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NUMERICAL REPRESENTATION OF *PQI* INTERVAL ORDERS

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Représentation numérique d'ordres d'intervalle *PQI*

Résumé

Nous considérons le problème de la représentation numérique des ordres d'intervalle *PQI*. Une structure de préférence sur un ensemble fini A qui contient les relations P (préférence stricte), Q (préférence faible) et I (indifférence) est un ordre d'intervalle *PQI* si chaque élément de l'ensemble A est représentable par un intervalle de façon que la relation P est vraie si un intervalle est complètement à droite de l'autre, la relation I est vraie si un intervalle est inclus dans l'autre et la relation Q est vraie si un intervalle est à droite de l'autre mais leur intersection n'est pas vide. Cette structure a été axiomatisée seulement récemment. Dans ce papier, nous analysons des concepts tels que la magnitude, la matrice caractéristique ou le graphe synthétique. Enfin, nous présentons deux algorithmes : le premier en $O(n^2)$ pour la détermination d'une représentation générale, le deuxième en $O(n)$ pour trouver une représentation minimale.

Mots clés : Intervalles, Ordres d'intervalle *PQI*, Représentation numérique, Représentation minimale.

Numerical representation of *PQI* interval orders

Abstract

We consider the problem concerning the numerical representation of *PQI* interval orders. A preference structure on a finite set A with three relations P , Q , I standing for "strict preference", "weak preference" and "indifference", respectively, is defined as a *PQI* interval order if there exists a representation of each element of A by an interval in such a way that P holds when one interval is completely to the right of the other, I holds when one interval is included to the other and Q holds when one interval is to the right of the other, but they do have a non empty intersection (Q modelling the hesitation between P and I). Only recently necessary and sufficient conditions for a *PQI* preference structured to be identified as a *PQI* interval order have been established. In this paper, we are interested in the problem concerning the representation of a *PQI* interval order, particularly numerical representations. We will investigate some concepts aiming at characterising a *PQI* interval order such as: magnitude, characteristic matrix, synthetic graph (SG). Finally, we present two algorithms, the first one in $O(n^2)$ to determine a general numerical representation, and the second one, in $O(n)$, to minimise this representation.

Keywords: Intervals, *PQI* interval orders, Numerical representation, Minimal representation.

1 Introduction

In preference modelling and decision support we often have to compare intervals instead of discrete values. This is due to the fact that the comparison of alternatives is usually realised through their evaluations on numerical scales, subject to the unavoidable lack of precision and certainty. The conventional structure adopted in order to compare two intervals, considers that “ x is preferred to y ” ($P(x, y)$) iff the interval associated to x is completely to the “right” (in the sense of the line representing the reals) of the interval associated to y . In all other cases “ x is indifferent to y ”. Such a model (where indifference is not transitive) may conceal the fact that “ x being to the right of y ” (the intersection being not empty) is a situation intuitively different from the case where one interval (let’s say x) is included in the other (let’s say y). The second case can be considered a “sure indifference” as much as can be considered a “sure preference” the case $P(x, y)$. Under such a perspective the first case is a situation of hesitation between preference and indifference which merits to be considered separately (see Tsoukiàs and Vincke, 1997). We may denote such a situation as “weak preference” and represented it as $Q(x, y)$.

The *PQI* interval order has been discussed since 1988 by Vincke. The problem of characterising such a structure was left open until recently. Tsoukiàs and Vincke, 2000, presented a theorem providing necessary and sufficient conditions for a *PQI* preference structure to be identified as a *PQI* interval order. The operational problem of detecting if a given *PQI* preference structure satisfies the conditions of the theorem was solved in Ngo The et al., 2000, through an algorithm which is demonstrated to run in polynomial time.

In this paper, we are interested in the problem of the numerical representation of a *PQI* interval order. For this purpose, our paper is dedicated to investigate some aspects of the representation of a *PQI* interval order (once detected). We introduce and study some concepts aiming to characterise a *PQI* interval order such as: magnitude, characteristic matrix, synthetic graph (SG). These theoretical results lead to two algorithms: the first one is to determine a general representation and the second one a minimal one.

The paper is organised as follows. Section 2 provides the basic notations and definitions. In section 3 we recall some definitions and previous results concerning the numerical representation of interval orders. Section 4 is dedicated to *PQI* interval orders. Section 5 gives the two algorithms to construct a general representation of a *PQI* interval order and to minimise this representation. Some conclusions are given at the end of the paper.

2 Basic notations, definitions and results

Further on, if not indicated differently, all the relations under consideration are binary relations defined on a finite set A and denoted by P, Q, I, R, S, T . The fact that $(x, y) \in S$ is denoted either by $S(x, y)$ or xSy . We adopt the following notation.

$$\begin{aligned}
 S^{-1} &= \{(x, y) : S(y, x)\} \\
 S^c &= \neg S = \{(x, y) : \neg S(x, y)\} \\
 S^d &= \neg S^{-1} = \{(x, y) : \neg S(y, x)\} \\
 S^\sim &= A^2 \setminus (S \cup S^{-1}) \\
 S^\approx &= \{(x, y) : \forall z, S(x, z) \Leftrightarrow S(y, z) \text{ and } S(z, x) \Leftrightarrow S(z, y)\} \\
 S \subset T &: \forall x, y, S(x, y) \Rightarrow T(x, y) \\
 S.T &= \{(x, y) : \exists z, S(x, z) \wedge T(z, y)\} \\
 S^2 &= S.S \\
 S \cup T &= \{(x, y) : S(x, y) \vee T(x, y)\} \\
 S \cap T &= \{S(x, y) \wedge T(x, y)\} \\
 S^+(a) &= \{x \in A : S(a, x)\}.
 \end{aligned}$$

If R is an equivalence relation on A then the equivalence class containing $a \in A$ is denoted by $[a]_R$. When there is no ambiguity, we can use simply $[a]$. A binary relation R on a finite set $A = \{a_1, a_2, \dots, a_n\}$ can be represented by an $n \times n$, 0 – 1 matrix M^R with $M_{ij}^R = 1$ iff $(a_i, a_j) \in R$. Further on we use the following definitions (see Roubens and Vincke, 1985).

