asymptotically converging to $\delta = 2/3$ for BP* ([19]).

8 Functional approximation-preserving reductions

We propose in what follows a new polynomial reduction, the *functional approximation-preserving reduction* (FP-reduction). This new reduction encapsulates several approximation-preserving reductions, for example the ones of [10, 40, 46].

Definition 7. FP-reduction.
Let $\Pi = (\mathcal{I}_\Pi, \mathrm{S}_\Pi, v_{I_\Pi}, \mathrm{opt}_\Pi)$ and $\Pi' = (\mathcal{I}_{\Pi'}, \mathrm{S}_{\Pi'}, v_{I_{\Pi'}}, \mathrm{opt}_{\Pi'})$ be two **NP** optimization problems. A FP-reduction from Π to Π' with expansion g, denoted by $\Pi \overset{g}{\prec} \Pi'$, is a triple (f, h, g) such that:

1. $f : \mathcal{I}_\Pi \rightarrow \mathcal{I}_{\Pi'}$ and, for any $I \in \mathcal{I}_\Pi$, $f(I) \in \mathcal{I}_{\Pi'}$ is computable in time polynomial in $|I|$;
2. $h : \mathcal{I}_\Pi \times \mathrm{S}_{\Pi'} \rightarrow \mathrm{S}_\Pi$ and, for any $I \in \mathcal{I}_\Pi$ and for any $S' \in \mathrm{S}_{\Pi'}(f(I))$, $h(I, S') \in \mathrm{S}_\Pi$ is computable in time polynomial in $\max\{|I|, |S'|\}$;

3. $g : \mathcal{P}_{\Pi'} \to \mathcal{P}_{\Pi}$ is a mapping such that, $\forall I \in \mathcal{I}_{\Pi}$ and for every PTAC $(\mathtt{A}'_k)_{k\in\mathbb{N}}$ guaranteeing approximation ratio $\rho : (f(I), k) \mapsto \rho(f(I), k)$, for Π', algorithm $\mathtt{A}_k = h \circ \mathtt{A}'_k \circ f$ guarantees approximation ratio $g(\rho)$ for Π. ∎

If $P \subset \mathcal{P}_{\Pi}$ and $P' \subset \mathcal{P}_{\Pi'}$ are approximation levels for Π and Π', respectively, such that $g(P') \subset P$, then *the FP-reduction transforms P' into P*. In particular, if $P = P'$, then FP-reduction *preserves level P* and will be called *level-preserving reduction.*

Proposition 1. *FP-reductions are transitive and compose.*

The L-reduction of [41] corresponds to FP-reduction for which

$$g : \rho(f(I), k) \mapsto 1 - (ab(1 - \rho(f(I), k)))$$

where a and b ($a.b < 1$) are constants appearing in the original definition of [41]. Here, $g(\mathbf{PTAS}) \subset \mathbf{PTAS}$, i.e., L-reduction preserves the level of the approximation by polynomial time approximation schemata. For minimization problems, $g(\mathbf{APX}) \subset \mathbf{APX}$ while, for maximization ones, $g(\mathbf{APX}) \not\subset \mathbf{APX}$, unless $\mathbf{P} = \mathbf{NP} \cap \mathbf{co} - \mathbf{NP}$ ([9]).

On the other hand, the continuous reduction of [46], is an FP-reduction with functional expansion

$$g : \begin{cases} \rho \mapsto e.\rho & \rho \in P^*_{\Pi} \\ \rho \mapsto 0 & \text{otherwise} \end{cases}$$

Here, $g(\mathrm{APX}) \subset \mathrm{APX}$.

In the literature ([10, 40, 41, 46]), most of the known approximation-preserving reductions are more restrictive than the one of definition 7, since they impose conditions on how optimal and approximated values are transformed. The value of the expansion (and, consequently, of the new ratio) is a consequence of these conditions. Moreover, several existing reductions use the very strong underlying hypothesis that Π' is solved by (fixed) constant-ratio approximation algorithms. As we will see in the sequel, such hypothesis is not used by FP-reduction.

A very interesting problem dealing with approximation-preserving reductions is how approximation algorithms devised for unweighted problems can be transformed to efficiently work for their weighted versions, in particular when these algorithms do not guarantee constant approximation ratios. There exist, to our knowledge, very few reductions between weighted and unweighted versions of the same problem inducing expansions leading to interesting approximation ratios for the latters. Unfortunately, works as the ones of [11, 42], etc., despite their interest, neither produce satisfying results, nor propose general tools for the design of such reductions. We think that FP-reductions can contribute to the design of such tools. In what follows, we describe some reductions guaranteeing expansions that contribute to the achievement of non-trivial approximation ratios for a number of **NP**-hard weighted problems.

Another equally interesting domain of FP-reductions is the investigation, for a given **NP**-hard problem, of the approximation links between its standard and differential approximations. In part III, we give sufficient conditions for transferring results between standard and differential approximation. Moreover we present results dealing with FP-reductions linking standard and differential approximation of specific problems.

Part III

Achieving approximation results in the new framework

9 Induced hereditary subgraph maximization problems

Consider a hereditary property π, an induced subgraph problem Π stated with respect to π and the weighted version WΠ of Π (we suppose that weights are positive). We propose in this section an FP-reduction between max_Π and max_WΠ. The underlying idea of this reduction is the following. Suppose, without loss of generality, that the subgraphs induced by all the singletons of vertices (every such subgraph is reduced to a single vertex) verify π. Partition the input-graph G into clusters $G[V^{(i)}]$, $V^{(i)} \subseteq V$, and, by ommitting the vertex-weights, compute the solution of an unweighted max_Π on any such cluster. Let $S^{(i)}$ be a solution for $G[V^{(i)}]$. Then, $G[V^{(i)}][S^{(i)}] = G[S^{(i)}]$ verifies π; hence $S^{(i)}$ is feasible for G. Next, by reconsidering the vertex-weights, choose the heaviest among the solutions obtained and $\mathrm{argmax}_{v_i \in V}\{w_i\}$ as the final solution for max_WΠ.

Theorem 1. max_W$\Pi \overset{g}{\preceq}$ max_Π, *with* $g : \rho \mapsto O(\rho/\log n)$.

Proof. Fix $M > 2$ and set, for $i = 1, \ldots,$

$$\begin{aligned}
V^{(i)} &= \left\{v_j \in V : \frac{w_{\max}(G)}{M^i} < w_j \leqslant \frac{w_{\max}(G)}{M^{i-1}}\right\} \\
x &= \sup\left\{\ell : \beta_w\left(G\left[\bigcup_{1 \leqslant i \leqslant \ell} V^{(i)}\right]\right) < \frac{\beta_w(G)}{2}\right\} \\
G_x &= G\left[\bigcup_{1 \leqslant i \leqslant x} V^{(i)}\right] \\
G_{x+1} &= G\left[\bigcup_{1 \leqslant i \leqslant x+1} V^{(i)}\right] \\
G_d &= G\left[V \setminus \bigcup_{1 \leqslant i \leqslant x} V^{(i)}\right].
\end{aligned}$$

Of course, $\beta_w(G_d) \geqslant \beta_w(G)/2$ and $\beta_w(G_{x+1}) \geqslant \beta_w(G)/2$.

Lemma 1. *There exists a PTAA for* max_WΠ *achieving approximation ratio* $M^x/(2n)$.

Proof of lemma 1. The algorithm claimed consists in simply taking $v^* \in \mathrm{argmax}_{v_i \in V}\{w_i\}$ as max_WΠ-solution. Then, $\beta'_w(G) = w_{\max}(G)$ and

$$\beta_w(G) \leqslant 2\beta_w(G_d) \leqslant 2\left|S^*(G_d)\right| w_{\max}(G_d) \leqslant 2\left|S^*(G_d)\right| \frac{w_{\max}(G)}{M^x}.$$

Consequently,

$$\frac{\beta'_w(G)}{\beta_w(G)} \geqslant \frac{M^x}{2\left|S^*(G_d)\right|} \geqslant \frac{M^x}{2n} \Longrightarrow \rho_{\max\_\mathrm{W}\Pi}(G) \geqslant \frac{M^x}{2n} \quad \blacksquare \tag{1}$$

Remark 1. For every $i \geqslant 1$, the weight of any max_Π-solution $S^{(i)}$ of $G[V^{(i)}]$ lies in the interval $[|S^{(i)}|w_{\max}(G)/M^i, |S^{(i)}|w_{\max}(G)/M^{i-1}]$: it is at least $|S^{(i)}|w_{\min}(G[V^{(i)}]) \geqslant |S^{(i)}|w_{\max}(G)/M^i$ and at most $|S^{(i)}|w_{\max}(G[V^{(i)}]) \leqslant |S^{(i)}|w_{\max}(G)/M^{i-1}$. $\blacksquare$

Let us now prove the following lemma which is the central part of the proof of the theorem.