Definition 2.1 A binary relation S is:

- a partial order iff it is asymmetric and transitive;
- a weak order iff it is asymmetric and negatively transitive;
- a linear order iff it is irreflexive, complete and transitive;
- an equivalence relation iff it is reflexive, symmetric and transitive.

It is easy to verify that

- $S^\sim = S^c \cap S^d$
 - $S^\approx = \{(x, y) : \forall z, S^\sim(x, z) \Leftrightarrow S^\sim(y, z)\}$
 - a weak order is also a partial order and a linear order is also a weak order.
- Let's introduce now the concept of rank function.

Definition 2.2 Let S be a linear order on a finite set A . Its rank function is defined as:

$$\begin{aligned}
 g : A &\mapsto \mathcal{N} \\
 g(a) &= |S^+(a)| + 1
 \end{aligned}$$

We have the two following fundamental results from Fishburn 1985:

Theorem 2.1 *If S is a partial order then*

- i) $S^\sim$ is an equivalence relation;
- ii) $S = S.S^\sim = S^\sim.S$;
- iii) $S^\sim(x, y) \Leftrightarrow \{z : S(x, z)\} = \{z : S(y, z)\}$ and $\{z : S(z, x)\} = \{z : S(z, y)\}$;
- iv) $(A/S^\sim, S)$ is a partial order;

Theorem 2.2 *If S is a partial order then the following are equivalent:*

- i) S is a weak order;
- ii) $S^\sim$ is transitive;
- iii) $S^\sim = S^\sim$;
- iv) $S = S.S^\sim = S^\sim.S$;
- v) $(A/S^\sim, S)$ is a linear order;

In addition, S is a linear order iff $S^\sim$ is the identity relation $I_0 = \{(x, x) : x \in A\}$

In this paper we will consider relations representing strict preference, weak preference and indifference, respectively denoted as P, Q, I . Such relations satisfy some “natural” properties announced in the following two definitions.

Definition 2.3 *A $\langle P, I \rangle$ preference structure on a set A is a couple of binary relations, defined on A , such that:*

- I is reflexive and symmetric;
- P is asymmetric;
- $I \cup P$ is complete;
- P and I are mutually exclusive ($P \cap I = \emptyset$).

By definition, a $\langle P, I \rangle$ preference structure is perfectly characterised by P or $S = P \cup I$. This means that we can represent it by a matrix just as the case of a binary relation.

Definition 2.4 *A $\langle P, Q, I \rangle$ preference structure is a triple of binary relations, defined on A , such that:*

- I is reflexive and symmetric;
- P and Q are asymmetric;
- $I \cup P \cup Q$ is complete;
- P, Q and I are mutually exclusive ($P \cap Q = P \cap I = Q \cap I = \emptyset$).

In this case, we have three relations. As they are mutually exclusive, we also can represent it by a matrix M^{PQI} with $M_{ij}^{PQI} = X$ where $X = P, Q, I$ and $X(a_i, a_j), \forall a_i, a_j \in A$.

The notion "ex æquo" is formally defined as follows:

Definition 2.5 *The equivalence relation associated to a set of relations $B = \{P, Q, R, \dots\}$ defined on a set A is the binary relation E , defined on the set A , such that, $\forall x, y \in A : E(x, y)$ iff $\forall z \in A : R(x, z) \Leftrightarrow R(y, z), R$ or $R^{-1} \in B$.*

All $x, y \in A$ such that $E(x, y)$ are called "ex æquo" w.r.t. B . When there is no ambiguity, B is not mentioned.

By definition, it is obvious that $E = P^\approx \cap Q^\approx \cap R^\approx \dots$. Particularly, $E = P^\approx$ in a $\langle P, I \rangle$ preference structure and $E = P^\approx \cap Q^\approx$ in a $\langle P, Q, I \rangle$ preference structure.

A useful tool to study the minimal numerical representation of preference structures is the potential function in a valued graph. Let $G = (A, U, v)$ be a valued graph on a finite set of nodes A ; a real value $v(a, b)$ is attached to each arc (a, b) of U .

Definition 2.6 *A potential function of the valued graph $G = (A, U, v)$ is a function $g : A \mapsto \mathcal{R}$ such that, $\forall (a, b) \in U, g(a) \geq g(b) + v(a, b)$.*

It is easy to see that if g is a potential function whose minimal value is 0, then $g(a)$ cannot be smaller than the maximal value of the paths starting from a . A fundamental result is the following (Roy 1969).

Theorem 2.3 *A valued graph admits potential functions iff there is no circuit of strictly positive value in the graph. The smallest non-negative potential function assigns to each node the maximal value of the paths starting from the node.*

3 Interval orders

Definition 3.1 *A $\langle P, I \rangle$ preference structure on a finite set A is an interval order iff $\exists l, r : A \mapsto \mathcal{R}^+$ such that, $\forall x, y \in A$:*

- i) $r(x) \geq l(x)$;
- ii) $P(x, y) \Leftrightarrow l(x) > l(y)$;
- iii) $I(x, y) \Leftrightarrow l(x) \leq r(y)$ and $l(y) \leq r(x)$;

Any couple (l, r) satisfying the above conditions is a general representation of the interval order.

For a finite set A , definition 3.1 is equivalent to the condition $P.I.P \subset P$ which is an alternative definition of an interval order (see Fishburn 1985).

Since A is finite, given a general representation (l, r) of an interval order, there exists a positive constant $\epsilon = \min_{(a,b) \in P} \{l(a) - r(b)\}$. The triple (l, r, ϵ) is called an ϵ -representation of the interval order. With an ϵ -representation, condition *ii* of definition 3.1 can be rewritten as:

$$P(x, y) \Leftrightarrow l(x) \geq l(y) + \epsilon.$$

Among all the possible ϵ -representations (with the same ϵ), the minimal ϵ -representation is of special interest. Naturally, it is defined as an ϵ -representation (l^*, r^*, ϵ) satisfying, for any other ϵ -representation (l, r, ϵ) , $\forall a \in A$, $l^*(a) \leq l(a)$ and $r^*(a) \leq r(a)$. The construction of the minimal representation is based on the following results.

Theorem 3.1 *Let $\langle P, I \rangle$ be an interval order on a finite set A , and let $T_l = P.I$, $T_r = I.P$. Then*

- i) T_l, T_r are weak orders on A ;*
- ii) $T_l^\sim, T_r^\sim$ are equivalence relations and T_l, T_r are linear orders on $A/T_l^\sim, A/T_r^\sim$;*
- iii) $E = T_l^\sim \cap T_r^\sim$.*

Proof See Fishburn 1985.

Let define two copies of A , say A_l , and A_r .

We define T_0 on $A_l \cup A_r$ as follows:

- $T_0(a_l, b_l) \Leftrightarrow T_l(a, b)$;
- $T_0(a_r, b_r) \Leftrightarrow T_r(a, b)$;
- $T_0(a_l, b_r) \Leftrightarrow P(a, b)$;
- $T_0(a_r, b_l) \Leftrightarrow I(a, b)$ or $P(a, b)$.

Theorem 3.2 *Let $\langle P, I \rangle$ be an interval order on a finite set A , and let T_l, T_r, T_0 defined as above. Then*

- i) T_0 is a weak order on $(A_l \cup A_r)$;*
- ii) $T_0^\sim$ is an equivalence relation and T_0 is a linear order on $(A_l \cup A_r)/T_0^\sim$;*
- iii) $(A_l \cup A_r)/T_0^\sim = (A_l/T_l^\sim) \cup (A_r/T_r^\sim)$;*
- iv) $x \in A_l/T_l^\sim \Rightarrow T_0(y, x)$ for some $y \in A_r/T_r^\sim$,
 $y \in A_r/T_r^\sim \Rightarrow T_0(y, x)$ for some $x \in A_l/T_l^\sim$,
 $T_0(x_1, x_2)$, $x_1, x_2 \in A_l/T_l^\sim \Rightarrow x_1 T_0 y T_0 x_2$ for some $y \in A_r/T_r^\sim$, and,
finally,
 $T_0(y_1, y_2)$, $y_1, y_2 \in A_r/T_r^\sim \Rightarrow y_1 T_0 x T_0 y_2$ for some $x \in A_l/T_l^\sim$.*

Proof See Fishburn 1985.

T_l (T_r) represents the order of the left (right) end points of the intervals associated to elements of A . Each equivalence class in $A/T_l^\sim$, ($A/T_r^\sim$) represents a group of elements whose left (right) end points can be identical. Two elements are ex æquo if both their two end points are identical. T_0 represents the order of all end points. Theorem 3.2 shows that the equivalence classes of left and right end points are alternative, i.e., after a class of left end points there is a class of right end points.

Theorem 3.3 *Let $\langle P, I \rangle$ be an interval order on a finite set A , and T_l, T_r, T_0 defined as above, then*

- i) $A/T_l^\sim$ and $A/T_r^\sim$ have the same cardinality, say m ;
- ii) If $A/T_l^\sim = \{A_m T_0 A_{m-1} T_0 \dots T_0 A_1\}$
and $A/T_r^\sim = \{B_m T_0 B_{m-1} T_0 \dots T_0 B_1\}$ then
($A_l \cup A_r/T_0^\sim$) = $\{B_m, A_m, \dots, B_1, A_1\}$,
and

$$B_m T_0 A_m T_0 B_{m-1} T_0 A_{m-1} \dots T_0 B_1 T_0 A_1$$

Proof See Fishburn 1985.

The construction of the minimal ϵ -representation of an interval order is direct from theorems 2.3, 3.3. The number m is called magnitude of the interval order. With $\epsilon = 1$, the minimal 1-representation is a representation on the smallest possible interval of the set of integer numbers.

Example 3.1

Let's consider the following $\langle P, I \rangle$ interval order.

	a	b	c	d	e	f	g	h
a		P	P	P	P	P	P	P
b			I	P	P	P	P	P
c				I	P	P	P	I
d					P	P	P	I
e						I	I	I
f							P	I
g								I
h								

$$A/T_l^\sim = \{A_1 = \{h, g, e\}, A_2 = \{f\}, A_3 = \{d, c\}, A_4 = \{b\}, A_5 = \{a\}\}$$

$$A/T_r^\sim = \{B_1 = \{g_r\}, B_2 = \{e_r, f_r\}, B_3 = \{d_r, h_r\}, B_4 = \{c_r, b_r\}, B_5 = \{a_r\}\}$$

The 1-minimal representation of the interval order is:

	a	b	c	d	e	f	g	h
l^*	5	4	3	3	1	2	1	1
r^*	5	4	4	3	2	2	1	3

4 PQI interval orders

First, we recall some definitions and fundamental results concerning PQI interval orders.