Lemma 2. *Assume $x > 0$.*

1. *Let $\beta_w(G)/2 \geqslant \beta_w(G_x) \geqslant ((M-2)/2M)\beta_w(G)$ and $p \in \operatorname{argmax}_{1\leqslant i\leqslant x}\{\beta_w(G[V^{(i)}])\}$. If* max_Π *is approximable on level $\rho_{\max\_\Pi}(G[V^{(p)}])$ in $G[V^{(p)}]$, then* max_WΠ *is approximable on level $((M-2)/2xM^2)\rho_{\max\_\Pi}(G)$.*

2. *Let $\beta_w(G_x) \leqslant ((M-2)/2M)\beta_w(G)$. If* max_Π *is approximable within $\rho_{\max\_\Pi}(G[V^{(x+1)}])$ in $G[V^{(x+1)}]$, then* max_WΠ *is approximable on level $(1/M^2)\rho_{\max\_\Pi}(G)$.*

Proof of item 1. Obviously,

$$\beta_w(G_x) \leqslant x\beta_w\left(G[V^{(p)}]\right) \overset{\text{(remark 1)}}{\leqslant} x\left|S^*\left(G\left[V^{(p)}\right]\right)\right|\frac{w_{\max}(G)}{M^{p-1}}$$

and, by the hypothesis of the item,

$$\beta_w(G) \leqslant \frac{2M}{M-2}\beta_w(G_x) \leqslant \frac{2M}{M-2}x\left|S^*\left(G\left[V^{(p)}\right]\right)\right|\frac{w_{\max}(G)}{M^{p-1}}.$$

On the other hand, application of a PTAA guaranteeing approximation ratio $\rho_{\max\_\Pi}(G) < 1$ for max_Π in $G[V^{(p)}]$ constructs a solution $S^{(p)}$ of max_WΠ of weight at least

$$\left|S^{(p)}\right| w_{\min}\left(G\left[V^{(p)}\right]\right) \geqslant \left|S^{(p)}\right|\frac{w_{\max}(G)}{M^p}.$$

Note that $S^{(p)}$ is max_Π-feasible for G. Moreover, starting from this solution, one can greedily augment it in order to finally produce a maximal max_WΠ-solution for G. This final solution verifies $\beta'_w(G) \geqslant |S^{(p)}|w_{\max}(G)/M^p$.

Combination of the above expressions for $\beta'_w(G)$ and $\beta_w(G)$ yields

$$\frac{\beta'_w(G)}{\beta_w(G)} \geqslant \left(\frac{M-2}{2xM^2}\right)\left(\frac{|S^{(p)}|}{|S^*(G[V^{(p)}])|}\right) \geqslant \frac{M-2}{2xM^2}\rho_{\max\_\Pi}\left(G\left[V^{(p)}\right]\right) \geqslant \frac{M-2}{2xM^2}\rho_{\max\_\Pi}(G)$$

and, consequently,

$$\rho_{\max\_W\Pi}(G) \geqslant \frac{M-2}{2xM^2}\rho_{\max\_\Pi}(G). \tag{2}$$

This concludes the proof of item 1. ∎

Proof of item 2. We now suppose that $\beta_w(G_x) \leqslant ((M-2)/2M)\beta_w(G)$. Note that since x is the largest ℓ for which $\beta_w(G[\cup_{1\leqslant i\leqslant \ell}V^{(i)}]) < \beta_w(G)/2$, set $V^{(x+1)}$ is non-empty.

Let $S^{\text{opt}}(G_{x+1})$ be an optimal max_WΠ-solution in G_{x+1} (i.e., $\beta_w(G_{x+1}) = w(S^{\text{opt}}(G_{x+1}))$). Let $S(G_x) = S^{\text{opt}}(G_{x+1})\cap V(G_x)$ (where $V(G_x)$ denotes the vertex-set of G_x) and $S(G[V^{(x+1)}]) = S^{\text{opt}}(G_{x+1})\cap V^{(x+1)}$ (in other words, $\{S(G_x), S(G[V^{(x+1)}])\}$ is a partition of $S^{\text{opt}}(G_{x+1})$). Since π is hereditary, sets $S(G_x)$ and $S(G[V^{(x+1)}])$, being subsets of $S^{\text{opt}}(G_{x+1})$, also verify π (and, consequently they are feasible max_WΠ-solutions for G_x and $G[V^{(x+1)}]$, respectively). We then have:

$$\begin{aligned}
w(S(G_x)) &\leqslant \beta_w(G_x) \\
w\left(S\left(G\left[V^{(x+1)}\right]\right)\right) &\leqslant \beta_w\left(G\left[V^{(x+1)}\right]\right) \\
\beta_w(G_{x+1}) &= w(S(G_x)) + w\left(S\left(G\left[V^{(x+1)}\right]\right)\right) \leqslant \beta_w(G_x) + \beta_w\left(G\left[V^{(x+1)}\right]\right) \\
&\leqslant \frac{M-2}{2M}\beta_w(G) + \beta_w\left(G\left[V^{(x+1)}\right]\right)
\end{aligned}$$

and also $\beta_w(G_{x+1}) \geqslant \beta_w(G)/2$. It follows from the above expressions that $\beta_w(G[V^{(x+1)}]) \geqslant \beta_w(G)/M$ and this together with remark 1 yield, after some easy algebra,

$$\beta_w(G) \leqslant \left|S^*\left(G\left[V^{(x+1)}\right]\right)\right| \frac{w_{\max}(G)}{M^{x-1}}. \tag{3}$$

As previously, suppose that a PTAA provides a solution $S^{(x+1)}$ for max_Π in $G[V^{(x+1)}]$, the cardinality of which is at least $\rho_{\max\_\Pi}(G[V^{(x+1)}])|S^*(G[V^{(x+1)}])|$. Then,

$$\beta'_w(G) \geqslant \left|S^{(x+1)}\right| w_{\min}\left(G[V^{(x+1)}]\right) \geqslant \frac{|S^{(x+1)}|\, w_{\max}(G)}{M^{x+1}} \tag{4}$$

Combination of expressions (3) and (4) yields

$$\frac{\beta'_w(G)}{\beta_w(G)} \geqslant \left(\frac{1}{M^2}\right)\left(\frac{|S^{(x+1)}|}{\left|S^*\left(G\left[V^{(x+1)}\right]\right)\right|}\right) \geqslant \frac{1}{M^2}\rho_{\max\_\Pi}\left(G\left[V^{(x+1)}\right]\right) \geqslant \frac{1}{M^2}\rho_{\max\_\Pi}(G).$$

Therefore,

$$\rho_{\max\_W\Pi}(G) \geqslant \frac{1}{M^2}\rho_{\max\_\Pi}(G) \tag{5}$$

and this concludes the proof of item 2 and of the lemma. ∎

Remark 2. For the case where $x = 0$, i.e., $\beta_w(G[V^{(1)}]) \geqslant \beta_w(G)/2$, arguments similar to the ones of the proof of item 2 in lemma 2 lead to $\rho_{\max\_W\Pi}(G) = \beta'_w(G)/\beta_w(G) \geqslant \rho_{max\_\Pi}(G)/2M$, better than the one of expression (5). ∎

Consider now the following algorithm where we take up the ideas of lemmata 1 and 2 and where, for a graph G', we denote by A(G') the solution-set provided by the execution of the Π-PTAA A on the unweighted version of G'.

```
BEGIN (*WA*)
      fix a constant M > 2;
      partition V in sets V^(i) ← {v_k : w_max/M^i < w_k ≤ w_max/M^(i-1)};
      S^(0) ← v* ∈ {argmax_{v_i∈V}{w_i}};
      OUTPUT argmax{w(S^(0)), w(A(G[V^(i)])), i = 1,...};
END. (*WA*)
```

Revisit expressions (1), (2) and (5). It is easy to see that

$$\rho_{\max\_W\Pi}(G) \geqslant \rho_{WA}(G) \geqslant \max\left\{\frac{M^x}{2n}, \min\left\{\frac{M-2}{2M^2x}\rho_{max\_\Pi}(G), \frac{1}{M^2}\rho_{max\_\Pi}(G)\right\}\right\} \tag{6}$$

By expression (1) and by the fact that the approximation ratio of any PTAA for max_WΠ must be less than 1 (max_WΠ being a maximization problem), $x \leqslant O(\log_M n)$. Taking this value for x into account in expression 6, concludes the proof of the theorem which, obviously, works also in the case where weights are exponential in n. ∎

10 Maximum independent set

10.1 A first improvement of the approximation of the maximum-weight independent set via theorem 1

It is well-known ([46]) that, $\forall k \geqslant 1$, the general weighted independent set problem polynomially reduces to max_PWIS(k) (the max_WIS-subproblem where the weights are bounded by n^k), by a simple scaling and rounding technique. This reduction preserves (within a factor of $(1-\epsilon)$) the ratios for max_WIS and max_PWIS(k), $\forall k \geqslant 1$, and works also for instance-depending ratios. On the other hand, the following approximation preserving reduction from max_PWIS to max_IS *working only for constant ratios* is established in [46].