Definition 4.1 A PQI preference structure on a finite set A is a PQI interval order iff $\exists l, r : A \mapsto \mathcal{R}^+$, such that $\forall x, y \in A$:

- i) $r(x) \geq l(x)$;
- ii) $P(x, y) \Leftrightarrow l(x) > r(y)$;
- iii) $Q(x, y) \Leftrightarrow r(x) > r(y) \geq l(x) > l(y)$;
- iv) $I(x, y) \Leftrightarrow r(x) \geq r(y) \geq l(y) \geq l(x)$ or $r(y) \geq r(x) \geq l(x) \geq l(y)$. A couple (l, r) satisfying these conditions is a general representation of the PQI interval order.

Theorem 4.1 A $\langle P, Q, I \rangle$ preference structure on a finite set A is a PQI interval order iff there exists a partial order I_l such that:

- i) $I = I_l \cup I_r \cup I_0$ where $I_0 = \{(x, x), x \in A\}$ and $I_r = I_l^{-1}$;
- ii) $(P \cup Q \cup I_l).P \subset P$;
- iii) $P.(P \cup Q \cup I_r) \subset P$;
- iv) $(P \cup Q \cup I_l).Q \subset P \cup Q \cup I_l$;
- v) $Q.(P \cup Q \cup I_r) \subset P \cup Q \cup I_r$.

Proof. See Tsoukiàs and Vincke, 2000.

An algorithm to detect a PQI interval order, i.e. to construct I_l , was presented in Ngo The et al., 2000. In this paper, we assume that I_l is known. From the above results, $I_l(x, y)$ iff $r(y) \geq r(x) \geq l(x) \geq l(y)$, and $I_r(x, y)$ iff $r(x) \geq r(y) \geq l(y) \geq l(x)$ with at least one strict inequation.

Since A is finite, there exists

$$\epsilon = \min \left\{ \min_{(a,b) \in P} \{l(a) - r(b)\}, \min_{(a,b) \in Q} \{r(a) - r(b), l(a) - l(b)\} \right\}$$

The triple (l, r, ϵ) is called an ϵ -representation of the PQI interval order. With an ϵ -representation, conditions ii, iii of definition 4.1 can be rewritten as $P(x, y) \Leftrightarrow l(x) \geq l(y) + \epsilon$ and $Q(x, y) \Leftrightarrow r(x) \geq r(y) + \epsilon$ and $r(y) \geq l(x) \geq l(y) + \epsilon$. The minimal ϵ -representation of a PQI interval order is defined similarly to that of the interval order.

The following theorem presents the interval order associated to a PQI interval through the reduction of the two relations I, Q into $\hat{I}$.

Theorem 4.2 *If $\langle P, Q, I \rangle$ is a PQI interval order and $\hat{I} = I \cup Q \cup Q^{-1}$ then $\langle P, \hat{I} \rangle$ is an interval order.*

Proof. See Tsoukiàs and Vincke, 2000.

Let's define the following relations:

$$\hat{T}_l = P.\hat{I};$$

$$\hat{T}_r = \hat{I}.P;$$

We introduce two copies of A , say A_l and A_r and we construct the relation $\hat{T}_0$ on $A_l \cup A_r$ as follows:

$$\hat{T}_0(a_l, b_l) \Leftrightarrow \hat{T}_l(a, b),$$

$$\hat{T}_0(a_r, b_r) \Leftrightarrow \hat{T}_r(a, b),$$

$$\hat{T}_0(a_l, b_r) \Leftrightarrow P(a, b),$$

$$\hat{T}_0(a_r, b_l) \Leftrightarrow \neg P(b, a).$$

Since $\langle P, \hat{I} \rangle$ is an interval order, we can apply theorems 3.1, 3.2, and 3.3 for the relations $\hat{T}_l, \hat{T}_r, \hat{T}_0$. We obtain:

- $m = |A_l/\hat{T}_l^\sim| = |A_r/\hat{T}_r^\sim|$ the magnitude of the interval order $\langle P, \hat{I} \rangle$;
- $(A_l \cup A_r)/\hat{T}_0^\sim = (A_l/\hat{T}_l^\sim) \cup (A_r/\hat{T}_r^\sim)$;
- $A_l/\hat{T}_l^\sim = \{A_m \hat{T}_0 A_{m-1} \hat{T}_0 \dots A_1\}$;
- $A_r/\hat{T}_r^\sim = \{B_m \hat{T}_0 B_{m-1} \hat{T}_0 \dots B_1\}$;
- $B_m \hat{T}_0 A_m \hat{T}_0 B_{m-1} \hat{T}_0 A_{m-1} \dots \hat{T}_0 B_1 \hat{T}_0 A_1$.

We extend now the relations $\hat{T}_l, \hat{T}_r, \hat{T}_0$ into T_l, T_r, T_0 as follows:

$$Q_l = Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l;$$

$$Q_r = Q \cup I_r.Q \cup Q.I_r \cup I_r.Q.I_r;$$

$$T_l = \hat{T}_l \cup Q_l;$$

$$T_r = \hat{T}_r \cup Q_r;$$

$$T_0(a_l, b_l) \Leftrightarrow T_l(a, b),$$

$$T_0(a_r, b_r) \Leftrightarrow T_r(a, b),$$

$$T_0(a_l, b_r) \Leftrightarrow P(a, b),$$

$$T_0(a_r, b_l) \Leftrightarrow \neg P(b, a).$$

It is obvious that $\hat{T}_0 \subset T_0$, as $\hat{T}_l \subset T_l$ and $\hat{T}_r \subset T_r$.

Proposition 4.1 *Let $\langle P, Q, I \rangle$ be a PQI interval order on a finite set A , and let $I_l, I_r, \hat{T}_l, \hat{T}_r, Q_l, Q_r, T_l, T_r$ defined as above. Then*

- i) $Q.I_l \subset Q \cup I_l$ and $I_r.Q \subset I_r \cup Q$;
- ii) $P.I_l \subset P \cup Q \cup I_l$ and $I_r.P \subset P \cup Q \cup I_r$;
- iii) $P.Q^{-1} \subset (P \cup Q \cup I_l)$ and $Q^{-1}.P \subset (P \cup Q \cup I_r)$;
- iv) $Q_l \cap \hat{T}_l^{-1} = Q_r \cap \hat{T}_r^{-1} = \emptyset$;
- v) $P \cup Q \subset T_l \subset I_l \cup P \cup Q$ and $P \cup Q \subset T_r \subset I_r \cup P \cup Q$;
- vi) $(P^{-1} \cup Q^{-1} \cup I_r) \subset \neg T_l \subset (P^{-1} \cup Q^{-1} \cup I_l \cup I_r)$, and $(P^{-1} \cup Q^{-1} \cup I_l) \subset \neg T_r \subset (P^{-1} \cup Q^{-1} \cup I_l \cup I_r)$.

- vii) $T_l.P \subset P$ and $P.T_r \subset P$
- viii) $P.T_l \subset T_l$ and $T_r.P \subset T_r$

Proof

We provide only the proofs for I_l (those of I_r are similar).

- i) $aQbI_lc \Rightarrow [(r(a) > r(b) \geq l(a) > l(b)) \text{ and } (r(c) \geq r(b) \geq l(b) \geq l(c))] \Rightarrow r(c) \geq l(a) > l(c) \Rightarrow (a, c) \in Q \cup I_l$.
- ii) $aPbI_lc \Rightarrow [(l(a) > r(b)) \text{ and } (r(c) \geq r(b) \geq l(b) \geq l(c))] \Rightarrow l(a) > l(c) \Rightarrow (a, c) \in P \cup Q \cup I_l$.
- iii) $aPbQ^{-1}c \Rightarrow [(l(a) > r(b)) \text{ and } (r(c) > r(b) \geq l(c) > l(b))] \Rightarrow l(a) > l(c) \Rightarrow (a, c) \in P \cup Q \cup I_l$.
- iv) Otherwise, $\exists x, (x, x) \in (Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l).\hat{T}_l$. By theorem 4.1 and i, ii we have $(Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l) \subset (Q \cup P \cup I_l)$ and $(Q \cup P \cup I_l).P \subset P$.
We have $(x, x) \in (Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l).\hat{T}_l \subset (Q \cup P \cup I_l).P.\hat{I} \subset P.\hat{I} = \hat{T}_l$, impossible as $\hat{T}_l$ asymmetric.
- v) As $P \subset P.I \subset \hat{T}_l \subset T_l$ and $Q \subset T_l$ then $P \cup Q \subset T_l$.
 $Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l \subset P \cup Q \cup I_l$ (theorem 4.1 and i).
 $\hat{T}_l = P.(I \cup Q \cup Q^{-1})$, $P.I \subset P$, $P.Q \subset P$ and $P.Q^{-1} \subset P \cup Q \cup I_l$ (by iii). Therefore, $T_l \subset P \cup Q \cup I_l$.
- vi) Direct consequence of v.
- vii) $T_l.P \subset P$ and $P.T_r \subset P$
 $T_l.P = P.\hat{I}.P \cup Q.P \cup I_l.Q.P \cup Q.I_l.P \cup I_l.Q.I_l.P \subset P$ (as $I_l.P \subset P$ and $Q.P \subset P$).
- viii) $P.T_l \subset T_l$ and $T_r.P \subset T_r$
 $P.T_l = P.P.\hat{I} \cup P.Q \cup P.I_l.Q \cup P.Q.I_l \cup P.I_l.Q.I_l \subset P.\hat{I} \cup P \cup P.(P \cup \hat{I}) \subset T_l$.

■

For the construction of the minimal ϵ -representation of a PQI interval order, we will extend theorems 3.1, 3.2, 3.3 using T_l, T_r, T_0 .

Theorem 4.3 Let $\langle P, Q, I \rangle$ be a PQI interval order on a finite set A , and let T_l, T_r defined as above. Then

- i) T_l, T_r are weak orders on A ;
- ii) $T_l^\sim, T_r^\sim$ are equivalence relations
and T_l, T_r are linear orders on $A/T_l^\sim, A/T_r^\sim$;
- iii) $T_l^\sim \cap T_r^\sim \subset E$.
- iv) $\forall a \in A: [a]_{T_l^\sim} \subset [a]_{\hat{T}_l^\sim}$ and $[a]_{T_r^\sim} \subset [a]_{\hat{T}_r^\sim}$.

Proof We consider only T_l (T_r is similar).

i) We show that T_l is asymmetric and negatively transitive.

- *Asymmetry.* We recall that if R, S are two asymmetric relations and $R \cap S^{-1} = \emptyset$ then $R \cup S$ is asymmetric.

P, Q, I_l are asymmetric and mutually exclusive $\Rightarrow (P \cup Q \cup I_l)$ is asymmetric $\Rightarrow Q_l \subset (P \cup Q \cup I_l)$ is asymmetric too.

As $\hat{T}_l$ and Q_l are asymmetric, furthermore $Q_l \cap \hat{T}_l^{-1} = \emptyset$ (proposition 4.1.iv), T_l is asymmetric.

- *Negative transitivity.* We recall that the formula:

$$\forall x, y, z \neg T_l(a, b) \wedge \neg T_l(b, c) \rightarrow \neg T_l(a, c)$$

can be reformulated (through simple logical equivalences) as:

$$\forall x, y, z \neg T_l(b, c) \wedge T_l(a, c) \rightarrow T_l(a, b).$$

We will demonstrate this second formulation.