Definition 8. Given a weighted graph $(G=(V,E),\vec{w})$, an unweighted graph $G_w=(V_w,E_w)$ can be constructed in the following way:

$$
\begin{aligned}
V_w &= \{(u,i) : u \in V, i \in \{1,\dots,w_u\}\} \\
E_w &= \{(u,i)(v,j) : i \in \{1,\dots,w_u\},\ j \in \{1,\dots,w_v\},\ u \neq v,\ uv \in E\}.
\end{aligned}
$$

In other words, every vertex u of V is replaced by an independent set W_u of size w_u in G_w and every edge uv of E corresponds in G_w to a complete bipartite graph between W_u and W_v. ∎

One can easily show that every independent set S of G of total weight $w(S)$ induces, in G_w, the independent set $\{(s,i) : s \in S, i \in \{1,\dots,w_s\}\}$ of size $w(S)$, and conversely, for every independent set S_w of G_w, the set $S=\{u \in V : \exists i \in \{1,\dots,w_u\}, (u,i) \in S_w\}$ is an independent set of weight $w(S) \geqslant |S_w|$. Consequently, $\alpha_w(G)=\alpha(G_w)$ and, by applying a $\rho(G)$-approximation max_IS-algorithm to G_w, one can derive an approximated max_WIS-solution of $(G,\vec{w})$ guaranteeing ratio $\rho(G_w)$.

By the above reduction, a ratio $\rho(n,\Delta(G))$, non-increasing in Δ, for max_IS transforms to a ratio $\rho(w(V), w_{\max}(G)\Delta(G_w)) \geqslant \rho(w(V), w_{\max}(G)\Delta(G))$ for max_WIS, i.e., except from the case of constant approximation ratios, the reduction above results in max_WIS-ratios depending on the weights. More precisely, the following result can be easily proved.

Proposition 2. *For every $\epsilon > 0$ and every constant $k > 0$, there exists an FP-reduction from* max_WIS *to* max_IS *transforming approximation level $\rho(n,\Delta,\mu)$ for the latter into approximation level converging to $\rho(n^{1+\epsilon}w(V)/w_{\max}(G), n^{1+\epsilon}\Delta, \mu_w) \geqslant \rho(n^{2+\epsilon}, n^{1+\epsilon}\Delta, \Delta)$ for the former.*

Unfortunately, the approximation results known for max_IS do not allow achievement of interesting approximation ratios for max_WIS using proposition 2.

Revisit expression (6) in the proof of theorem 1 and let Π be max_IS. Set, for every k, $M=6$. Then:

- if $x \geqslant k \log\log n/\log M$, then $\rho_{\max\_\mathrm{WIS}}(G) \geqslant \log^k n/2n$;
- if $x \leqslant k \log\log n/\log M$, then $\rho_{\max\_\mathrm{WIS}}(G) \geqslant 0.099\rho_{\max\_\mathrm{IS}}(G)/(k\log\log n)$

and the following theorem holds.

Theorem 2. *For every fixed ℓ, every approximation level $\rho_{\max\_\mathrm{IS}}(G)$ for max_IS can be transformed into approximation level*

$$\rho_{\max\_\mathrm{WIS}}(G) \geqslant \min\left\{\frac{\log^\ell n}{2n}, \frac{0.099\rho_{\mathrm{IS}}(G)}{\ell\log\log n}\right\}.$$

In terms of n, the best-known approximation ratio for max_IS is, to our knowledge, $O(\log^2 n/n)$ achieved by the max_IS-PTAA of [7]. Embedding it in $\rho_{\max\_\mathrm{IS}}(G)$-expression of theorem 2, we obtain the following concluding theorem.

Theorem 3.

$$\rho_{\max\_\mathrm{WIS}}(G) \geqslant O\left(\frac{\log^2 n}{n\log\log n}\right).$$

The above result improves by a factor $O(\log\log n)$ the best-known approximation ratio function of n for max_WIS ($O(\log^2 n/(n\log^2\log n))$, due to [27]).

10.2 Approximation chains for maximum independent set with ratios functions of graph-degree

10.2.1 An approximation chain with ratio function of $\Delta(G)$

The main result of this section is based upon the following facts; fact 1 is proved in [18] while fact 2 is proved in [1, 30].

Fact 1. If there exists an algorithm guaranteeing, for every $\ell \in \mathbb{N}$ and every graph without clique of order ℓ, an approximation ratio ρ_ℓ for max_IS, then, for every graph G, for every $\epsilon > 0$ and for every $\lambda > 0$, there exists an algorithm guaranteeing for max_IS approximation ratio bounded below by

$$\min\left\{\lambda, \epsilon'(1-\lambda)\rho_\ell, \frac{2(\ell-\epsilon)(1-\lambda)}{\Delta(G)+2}\right\}$$

where ϵ' is such that $1/(\ell-\epsilon) = (1/\ell) + \epsilon'$. ∎

Fact 2. There exists a fixed constant c such that, for every constant ℓ, there exists a polynomial time approximation algorithm L_FREE(ℓ, G) such that, for every graph G of order n without cliques of order ℓ, it provides an independent set of cardinality greater than, or equal to, $cn[\log[(\log\mu(G))/\ell]]/\mu(G)$. ∎

Consider now the following algorithm of time-complexity $O(\max\{n|E|, T(\ell,n), \Delta(G)^{\ell-2}|E|\})$ where by $T(\ell,n)$ we denote the complexity of L_FREE.

```
BEGIN (*MDCHAIN*)
    S ← any non-empty independent set;
    REPEAT
        b1 ← FALSE;
        b2 ← FALSE;
        IF ∃v ∈ V \ S : Γ(v) ∩ S = ∅
           THEN S ← S ∪ {v};
           ELSE b1 ← TRUE;
        FI
        IF ∃{u,v} ⊂ V \ S : (Γ(u) ∪ Γ(v)) ∩ S = {s}, uv ∉ E
           THEN S ← (S \ {s}) ∪ {u,v};
           ELSE b2 ← TRUE;
        FI
    UNTIL b1 AND b2;
    Ṽ ← {v ∈ V \ S : δ_S(v) ⩾ 2} ∪ S;
    compute a maximal collection C_ℓ of disjoint ℓ-cliques in G[Ṽ];
    X_ℓ ← {v ∈ Ṽ : v ∉ ∪_{C∈C_ℓ} C};
    S_ℓ ← L_FREE(ℓ, G[X_ℓ]);
    OUTPUT S ← argmax{|S|, |S_ℓ|};
END (*MDCHAIN*)
```

Consider a constant κ and set $\ell = \lceil(\kappa/2)\rceil + 1$. Then, using facts 1 and 2 and appropriately choosing λ and ϵ, the following theorem holds.

Theorem 4. *([18]) For every fixed integer constant κ, algorithm MDCHAIN is a PTAC guaranteeing asymptotic (with respect to hardness threshold $\Delta(G)$) approximation ratio $\kappa/\Delta(G)$ for* max_IS *in time $O(n^{\lceil\kappa/2\rceil})$.*

10.2.2 An approximation chain with improved ratio function of $\mu(G)$

We first recall the well-known greedy max_IS-algorithm, called GREEDY in the sequel.

```
BEGIN (*GREEDY*)
      S ← ∅;
      REPEAT
            v ← argmin_{v_i∈V}{δ(v_i)};
            S ← S ∪ {v};
            V ← V \ ({v} ∪ Γ(v));
            G ← G[V];
      UNTIL V = ∅;
END. (*GREEDY*)
```

The improvement of the result of theorem 4 is based upon the following facts, the first proved in [18] and the second in [47].

Fact 4. Consider, for every fixed integer constant ℓ, the simultaneous existence of L_FREE(ℓ, G) and of an algorithm computing, for every graph G, a maximal independent set of size at least $nf(G)$. Then, there exists a PTAC solving max_IS within approximation ratio bounded below by $\min\{\epsilon'\rho_\ell, (\ell-\epsilon)f(G)\}$, where ρ_ℓ is the approximation ratio of L_FREE and ϵ' is such that $1/(\ell-\epsilon) = (1/\ell) + \epsilon'$. ∎

Fact 5. GREEDY computes, in a graph G, a maximal independent set of size at least $n/(\mu(G)+1)$. ∎

```
BEGIN (*ADCHAIN*)
      S ← GREEDY(G);
      compute a maximal collection C_ℓ of disjoint ℓ-cliques in G;
      X_ℓ ← {v ∈ V : v ∉ ∪_{C∈C_ℓ} C};
      S_ℓ ← L_FREE(ℓ, G[X_ℓ]);
      OUTPUT S ← argmax{|S|, |S_ℓ|};
END. (*ADCHAIN*)
```

Using facts 4 and 5 with $\ell = \kappa$ and $\epsilon = 1$, the following theorem holds.

Theorem 5. *([18]) For every fixed integer constant κ, ADCHAIN guarantees, in $O(n^\kappa)$, asymptotic (with respect to hardness threshold $\mu(G)$) approximation ratio bounded below by*

$$\min\left\{\frac{\kappa}{\mu(G)}, \kappa'\frac{\log\log\Delta(G)}{\Delta(G)}\right\}$$

where $\kappa' = c'/(\kappa(\kappa-1))$, for a fixed constant c'.

The result of theorem 5 further improves (sometimes quite largely) the result of theorem 4.