By proposition 4.1.vi, we have $(b, c) \in \neg T_l \Rightarrow (c, b) \in (P \cup Q \cup I_l \cup I_r)$.

Since $T_l \subset P \cup Q \cup I_l$, we consider three cases.

1 - $(a, c) \in P$. Then, if $(c, b) \in (P \cup Q \cup I_r)$, we have $(a, b) \in P \subset T_l$. If $(c, b) \in I_l$ then $(a, b) \in P.I_l \subset \hat{T}_l \subset T_l$.

2 - $(a, c) \in Q$. Then, if $(c, b) \in (P \cup Q)$ we have $(a, b) \in (P \cup Q) \subset T_l$. If $(c, b) \in I_l \Rightarrow (a, b) \in Q.I_l \subset T_l$. If $(c, b) \in I_r \Rightarrow (a, b) \in (P \cup Q \cup I_r)$. If $(a, b) \in (P \cup Q) \Rightarrow (a, b) \in T_l$, otherwise, $(b, a) \in I_l \Rightarrow (b, c) \in I_l.Q \subset T_l$, impossible as $(b, c) \in \neg T_l$.

3 - $(a, c) \in (T_l \setminus (P \cup Q)) \subset I_l$. We also have $(a, c) \in \hat{T}_l \cup Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l \setminus (P \cup Q) \subset P.Q^{-1} \cup P.I_l \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l$ (theorem 4.1, proposition 4.1).

Let's consider different possibilities of (c, b) .

- $(c, b) \in (P \cup Q)$ then $(a, b) \in (I_l.P \cup I_l.Q) \subset (P \cup I_l.Q) \subset T_l$.

- $(c, b) \in I_l$.

We have $(a, b) \in (P.Q^{-1} \cup P.I_l \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l).I_l \subset (P.Q^{-1}.I_l \cup P.I_l.I_l \cup I_l.Q.I_l \cup Q.I_l.I_l \cup I_l.Q.I_l.I_l)$.

By proposition 4.1, $I_r.Q \subset I_r \cup Q \Rightarrow Q^{-1}.I_l \subset Q^{-1} \cup I_l$.

Therefore, $P.Q^{-1}.I_l \subset P.Q^{-1} \cup P.I_l \subset T_l$.

$P.I_l.I_l \subset P.I_l \subset T_l$.

$I_l.Q.I_l \cup Q.I_l.I_l \cup I_l.Q.I_l.I_l \subset I_l.Q.I_l \cup Q.I_l \cup I_l.Q.I_l \subset T_l$.

- $(c, b) \in I_r$.

We consider five possibilities for (a, c) .

- * $(a, c) \in P.Q^{-1} \Rightarrow \exists x \in A, \text{ s.t. } (aPxQ^{-1}c) \text{ and } bI_l cQx \Rightarrow (b, x) \in (P \cup Q \cup I_l)$.

If $(b, x) \in P \Rightarrow bPxQ^{-1}c \Rightarrow (b, c) \in T_l$, impossible as $(b, c) \in \neg T_l$.
 If $(b, x) \in Q \Rightarrow aPxQ^{-1}b \Rightarrow (a, b) \in T_l$.
 If $(b, x) \in I_l \Rightarrow aPxI_r b \Rightarrow aPb \Rightarrow aT_l b$.
 * $(a, c) \in P.I_l \Rightarrow \exists x \in A, \text{ s.t. } (aPxI_l c)$
 If $(b, x) \in P \Rightarrow bPxQ^{-1}c \Rightarrow (b, c) \in T_l$, impossible.
 If $(b, x) \in (Q \cup I \cup P^{-1} \cup Q^{-1}) \Rightarrow aPx[P \cup (Q \cup Q^{-1} \cup I)]b \Rightarrow aPx(P \cup \hat{I})b \Rightarrow (a, b) \in (P.P \cup \hat{T}_l) \subset T_l$.
 * $(a, c) \in I_l.Q \Rightarrow \exists x \in A, \text{ s.t. } (aI_l xQc)$
 If $(b, x) \in (P \cup Q \cup I_l) \Rightarrow (b, c) \in (P \cup Q \cup I_l).Q \subset (P \cup Q \cup I_l.Q) \subset T_l$, impossible.
 If $(b, x) \in I_r \Rightarrow xI_l bI_l c \Rightarrow xI_l c$, impossible as $(x, c) \in Q$.
 If $(x, b) \in (P \cup Q) \Rightarrow aI_l x(P \cup Q)b \Rightarrow (a, b) \in I_l.P \cup I_l.Q \subset P \cup I_l.Q \subset T_l$.
 * $(a, c) \in Q.I_l \Rightarrow \exists x \in A, \text{ s.t. } (aQxI_l c)$
 If $(b, a) \in (P \cup Q \cup I_l) \Rightarrow (b, c) \in (P \cup Q \cup I_l).Q.I_l \subset (P \cup Q \cup I_l).Q.I_l$. But we have $P.Q.I_l \subset P.I_l \subset T_l$, and $Q.Q.I_l \subset (P \cup Q).I_l \subset T_l$ and $I_l.Q.I_l \subset T_l$, then $(b, c) \in T_l$, impossible.
 If $(b, a) \in I_r \Rightarrow (b, x) \in I_r.Q \subset I_r \cup Q$. If $(b, x) \in I_r \Rightarrow (x, b) \in I_l \Rightarrow aQxI_l b \Rightarrow (a, b) \in T_l$, otherwise $(b, x) \in Q \Rightarrow bQxI_l c \Rightarrow (b, c) \in T_l$, impossible.
 Therefore $(a, b) \in P \cup Q \subset T_l$.
 * $(a, c) \in I_l.Q.I_l \Rightarrow \exists x, y \in A, \text{ s.t. } (aI_l xQyI_l c)$
 If $b(P^{-1} \cup Q^{-1})a \Rightarrow aT_l b$.
 If $(b, a) \in (P \cup Q \cup I_l) \Rightarrow (b, c) \in (P \cup Q \cup I_l).I_l.Q.I_l \subset P.I_l.Q.I_l \cup Q.I_l.Q.I_l \cup I_l.Q.I_l$. We have $I_l.Q.I_l \subset T_l$, and $P.(I_l.Q).I_l \subset P.(P \cup Q \cup I_l).I_l \subset P.P.I_l \cup P.Q.I_l \cup P.I_l.I_l \subset P.I_l \subset T_l$, and $Q.(P \cup Q \cup I_l).I_l \subset Q.P.I_l \cup Q.Q.I_l \cup Q.I_l \subset P.I_l \cup Q.I_l \subset T_l$, i.e., $(b, c) \in T_l$, impossible.
 If $(b, a) \in I_r$, we continue to consider the five possibilities of (b, y) .
 • $b(P \cup Q)yI_l \Rightarrow (b, c) \in T_l$, impossible.
 • $bI_l y$.
 If $b(P \cup P^{-1})x$ then we have either (bPa) , impossible as $(bI_r a)$, or (xPa) , impossible too as $xI_r a$.
 If $b(Q \cup I_l)xQyI_l c \Rightarrow (b, c) \in Q.Q.I_l \cup I_l.Q.I_l \subset T_l$, impossible.
 If $bQ^{-1}x \Rightarrow aI_l xQb \Rightarrow aT_l b$.
 If $bI_r x \Rightarrow xI_l bI_l y \Rightarrow xI_l y$, impossible as xQy .
 • $b(P^{-1} \cup Q^{-1})y \Rightarrow y(P \cup Q)b \Rightarrow aI_l xQy(P \cup Q)b \Rightarrow$