10.3 Towards $\Omega(1/\Delta)$ -approximations

In this section we show how one can use the clique-removal method of [7] to obtain, for every Δ, $\Omega(1/\Delta)$-approximations for max_IS and max_WIS. The authors of [7] repeatedly call a procedure computing either a k-clique (clique of order k), or an independent set of expective size. At most n/k cliques can so be detected, while at each clique-deletion the independence number decreases no more than 1. If the independence number of the initial graph is large enough, a large independent set is necessarily detected during one execution of the procedure. In all, the following theorem summarizes the thought process of [7].

then the ratio guaranteed is largely superior to the best-known n-depending ratio for IS (for instance, consider the case $\ell(n) = \log\log n$). In fact, if $K \log^{\ell(n)} n/n \leqslant \log n/(\ell(n)(\mu+1)\log\log n)$ and $n \geqslant C_0$, for a fixed C_0, then $n/\mu \geqslant K\ell(n)(\log\log n)\log^{\ell(n)-1} n$ and in this case GREEDY guarantees ratio bounded below by $\log^{\ell(n)-1} n/n$. In all, the following corollary can be deduced.

Corollary 1. *Given a graph G and ℓ such that, $\forall n$, $2 < \ell(n) \leqslant \log\log n$, at least one of the two following conditions holds:*

1. *GREEDY guarantees ratio bounded below by $\log^{\ell(n)-1} n/n$ (improving the best known $\rho(n)$-ratio if $\ell > 3$);*
2. *STABLE guarantees ratio bounded below by $\log n/(\ell(n)(\mu+1)\log\log n)$ (achieving so ratio $\Omega(1/\mu)$).*

The discussion above draws an interesting remark about the instance-parameters expressing non-constant approximation ratios. Until now, studies about the approximation of max_IS were limited in expressing ratio using only one instance-parameter (either the size or the degree). The results of this section show that considering both parameters, it is possible to reach tighter and more interesting approximation ratios.

Combination of theorems 8 and 2 allows the achievement of important approximation results also for max_WIS.

Theorem 9. *For every constant k,* max_WIS *can be approximated within ratios on the level*

$$\min\left\{\frac{\log^k n}{n}, \frac{0.099\log n}{k(k+1)(\mu+1)\log^2\log n}\right\} \geqslant \min\left\{\frac{\log^k n}{n}, \frac{0.099\log \mu}{k(k+1)(\mu+1)\log^2\log \mu}\right\}.$$

In other words, theorem 9 guarantees for max_WIS the existence of PTAAs achieving either n-depending ratios much better than the ones known for max_IS (and comparable with the ones of corollary 1), or the first $\Omega(1/\mu)$ ratios for max_WIS (recall that the best-known $\rho(\Delta)$-ratio for max_WIS – without restrictions on Δ – was, until now, the one of [27], bounded below by $3/(\Delta+2)$).

10.4 Further improvements

In this section, we propose a polynomial approximation result for max_WIS not deduced by reduction from the unweighted case. It improves all the approximation results of section 10.3, for both weighted and unweighted cases, but the corresponding complexity is higher.

10.4.1 A weighted version of Turán's theorem

Recall that $\mu_w(G) = \sum_{v\in V} w(\Gamma(v))/w(V)$ (note that $\mu_w(G) \leqslant \Delta(G)$) and consider the following algorithm for max_WIS.

```
BEGIN (*WGREEDY*)
      S ← ∅;
      WHILE V ≠ ∅ DO
            v ← argmin_V{w(Γ(v))/w(v)};
            S ← S ∪ {v};
            V ← V \ ({v} ∪ Γ(v));
            update E;
      OD
      OUTPUT S; END. (*WGREEDY*)
```

Then, the following easy theorem holds (an unweighted version of it – see also [29] – is the famous Turán's theorem).

Theorem 10. *For every weighted-graph with maximum degree Δ, algorithm `WGREEDY` computes an independent set of weight at least $w(V)/(\mu_w(G)+1) \geqslant w(V)/(\Delta+1)$.*

Proof. Consider algorithm `WGREEDY` and suppose that its `WHILE` loop is executed t times. For $i \in \{1, \dots t\}$ let us denote by $G_i = (V_i, E_i)$ the surviving graph at the beginning of iteration i, by v_i the vertex selected during the ith execution and by $\Gamma_i(v)$ the neighborhood of v in G_i. Then, $\forall i \in \{1, \dots t\}$,

$$\sum_{v \in (\{v_i\} \cup \Gamma(v_i))} w(\Gamma(v)) \geqslant \sum_{v \in (\{v_i\} \cup \Gamma_i(v_i))} w(\Gamma_i(v)) \geqslant \frac{w(\Gamma_i(v_i))}{w_{v_i}} \left(w_{v_i} + w(\Gamma_i(v_i))\right) \tag{7}$$

and, consequently, by adding side-by-side the expressions (7) above, for $i = 1 \dots t$, we get (note that $V = \cup_{i=1\dots t}\{\{v_i\} \cup \Gamma_i(v_i)\}$),

$$w(V)\mu_w \geqslant \sum_{i=1}^{t} \frac{w(\Gamma_i(v_i))}{w_{v_i}} \left(w_{v_i} + w(\Gamma_i(v_i))\right). \tag{8}$$

On the other hand,

$$w(V) = \sum_{i=1}^{t} \left(w_{v_i} + w(\Gamma_i(v_i))\right) \tag{9}$$

and using expressions (8) and (9), we get by the Cauchy-Schwarz inequality

$$w(V)(\mu_w + 1) \geqslant \sum_{i=1}^{t} \frac{\left(w_{v_i} + w(\Gamma_i(v_i))\right)^2}{w_{v_i}} \geqslant \frac{(w(V))^2}{\sum_{i=1}^{t} w_{v_i}}$$

which concludes the proof since the weight of the greedy solution is $\sum_{i=1}^{t} w_{v_i}$. ∎

An easy corollary of theorem 10 is that whenever $\alpha_w(G) \leqslant w(V)/k$, then `WGREEDY` achieves in G ratio $k/(\Delta+1)$ for max_WIS. Consequently, in order to devise $O(k/\Delta)$-approximations for every graph, one can focus him/herself on the approximation of max_WIS in graphs with large weighted independence number. For an integer function $k = k(n) \leqslant n$, we denote by max_WIS$_k$ the class of graphs with $\alpha_w(G) > w(V)/k$; for reasons of simplicity we assimilate this class with the corresponding max_WIS-subproblem. The work of [7] and its improvement by [3] show how max_IS$_k$ (the unweighted version of max_WIS$_k$) can be approximated within $O(n^{\epsilon_k - 1})$ where ϵ_k depends only on k (note that using the reduction of [46], one immediately gets a ratio $O(w(V)^{\epsilon_k - 1})$ for max_WIS).

10.4.2 Graph coloring and the approximation of max_WIS$_k$

In this section, we adapt the method of [3] (originally devised for the unweighted max_IS) to the weighted case and relate the approximation of max_WIS$_k$ to the coloring of a class of graphs (called $\mathcal{G}_{k+1}$ in the sequel) containing the $(k+1)$-colorable graphs. We first recall two notions very closely related, the *Lovász θ-function* and the *orthonormal representation* of a graph.

Definition 9. ([37]) Consider a graph $G = (V, E)$, $V = \{1, \dots n\}$:

the Lovász θ-function $\theta(G)$ of G is the maximum value of $\sum_{i,j=1}^{n} b_{ij}$ where $B = (b_{ij})_{i,j=1\dots n}$ ranges over all positive semidefinite symmetric matrices with trace 1 and such that $b_{ij} = 0$ for every pair (i, j), $i \neq j, ij \in E$;

the orthonormal representation of G is a system of n unit vectors $(\vec{u_i})_{i=1\dots n}$ in a n-dimensional Euclidean space, such that for every (i,j) such that $ij \notin E$, $\vec{u_i}$ and $\vec{u_j}$ are orthogonal. ∎

Proposition 3. *([37]) Given a graph G, the following holds:*

- *given an orthonormal representation $(\vec{u_i})_{i=1\dots n}$ of G, and a unit vector $\vec{d}$, there exists $i \in \{1,\dots n\}$ such that $\theta(G) \leqslant 1/(\vec{d}\cdot\vec{u_i})^2$;*
- $\alpha(G) \leqslant \theta(G) \leqslant \chi(\bar{G})$.

For an integer function $\ell = \ell(n) \leqslant n$, set $\mathcal{G}_\ell = \{G : \theta(\bar{G}) \leqslant \ell\}$. By proposition 3, every ℓ-colorable graph belongs to $\mathcal{G}_\ell$. In [34] it is shown that every graph in $\mathcal{G}_\ell$ with order n and maximum degree Δ can be colored with $O(\min\{\Delta^{1-2/\ell}\log^{3/2} n, n^{1-3/(\ell+1)}\log^{1/2} n\})$ colors in randomized polynomial time ([34]). The algorithm of [34] has been derandomized later in [38].