$$(a, b) \in I_l Q . P \cup I_l . Q . Q \subset I_l . P \cup I_l . Q \subset T_l.$$

$$b I_r y \Rightarrow y I_l b \Rightarrow a I_l x Q y I_l b \Rightarrow a T_l b.$$

ii) Immediate from theorems 2.1, 2.2 and i.

iii) $T_l^\sim \cap T_r^\sim \subset E$. If $(x, y) \in T_l^\sim \cap T_r^\sim \Rightarrow (x, y) \notin T_l \cup T_l^{-1} \cup T_r \cup T_r^{-1}$.

Suppose that $(x, y) \notin E$ then $\exists z \in A$, $z R_1 x$ and $z R_2 y$ with $R_1 \neq R_2$.

Consider, for example, $R_1 = P$, we have:

$$z P^{-1} y \Rightarrow y P z P x \Rightarrow y T_l x, \text{ impossible.}$$

$$z Q y \Rightarrow y Q^{-1} z P x \Rightarrow y \hat{I} . P x \Rightarrow y T_r x, \text{ impossible.}$$

$$z Q^{-1} y \Rightarrow y Q z P x \Rightarrow y T_l x, \text{ impossible.}$$

The other cases are quite similar.

iv) Immediate from $\hat{T}_l \subset T_l$ and $\hat{T}_r \subset T_r$.

■

Remark 4.1 In general, we don't have $E = T_l^\sim \cap T_r^\sim$. For example, consider the following PQI interval order:

	a	b	c	d
a		I	I	I
b			Q	I
c				I
c				

One of its representations is:

	a	b	c	d
a		L	L	L
b			Q	L
c				L
c				

We have $(a, b) \in E \setminus T_l^\sim \cap T_r^\sim$.

If we assume the absence of ex æquo, then obviously, $T_l^\sim \cap T_r^\sim = E = I_0$.

Theorem 4.4 Let $\langle P, Q, I \rangle$ be a PQI interval order on a finite set A , and $\hat{T}_l, \hat{T}_r, \hat{T}_0, T_l, T_r, T_0$ defined as above, then

i) T_0 is a weak order on $(A_l \cup A_r)$;

ii) $T_0^\sim$ is an equivalence relation and T_0 is a linear order on $(A_l \cup A_r)/T_0^\sim$;

iii) $(A_l \cup A_r)/T_0^\sim = (A_l/T_l^\sim) \cup (A_r/T_r^\sim)$;

Proof.

i) We first demonstrate that T_0 is asymmetric and negatively transitive.

– *Asymmetry.*

$T_0 = (T_0 \cap A_l \times A_l) \cup (T_0 \cap A_r \times A_r) \cup (\hat{T}_0 \cap (A_l \times A_r \cup A_r \times A_l))$, where $(T_0 \cap A_l \times A_l)$ (resp. $(T_0 \cap A_r \times A_r)$) is in fact isomorph to T_l (resp. T_r). As each component of T_0 is asymmetric and belongs to, respectively, $A_l \times A_l$, $A_r \times A_r$, $A_l \times A_r \cup A_r \times A_l$ which are mutually exclusive, T_0 is asymmetric.

– *Negative transitivity.*

$\neg T_0(x, y), \neg T_0(y, z)$. We have the following case with x, y, z .

	x	y	z
1	a_l	b_l	c_l
2	a_l	b_l	c_r
3	a_l	b_r	c_l
4	a_l	b_r	c_r
5	a_r	b_l	c_l
2	a_r	b_l	c_r
3	a_r	b_r	c_l
4	a_r	b_r	c_r

We only have to provide a proof for cases 1,2,3,4.