Theorem 11. *Let $k = k(n) \leqslant n$ be an integer function and $f_k(x,y)$ be a function from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$, non-decreasing with respect to both x and y. If there exists a $O(T(n))$ algorithm A computing, for every graph $G \in \mathcal{G}_{k+1}$, $|V| \leqslant n$, a $f_k(n,\Delta)$-coloring, then there exists a $O(\max\{n^3, T(n)\})$ PTAA for* max_WIS$_k$ *guaranteeing ratio $1/(k(k+1)f_k(n,\Delta))$.*

Proof. Let $(G = (V,E), \vec{w})$ be an instance of WIS$_k$ of order n and G_w be as in definition 8 of section 9. Since the weights are supposed to be integral and $\alpha_w(G) > w(V)/k$, we have

$$\alpha_w(G) = \alpha(G_w) \geqslant \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)} + \frac{1}{k}.$$

In [25] it is shown that:

- $\theta(G_w)$ is equal to the maximum value of $\sum_{i,j=1}^n \sqrt{w_i w_j} b_{ij}$, where b_{ij} are as in definition 9;
- one can compute, in $O(n^3)$ by the ellipsoid method, a positive semidefinite symmetric matrix $B = (b_{ij})_{i,j=1\dots n}$ satisfying

$$\begin{array}{ll} \sum_{i=1}^n \sum_{j=1}^n \sqrt{w_i w_j} b_{ij} \geqslant \theta(G_w) - \frac{1}{2k} > 0 & ij \notin E \\ b_{ij} = 0 & ij \in E \end{array} \tag{10}$$

Given that B is symmetric and positive semidefinite, there exist n vectors $\hat{u}_i \in \mathbb{R}^n$, $i \in \{1,\dots n\}$ ($\mathbb{R}^n$ being seen as Euclidean space) such that, $\forall (i,j) \in \{1,\dots n\}^2$, $\hat{u}_i \cdot \hat{u}_j = b_{ij}$. In particular, we have $b_{ii} \geqslant 0$. For our purpose we just need to compute n vectors $\vec{u_i}$ satisfying the following expression (11) for $\epsilon = 1/(2knw(V))$:

$$\begin{array}{l} \vec{u_i}\cdot\vec{u_j} = b_{ij}, \quad i \neq j \\ |\vec{u_i}|^2 = b_{ii} + \epsilon > 0 \\ \sum_{i=1}^n |\vec{u_i}|^2 = \mathrm{Tr}(B) + n\epsilon = 1 + n\epsilon \\ \left|\sum_{i=1}^n \sqrt{w_i}\vec{u_i}\right|^2 \geqslant \sum_{i=1}^n \sum_{j=1}^n \sqrt{w_i w_j} b_{ij} \end{array} \tag{11}$$

Such vectors can be seen as non-zero approximations of $\hat{u}_i$, $i = 1\dots n$. The system of vectors $(\vec{u_i})_{i=1\dots n}$ can be computed ([24]) by applying Cholesky's decomposition to the symmetric positive definite matrix $B + \epsilon \mathrm{I}$ (where I is the identity matrix); one gets (in $O(n^3)$) an n-dimensional triangular matrix U such that $B = {}^tUU - \epsilon \mathrm{I}$. Let then $(\vec{u_i})_{i=1\ldots n}$ be the columns of U; they clearly satisfy expression (11).

Set, for $i \in \{1, \ldots n\}$,

$$\begin{aligned} \vec{d} &= \frac{\sum_{i=1}^{n} \sqrt{w_i}\vec{u_i}}{\left|\sum_{i=1}^{n} \sqrt{w_i}\vec{u_i}\right|} \\ \vec{z_i} &= \frac{\vec{u_i}}{|\vec{u_i}|} \end{aligned}$$

and note that $\vec{z_i}$ constitutes an orthonormal representation of $\bar{G}$. Without loss of generality, we can assume that $(\vec{d} \cdot \vec{z_1})^2 \geqslant (\vec{d} \cdot \vec{z_2})^2 \geqslant \ldots (\vec{d} \cdot \vec{z_n})^2$.

Lemma 3. *Set*

$$\begin{aligned} j &= \max\left\{ i \in \{1, \ldots, n\} : \sum_{l=1}^{i-1} w_l \leqslant \frac{w(V)}{k(k+1)} \right\} \\ K &= G\left[\{v_i : i = 1, \ldots, j\}\right]. \end{aligned} \tag{12}$$

Then, $(\vec{d} \cdot \vec{z_j})^2 \geqslant 1/(k+1)$ *which implies* $K \in \mathcal{G}_{k+1}$.

Proof of lemma 3. By Cauchy-Schwarz inequality and expressions (10) and (11), we get:

$$\begin{aligned} (1 + n\epsilon) \sum_{i=1}^{n} w_i \left(\vec{d} \cdot \vec{z_i}\right)^2 &= \left(\sum_{i=1}^{n} |\vec{u_i}|^2\right) \sum_{i=1}^{n} w_i \left(\vec{d} \cdot \vec{z_i}\right)^2 \geqslant \left(\sum_{i=1}^{n} \vec{d} \cdot \sqrt{w_i}\vec{u_i}\right)^2 = \left|\sum_{i=1}^{n} \sqrt{w_i}\vec{u_i}\right|^2 \\ &\geqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \sqrt{w_i w_j} b_{ij} \geqslant \theta(G_w) - \frac{1}{2k} \geqslant \alpha(G_w) - \frac{1}{2k} \\ &\geqslant \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)} + \frac{1}{2k} \end{aligned} \tag{13}$$

On the other hand, since for $i \in \{1, \ldots n\}$, $\vec{d} \cdot \vec{z_i} \leqslant 1$, (recall that $\vec{d}$ and $\vec{z_i}$, $i = 1 \ldots n$ are unit vectors),

$$(1 + n\epsilon) \sum_{i=1}^{n} w_i \left(\vec{d} \cdot \vec{z_i}\right)^2 \leqslant \sum_{i=1}^{n} w_i \left(\vec{d} \cdot \vec{z_i}\right)^2 + \frac{1}{2k}. \tag{14}$$

Consequently, combining expressions (13) and (14), we get

$$\sum_{i=1}^{n} w_i \left(\vec{d} \cdot \vec{z_i}\right)^2 \geqslant \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)}. \tag{15}$$

Recall that $(\vec{d} \cdot \vec{z_1})^2 \geqslant (\vec{d} \cdot \vec{z_2})^2 \geqslant \ldots (\vec{d} \cdot \vec{z_n})^2$. We have $(\vec{d} \cdot \vec{z_j})^2 \geqslant 1/(k+1) > 0$ since, in the opposite case,

$$\sum_{i=1}^{n} w_i \left(\vec{d} \cdot \vec{z_i}\right)^2 \leqslant \sum_{i=1}^{j-1} w_i + \sum_{i=j}^{n} w_i \left(\vec{d} \cdot \vec{z_j}\right)^2 < \frac{w(V)}{k+1} + \frac{w(V)}{k(k+1)}$$

which contradicts expression (15).

As noticed in [3], $(\vec{d}\cdot\vec{z_j})^2 \geqslant 1/(k+1) > 0$ implies that the subgraph K of G induced by vertices v_i, $i = 1\dots j$, satisfies $\theta(\bar{K}) \leqslant k+1$; this follows from the fact that $\vec{z_i}$, $i \in \{1,\dots j\}$, is an orthonormal representation of $\bar{K}$ with value less than $1/(\vec{d}\cdot\vec{z_j})^2 \leqslant k+1$ (see [37]). Note that this expression holds for the unweighted θ-function of G; so, $K \in \mathcal{G}_{k+1}$. This completes the proof of lemma 3. ∎

Lemma 3 is originally proved in [3] for the case of (unweighted) max_IS. As one can see from the above proof, extension of it for max_WIS is non-trivial.

Let us now continue the proof of theorem 11. By lemma 3, the algorithm A (claimed in the statement of theorem 11) computes a $f_k(n,\Delta)$-coloring of K. Then, the maximum-weight color is an independent set of $K = (V_K, E_K)$ (and of G) of total weight at least $w(V_K)/f_k(j,\Delta(K))$. Since $w(V_K) \geqslant w(V)/(k(k+1))$ (see expression 12), $j \leqslant n$ and $\Delta(K) \leqslant \Delta$, the maximum-weight color is an independent set of G of weight at least $w(V)/(k(k+1)f_k(n,\Delta))$. On the other hand $\alpha_w(G) \leqslant w(V)$, and consequently the following algorithm WIS_k guarantees ratio $1/(k(k+1)f_k(n,\Delta))$ for max_WIS$_k$.

```
BEGIN (*WIS_k*)
(1)   compute the matrix B = (b_ij)_{i,j=1,...n} by the ellipsoid method;
(2)   ε ← 1/(2knw(V));
(3)   compute the Cholesky's decomposition of B + εI;
(4)   compute vectors d⃗ and z⃗_i, i ∈ {1,...n};
(5)   sort vertices in decreasing order with respect to (d⃗·z⃗_i)²;
(6)   compute j and K;
(7)   IF (d⃗·z⃗_j)² ⩾ 1/(k+1)
(8)      THEN
(9)           call A to compute a coloring (C_1,...C_l) of K;
(10)          S ← argmax{w(C_i) : i = 1...l};
(11)          complete S to obtain a maximal independent set of G;
(12)     ELSE S ← ∅;
(13)  FI
(14)  OUTPUT S;
END. (*WIS_k*)
```

To conclude the proof of the theorem, let us note that lines (1) and (3) are executed in $O(n^3)$, line (5) in $O(n\log n)$ and line (11) in $O(n^2)$. Finally, the time-complexity of line (9) is bounded above by $T(n)$. ∎

Of course, unless **P=NP**, inclusion of a graph in max_WIS$_k$ cannot be polynomially decided. However, algorithm WIS_k runs on every graph within the same complexity. In fact (by instruction (7)), if $K \in \mathcal{G}_{k+1}$, then algorithm A returns a non-empty independent set S; otherwise, $S = \emptyset$. Consequently, if $S \neq \emptyset$ (in particular if $G \in$ max_WIS$_k$), algorithm WIS_k guarantees the ratio claimed by theorem 11. In the opposite case, the input-graph does not belong to max_WIS$_k$.

Using for A, the derandomized version of [34] presented in [38], theorem 11 leads to the following approximation result for max_WIS$_k$, for $2 \leqslant k(n) \leqslant n$.

Corollary 2. max_WIS$_k$ *is approximable within ratios on approximation level*

$$\rho_{\mathrm{WIS}_k}(G) \geqslant \frac{1}{k(k+1)\Delta^{1-2/(k+1)}\log^{3/2} n}.$$

10.4.3 The main result

Consider now the following algorithm, the worst-time complexity of which is the same as the one of WIS_k.

```
BEGIN (*WIS*)
      OUTPUT S ← argmax{w(WIS_k(G)), w(WGREEDY(G))};
END. (*WIS*)
```

By theorems 10 and 11, $\exists c'$ such that

$$\rho_{\max\_\mathtt{WIS}}(G) \geqslant \min\left\{\frac{k}{\Delta+1}, \frac{c'}{k(k+1)\Delta^{1-\frac{3}{k+1}}\log^{\frac{3}{2}} n}\right\}. \tag{16}$$

By an easy but somewhat tedious algebra, one can prove that the right-hand side of expression (16) is at least as large as

$$\min\left\{\frac{k}{\Delta+1}, \frac{c^k}{(k+1)^{\frac{1+3k}{2}}\log^{\frac{3(k+1)}{4}} n}\right\} \tag{17}$$

for a constant c. Let us suppose k constant. Then, the following theorem holds.

Theorem 12. *For any fixed integer $k \geqslant 2$ and for $t = 3(k+1)/4$,* max _ WIS *is approximable within ratios on approximation level*

$$\rho_{\max\_\mathtt{WIS}}(G) \geqslant \min\left\{\frac{k}{(\Delta+1)}, O\left(\log^{-t} n\right)\right\}.$$

Furthermore, in the case where $\min\{k/(\Delta+1), O(\log^{-t} n)\} = O(\log^{-t} n)$, $\Delta+1 \leqslant O(\log^{t} n)$ and then algorithm WGREEDY already guarantees a wonderful (given the result in [31]) approximation ratio.

Corollary 3. *Consider a graph G and $k \geqslant 2$. Then, there exists $t > 0$ such that at least one of the two following conditions holds:*

1. *algorithm WGREEDY achieves ratio bounded below by $O(\log^{-t} n)$;*
2. *algorithm WIS is a PTAC achieving ratio bounded below by $k/(\Delta+1)$.*

Revisit expression (17) and set $k = \log n/(3\log\log n)$; then,

$$\exists n_0 \geqslant 1, \forall n \geqslant n_0, \quad \frac{1}{c^k}(k+1)^{\frac{1+3k}{2}}\log^{\frac{3(k+1)}{4}} n \leqslant n^{4/5}$$

and, since instances with $n \leqslant n_0$ can be solved by exhaustive search in constant time, the following theorem holds and concludes the section.

Theorem 13. max _ WIS *is approximable within ratios on approximation level*

$$\rho_{\max\_\mathtt{WIS}}(G) \geqslant \min\left\{\frac{\log n}{3(\Delta+1)\log\log n}, O\left(n^{-4/5}\right)\right\}.$$

11 Maximum-weight clique

We describe in this section a new FP-reduction, working for both weighted and unweighted cases, between independent set and clique. Let us note that the classical reduction "independent set in G — clique in $\bar{G}$" preserves constant ratios as well as ratios depending only on n but it does not preserve ratios depending on Δ.

Let $G = (V, E)$ be an instance of max_KL and let $V = \{1, \dots, n\}$. We consider the n graphs $G_i = G[\{i\} \cup \Gamma(i)]$, $i = 1, \dots, n$ and denote by n_i their respective orders. We also consider the n graphs $\bar{G}_i$. Then, the following proposition holds (recall that in every graph G, the size of a maximum clique is never greater than $\Delta(G) + 1$).

Proposition 4. *For every $i \in \{1, \dots, n\}$, the following three facts hold:*

1. *$n_i \leqslant \Delta(G) + 1$ and $\Delta(\bar{G}_i) \leqslant \Delta(G)$;*
2. *cliques (resp., independent sets) of G_i (resp., $\bar{G}_i$) are also cliques (resp., independent sets) of G (resp $\bar{G}$); moreover, $\exists i^*$ such that a maximum clique (resp., independent set) of G_{i^*} (resp., $\bar{G}_{i^*}$) is exactly a maximum clique (resp., independent set) of G (resp., $\bar{G}$);*
3. *items 1 and 2 hold also for* max_WKL *and* max_WIS *if we consider weighted cliques and independent sets.*

Let A be any max_WIS-PTAA achieving ratio $\rho(n, \Delta)$; we shall use it to produce a max_WKL-solution for G. This can be done by the following algorithm.

```
BEGIN (*WKL*)
      OUTPUT K ← argmax{w(A(Ḡ_i)) : i = 1,...,n};
END. (*WKL*)
```

Since K is an independent set of $\bar{G}$, it is a clique (of the same weight) in G. Moreover, by items 1 and 2 of proposition 4, algorithm WKL achieves, in polynomial time, approximation ratio $\rho(\Delta + 1, \Delta)$ for max_WKL.

The reduction just described is, to our knowledge, the first one preserving $\rho(\Delta)$ ratios between independent set and clique (in both weighted and unweighted cases).

Given the IS result of [7] and the one of theorem 3 we conclude the following.

Theorem 14.

1. max_KL *is approximable within ratios on level*

$$\rho_{\max\_\mathrm{KL}} \geqslant O\left(\frac{\log^2 \Delta}{\Delta}\right);$$

2. max_WKL *is approximable within ratios on level*

$$\rho_{\max\_\mathrm{WKL}} \geqslant O\left(\frac{\log^2 \Delta}{\Delta \log \log \Delta}\right).$$

The results of theorem 14 represent, to our knowledge, the first non-trivial $\rho(\Delta)$ ratios for the clique-problem.

The discussion above shows that, when dealing with max_KL, *every approximation ratio $f(n)$ can be transformed into approximation ratio $f(\Delta)$ and vice-versa.* Such a result remains still an open problem when dealing with max_IS.

12 Maximum ℓ-colorable induced subgraph

Let us note that we can assume $\ell < \Delta$. If not, then $G' = G$ (see the definition of max_Cℓ in section 2 and recall that the polynomial time coloring-algorithm of [36] always guarantees a Δ-coloring of G, unless G is a $(\Delta+1)$-clique). Consider the graph ${}^{\ell}G = ({}^{\ell}V, {}^{\ell}E)$ defined as follows:

$$\begin{aligned} {}^{\ell}V &= V \times \{1, \ldots, \ell\} \\ {}^{\ell}E &= \{(v,i)\,(v',j) : [[(i=j) \wedge vv' \in E] \vee [(i \neq j) \wedge (v = v')]]\} \end{aligned}$$

For max_WCℓ the weight of vertex (v,i) equals w_v, $\forall i = 1, \ldots, \ell$. The following holds for ${}^{\ell}G$:

$$\left\{ \begin{array}{lcl} \left|{}^{\ell}V\right| & = & \ell n \\ \Delta\left({}^{\ell}G\right) & = & \Delta(G) + \ell - 1 \\ \mu\left({}^{\ell}G\right) & = & \mu(G) + \ell - 1 \end{array} \right. \tag{18}$$

(for all $(v_k, i) \in {}^{\ell}V$, $v_k \in V$, $i \leqslant \ell$, the degree of (v_k, i) equals the degree of v_k plus $\ell - 1$). Moreover,

1. if $S \subset {}^{\ell}V$ is an independent set of ${}^{\ell}G$, then the family $S_i = \{v \in V : (v,i) \in S\}$, $i = 1, \ldots, \ell$, is a collection of mutually disjoint independent sets of G; so, the graph $G[\cup_i S_i]$ is ℓ-colorable;

2. conversely, for every ℓ-colorable subgraph $G' = (V', E')$ of G and for every ℓ-coloring $(S_1, \ldots, S_\ell)$ of G', the set $S = \{(v,i) : i \in \{1, \ldots, \ell\}, v \in S_i\}$ is an independent set of ${}^{\ell}G$.

Consequently, every independent set (resp., maximum independent set) of a certain size in ${}^{\ell}G$ corresponds to an ℓ-colorable induced subgraph (resp., maximum-order ℓ-colorable induced subgraph) of the same order in G and vice-versa. The same correspondence holds between max_WIS and max_WCℓ if one considers weights instead of sizes.

12.1 ℓ is a fixed constant

By theorems 8 and 13, the following concluding theorem holds for max_Cℓ and max_WCℓ.

Theorem 15. *Consider f such that, $\forall x > 0$, $f(x) \leqslant \log\log x$. Then,*

1. max_Cℓ *is approximable within ratios on level*

$$\rho_{\max\_C\ell}(G) \geqslant \min\left\{ O\left(\frac{\log^{f(\ell n)} n}{n}\right), \frac{\log n}{f(\ell n)(\mu+1)\log\log n} \right\};$$

2. max_WCℓ *is approximable within ratios on level*

$$\rho_{\max\_WC\ell}(G) \geqslant \min\left\{ \frac{\log n}{3(\Delta+1)\log\log n}, O\left(n^{-4/5}\right) \right\}.$$

12.2 ℓ depends on graph-parameters

For max_WCℓ, let us consider the following algorithm where by `Δ_COLOR` we denote the Δ-coloring algorithm of [36] and by `WIS` the algorithm of section 10.4.3.

```
BEGIN (*WCL*)
(1)   IF ℓ ⩽ Δ^{1/9}
(2)      THEN construct ^ℓG;
(3)           WIS(^ℓG);
(4)           OUTPUT G[∪_i S_i] (item 1 just above);
(5)      ELSE Δ_COLOR(G);
(6)           OUTPUT the subgraph induced by the ⌈Δ^{1/9}⌉ heaviest colors;
(7)   FI
END. (*WCL*)
```

Theorem 16. max _ WCℓ *is approximable within ratios on level*

$$\rho_{\max\_\mathrm{WC}\ell}(G) \geqslant \min\left\{\frac{\log n}{\Delta \log\log n}, O\left(n^{-8/9}\right)\right\}.$$

Proof. If $\ell \leqslant \Delta^{1/9}$, then following expression (18), the solution computed by algorithm WCL in line (4) guarantees ratios on approximation level

$$\min\left\{\frac{\log n + \log \Delta^{1/9}}{3\left(\Delta + \Delta^{1/9}\right)\log\left(\log n + \log \Delta^{1/9}\right)}, O\left(\frac{1}{n^{4/5}n^{\frac{1}{9}\frac{4}{5}}}\right)\right\} \overset{n \geqslant n_0}{\geqslant} \min\left\{\frac{\log n}{6\Delta \log\log n}, O\left(\frac{1}{n^{8/9}}\right)\right\} \quad (19)$$

On the other hand, if $\ell > \Delta^{1/9}$, let us prove that the total weight of the solution computed at line (6) verifies

$$\lambda(G) \geqslant \Delta^{1/9}\frac{w(V)}{\Delta} \quad (20)$$

Consider first that the heaviest of the non-chosen colors (let us denote by $w^r_{\max}$ its weight) has weight at least $w(V)/\Delta$. Then, since the colors chosen by WCL are heavier than it, expression (20) holds. Suppose now that $w^r_{\max} \leqslant w(V)/\Delta$ and denote by W the total weight of the colors chosen at line (6). Then, it is easy to see that $w(V) \leqslant W + (\Delta - \Delta^{1/9})w^r_{\max}$ and on the hypothesis $w^r_{\max} \leqslant w(V)/\Delta$ this implies expression (20) which together with the fact that $\beta(G) \leqslant w(V)$, introduces an approximation ratio $\Delta^{-8/9}$. Then, comparison of this ratio with the one of expression (19) concludes the proof of the theorem. ∎

Of course, the results of theorem 16 hold also for max_Cℓ.

13 Minimum chromatic sum

Let us consider the following standard "excavation schema" originally introduced in [32]. In such a schema one solves a minimization graph-problem $\Pi_{\min}$ by iteratively solving a maximization sub-problem $\Pi_{\max}$ and by removing from the input-graph the subsequent solutions of the latter.

Consider min_CHS instead of $\Pi_{\min}$ and max_IS instead of $\Pi_{\max}$ and let IS be any max_IS-algorithm. We then have the following instantiation of the excavation schema.

```
BEGIN (*EXCAVATION*)
    i ← 0;
    WHILE V ≠ ∅ DO
        i ← i + 1;
        C_i ← IS(G);
        V ← V \ C_i;
        remove from E the edges adjacent to vertices of C_i;
    OD
```

14 Minimum coloring

In what follows, we denote by $\chi(G)$ the chromatic number of G. It is well-known ([6]) that

$$\alpha(G)\chi(G) \geqslant n \tag{21}$$

14.1 An algorithmic chain for minimum coloring with improved standard-approximation ratio

Revisit the excavation schema presented in section 13 and consider min_C instead of $\Pi_{\min}$ and max_IS instead of $\Pi_{\max}$. As it is proved in [32], for the case where $\Pi_{\max}$ is optimally solved, and in [2], for the case where it is approximately solved, if $\sigma_{\min}(G)$ and $\sigma_{\max}(G)$ are the standard-approximation ratios of $\Pi_{\min}$ and $\Pi_{\max}$, respectively ($\sigma_{\max}(G) = 1$ for the case where the latter is optimally solved), then

$$\sigma_{\min}(G) \geqslant \frac{\sigma_{\max}(G)}{\ln n} \tag{22}$$

Revisit also theorem 7 of section 10.3, consider $\ell(n)$ constant and denote it by ℓ. Denote by EXHAUST, an exhaustive-search algorithm for min_C and, as in section 12.2, by Δ(G)_COLOR the $\Delta(G)$-coloring algorithm of [36]. Without loss of generality we suppose that vertices are colored by $1, 2, \ldots$ Moreover, let K and C be as in theorem 7 and denote by $|G|$ the quantity $|V(G)|$.

```
BEGIN (*COLOR*)
    IF n ⩽ C THEN OUTPUT EXHAUST(G) FI
    S ← LARGEIS(G);
    i ← 1;
    X̂ ← ∅;
    WHILE |S| ⩾ Klog^ℓ|G| DO
          color S by color i;
          X̂ ← X̂ ∪ {i};
          i ← i + 1;
          G ← G[V \ S];
          IF G = ∅ THEN OUTPUT X̂ FI
          S ← LARGEIS(G);
    OD
    X̄ ← Δ_COLOR(G);
    OUTPUT X ← X̂ ∪ X̄;
END. (*COLOR*)
```

It is easy to see that the WHILE-loop of algorithm COLOR is nothing else than an application of algorithm EXCAVATION of section 13 with LARGEIS in the place of IS. Observe also that, for every iteration i of the WHILE-loop, if we denote by G_i the graph – input of iteration i ($G_1 = G$) and by n_i its order, then

$$\sigma_{\texttt{LARGEIS}}(G_i) \geqslant \frac{K \log^{\ell} n_i}{n_i} \tag{23}$$

Denote now by $\hat{G}$ the subgraph of G induced by the union of the independent sets S colored during the executions of the WHILE-loop, and by $\hat{n}$ its order. Then, by expressions (22) and (23):

$$\sigma_{\texttt{WHILE}}\left(\hat{G}\right) \geqslant \frac{K \log^{\ell-1} \hat{n}}{\hat{n}} \tag{24}$$

Let $\tilde{G}$ be the subgraph of G input of algorithm Δ_COLOR (i.e., $\tilde{G} = G[V \setminus V(\hat{G})]$), $\tilde{n}$ be its order and $\Delta(\tilde{G})$ be its maximum degree. Observe that, following theorem 7,

$$\alpha\left(\tilde{G}\right) < \frac{\ell \tilde{n} \log\log \tilde{n}}{\log \tilde{n}} \overset{(21)}{\Longrightarrow} \chi\left(\tilde{G}\right) \geqslant \frac{\log \tilde{n}}{\ell \log\log \tilde{n}} \tag{25}$$

Application of Δ_COLOR in $\tilde{G}$ will compute a set $\tilde{X}$ of colors verifying (using expression (25))

$$\sigma_{\Delta\_\mathtt{COLOR}}\left(\tilde{G}\right) \geqslant \frac{\frac{\log \tilde{n}}{\ell \log\log \tilde{n}}}{\Delta\left(\tilde{G}\right)} \geqslant \frac{\log \Delta\left(\tilde{G}\right)}{\Delta\left(\tilde{G}\right) \ell \log\log \Delta\left(\tilde{G}\right)} \tag{26}$$

Using expressions (24) and (26), the following holds for the set X of colors computed by algorithm COLOR:

$$\begin{aligned}
|X| &= \left|\hat{X}\right| + \left|\tilde{X}\right| \leqslant \frac{\chi\left(\hat{G}\right)}{\sigma_{\mathtt{WHILE}}\left(\hat{G}\right)} + \frac{\chi\left(\tilde{G}\right)}{\sigma_{\Delta_\mathtt{COLOR}}\left(\tilde{G}\right)} \\
&\leqslant \max\left\{\frac{1}{\sigma_{\mathtt{WHILE}}\left(\hat{G}\right)}, \frac{1}{\sigma_{\Delta_\mathtt{COLOR}}\left(\tilde{G}\right)}\right\}\left(\chi\left(\hat{G}\right) + \chi\left(\tilde{G}\right)\right) \\
&\leqslant \max\left\{\frac{1}{\sigma_{\mathtt{WHILE}}\left(\hat{G}\right)}, \frac{1}{\sigma_{\Delta_\mathtt{COLOR}}\left(\tilde{G}\right)}\right\} 2\chi(G)
\end{aligned} \tag{27}$$

Hence, by expression (27) one gets

$$\sigma_{\mathtt{COLOR}}(G) \geqslant \min\left\{\frac{K \log^{\ell-1} n}{2n}, \frac{\log \Delta(G)}{2\Delta(G)\ell \log\log \Delta(G)}\right\}.$$

In all, the following theorem has been proved in this section.

Theorem 20. $\exists k > 0$, *such that for every fixed constant* $\ell > 0$, min_C *can be approximately solved by a PTAC within ratios on approximation level*

$$\sigma_{\mathsf{min\_C}}(G) \geqslant \min\left\{\frac{k \log^{\ell-1} n}{n}, \frac{\log \Delta(G)}{2\ell\Delta(G) \log\log \Delta(G)}\right\}.$$

Let $\ell > 5$. In both cases theorem 20 improves either, by a factor $O((\log^2 \log n) \log^{\ell-4} n)$, the ratio of [26], or, by a factor $O(\log \Delta(G) / \log\log \Delta(G))$, the ratio of [36].

One can further improve the ratio in theorem 20 by the following way. Denote by COLORING the algorithm of [26] achieving approximation ratio $O(\log^3 n/(n \log^2 \log n))$ for min_C and recall that n is the order of the input-graph of COLOR. Replace line (13) of algorithm COLOR by the following instruction-block.

```
IF |G| ⩾ n/log^(ℓ-3)n
   THEN X̃ ← Δ_COLOR(G);
   ELSE X̃ ← COLORING(G);
FI
```

Corollary 6. *$\forall \epsilon > 0$, min_C can be solved by a SPTAC within ratio on level*

$$\sigma_{\text{min\_C}}(G) \geqslant O\left(\min\left\{\frac{1}{n^{\epsilon}}, \frac{\log n}{\Delta(G)\log\log n}\right\}\right)$$

The left-hand member of the ratio-expression obtained in corollary 6 is slightly better than the one in corollary 4. On the other hand, the time-complexity of algorithm COLOR in section 14.1 is much smaller than the one of C given that the de-randomization techniques of [38] imply important execution times.

15 FP-reductions between standard and differential approximation

As we have shown in many papers ([14, 16, 13, 39]) results obtained in standard and differential approximations can be very different the ones from the others, even for the same problem. However, this does not mean that bridges between the two thought processes do not exist. Such bridges exist and allow transfers of positive or negative approximation results from one theory to the other. They can be seen as FP-reductions preserving, or prohibiting, some approximation levels.

15.1 Sufficient conditions for transferring results between standard and differential approximation

We first note that positive (resp., inapproximability) approximation results are immediately transferred from differential (resp., standard) to standard (resp., differential) approximation in the case of maximization problems. In fact, consider any differential PTAA A guaranteeing differential-approximation ratio δ for every instance I of a maximization problem Π. Then,

$$\frac{\lambda_{\mathtt{A}}(I) - \omega(I)}{\beta(I) - \omega(I)} \geqslant \delta \Longrightarrow \lambda_{\mathtt{A}}(I) \geqslant \delta\beta(I) + (1-\delta)\omega(I) \overset{\substack{\delta \leqslant 1 \\ \omega(I) \geqslant 0}}{\Longrightarrow} \frac{\lambda_{\mathtt{A}}(I)}{\beta(I)} \geqslant \delta.$$

In what follows, we refine the above by giving some sufficient conditions in order that approximation results are transferred between the two thought processes considered, even in the case of minimization problems. We first prove the following theorem.

Theorem 22. *Consider any NP-hard problem $\Pi = (\mathcal{I}, \mathrm{S}, v_I, \mathrm{opt})$ and an instance $I \in \mathcal{I}$. Let $\mathcal{D}$ and $\mathcal{S}$ draw the sets of possible approximation levels for the differential and standard approximations, respectively.*

1. *If $\exists \epsilon > 0$ such that $|\omega(I) - \beta(I)| \leqslant \epsilon \min\{\omega(I), \beta(I)\}$, then there exists an FP-reduction $\Pi \overset{g}{\prec} \Pi$ such that f and h are the identity functions and*

$$g : \begin{cases} \mathcal{D} & \rightarrow & \mathcal{S} \\ \delta & \mapsto & \begin{cases} \frac{1}{1+\epsilon(1-\delta)} & \mathrm{opt} = \min \\ \frac{1+\epsilon\delta}{1+\epsilon} & \mathrm{opt} = \max \end{cases} \end{cases}$$

2. *If $\exists \epsilon > 0$ such that $|\omega(I) - \beta(I)| \geqslant \epsilon \max\{\omega(I), \beta(I)\}$, then there exists an FP-reduction $\Pi \overset{g}{\prec} \Pi$ such that f and h are the identity functions and*

$$g : \begin{cases} \mathcal{S} & \rightarrow & \mathcal{D} \\ \sigma & \mapsto & \begin{cases} \frac{\sigma+\epsilon-1}{\sigma\epsilon} & \mathrm{opt} = \min \\ \frac{\sigma+\epsilon-1}{\epsilon} & \mathrm{opt} = \max \end{cases} \end{cases}$$

Proof. We prove the theorem in the case of a minimization problem, the case of maximization being completely analogous. For item 1, after some easy algebra, one can easily see that

$$\frac{\omega(I) - \lambda(I)}{\omega(I) - \beta(I)} \geqslant \delta \Longrightarrow \frac{\beta(I)}{\lambda(I)} \geqslant \frac{1}{\delta + (1-\delta)\frac{\omega(I)}{\beta(I)}} \tag{30}$$

Using the hypothesis of item 1, one gets the standard-approximation level claimed.

Item 2 is proved with exactly the same arguments. ∎

For example, we devise in [39] a differential PTAA for max_TSP12 guaranteeing differential-approximation ratio $\delta \geqslant 3/4$. Observe that, for every instance K_n of max_TSP12, $\beta(K_n) - \omega(K_n) \leqslant 2n - n = n$. Then, application of item 1 of theorem 22 with $\delta = 3/4$ and $\epsilon = 1$ leads to the following corollary improving, for the case of max_TSP12, the standard ratio 3/4 presented in [45].

Corollary 7. *([39]).* max_TSP12 *is approximable within standard-approximation ratio bounded below by* 7/8.

The counter-part of item 1 (resp., item 2) of theorem 22 is that if there exists $\epsilon' > 0$ such that no algorithm for Π guarantees standard- (resp., differential-) approximation ratios in level $1 - \epsilon'$ (ϵ' is commonly called approximation threshold), then there exists $\epsilon'' > 0$ such that no algorithm guarantees differential- (resp., standard) approximation ratios in level $1 - \epsilon''$. For instance, if opt = min, then, by item 1, a standard-approximation threshold ϵ' implies a differential-approximation threshold $\epsilon'' = 1 - (\epsilon'/(\epsilon(1-\epsilon')))$.

For min_TSP12, using the standard-approximation result of [20][2] and item 1 with $\epsilon = 1$, the following holds.

Corollary 8. *([39]). Unless* **P=NP**, *no DPTAC can approximately solve* min_TSP12 *within ratio converging to* 5379/5380.

15.2 Bin packing

In [15], we propose another FP-reduction (not implied by theorem 22) between standard and differential approximations for BP. Next, based upon this reduction, we devise a differential PTAS for BP (recall that in standard approximation process, BP can be approximated only by *asymptotic* PTAS ([22])). In all, the following is proved in [15].

Theorem 23. *([15]).*

- *There exists an FP-reduction for* BP *transforming any constant standard-approximation ratio σ into differential-approximation ratio* $1/(2-\sigma)$. *Consequently, it transforms approximation level* ***S-APX*** *into approximation level* ***D-APX***;
- BP ∈ D − PTAS.

15.3 Minimum vertex-covering

Let us now denote by min_VC($2n/3$), the restriction of min_VC in graphs where the size of a minimum vertex cover is bounded above by $2n/3$ (of course, such graphs are not recognizable in polynomial-time). The following theorem is proved in [17].

[2] $\forall\epsilon' > 0$, no PTAA can guarantee standard-approximation ratio less than, or equal to, $5380/5381 - \epsilon'$ unless **P=NP**.

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