- * Case 1: $a_l \neg T_0 b_l \neg T_0 c_l \Rightarrow a_l \neg T_l b_l \neg T_l c_l$ (by definition).
 $\Rightarrow a_l \neg T_l c_l$, (T_l is a weak order).
 $\Rightarrow a_l \neg T_0 c_l$, (by definition).
- * Case 2: $a_l \neg T_0 b_l \neg T_0 c_r \Rightarrow a_l \neg T_0 c_r$
i.e. $a \neg T_l b, \neg P(b, c) \Rightarrow \neg P(a, c)$
i.e. $P(a, c), \neg P(b, c) \Rightarrow T_l(a, b)$
where $\neg P = P^{-1} \cup Q \cup Q^{-1} \cup I = P^{-1} \cup \hat{I}$
 $P(a, c), \neg P(b, c) \Rightarrow (a, b) \in P.(P \cup \hat{I}) \subset T_l$
- * Case 3: $a_l \neg T_0 b_r \neg T_0 c_l \Rightarrow a_l \neg T_0 c_l$
i.e. $\neg P(a, b), P(c, b) \Rightarrow \neg T_l(a, c)$
i.e. $T_l(a, c), P(c, b) \Rightarrow P(a, b)$
 $T_l(a, c), P(c, b) \Rightarrow (a, b) \in T_l.P \subset P$, (by proposition 4.1.vii).
- * Case 4: $a_l \neg T_0 b_r \neg T_0 c_r \Rightarrow a_l \neg T_0 c_r$
i.e. $\neg P(a, b), \neg T_r(b, c) \Rightarrow \neg P(a, c)$
Similar to case 2.

ii) Immediate from theorems 2.1, 2.2 and i.

- iii) Consider $[x]_{T_0^\sim}, x \in A_l \cup A_r$. We will demonstrate that “if $x = a_l (x = a_r)$ for some $a \in A$ then $[x]_{T_0^\sim} = [a_l]_{T_l^\sim} ([x]_{T_0^\sim} = [a_r]_{T_r^\sim})$ ”.
 By construction of T_0 , if $\neg T_0(x, y)$ and $\neg T(y, x)$ then $(x, y) \notin A_l \times A_r \cup A_r \times A_l$.
 Suppose now that $x = a_l$, if $y \in [x]_{T_0^\sim}$ then $y = b_l$ for some $b \in A$, and $\neg T_0(a_l, b_l)$ and $\neg T_0(b_l, a_l)$
 $\Leftrightarrow \neg T_l(a_l, b_l)$ and $\neg T_l(b_l, a_l) \Leftrightarrow b_l \in [a_l]_{T_l^\sim}$.
 The case $x = a_r$ is similar.

■

Theorem 4.5 Let $\langle P, Q, I \rangle$ be a PQI interval order on a finite set A , and $\hat{T}_l, \hat{T}_r, \hat{T}_0, T_l, T_r, T_0, m = |A/\hat{T}_l^\sim|, l = |A/T_l^\sim|, r = |A/T_r^\sim|, A/\hat{T}_l^\sim = \{A_i, i = 1..m\}, A/\hat{T}_r^\sim = \{B_i, i = 1..m\}$ defined above, then

i) classes of $A_l/T_l^\sim, A_r/T_r^\sim$ can be arranged in such a way that

$$A_l/T_l^\sim = \underbrace{\{X_l T_0 X_{l-1} T_0 \dots X_{l_1} T_0 X_{l_1-1} T_0 X_{l_1-2} T_0 \dots X_{l_2} \dots}_{A_m} \underbrace{X_{l_{m-1}-1} T_0 X_{l_{m-1}-2} T_0 \dots X_1\}_{A_{m-1}},$$

$$A_r/T_r^\sim = \underbrace{\{Y_r T_0 Y_{r-1} T_0 \dots Y_{r_1} T_0 Y_{r_1-1} T_0 Y_{r_1-2} T_0 \dots Y_{r_2} \dots}_{B_m} \underbrace{Y_{r_{m-1}-1} T_0 Y_{r_{m-1}-2} T_0 \dots Y_1\}_{B_{m-1}};$$

ii) with this arrangement, the linear order T_0 on $(A_l \cup A_r)/T_0^\sim$ becomes:

$$\underbrace{Y_r T_0 Y_{r-1} \dots Y_{r_1} T_0}_{B_m} \underbrace{X_l T_0 X_{l-1} T_0 \dots X_{l_1} T_0}_{A_m} \dots \underbrace{Y_{r_{m-1}-1} T_0 Y_{r_{m-1}-2} T_0 \dots Y_1}_{B_1} \underbrace{T_0 X_{l_{m-1}-1} T_0 X_{l_{m-1}-2} T_0 \dots X_1}_{A_1}.$$

Proof.

i) Immediate from $\forall a \in A, [a]_{T_l^\sim} \subset [a]_{\hat{T}_l^\sim}, T_l \cup T_r \subset T_0$

ii) Immediate from i and theorem 3.3.

■

We can now arrange the elements of $\langle (A_l \cup A_r)/T_0^\sim, T_0 \rangle$ using its rank function (i is the rank of Z_i).

$$\langle (A_l \cup A_r)/T_0^\sim, T_0 \rangle = \{Z_{l+r} T_0 Z_{l+r-1} T_0 \dots Z_1\}.$$

The relation between T_0 and an ϵ -representation is shown in the following proposition.

Proposition 4.2 *Let (l, r, ϵ) be an ϵ -representation of a PQI interval order on a finite set. We have:*

- i) $T_0(a_l, b_l) \Rightarrow l(a) \geq l(b) + \epsilon$;
- ii) $T_0(a_r, b_r) \Rightarrow r(a) \geq r(b) + \epsilon$;
- iii) $T_0(a_l, b_r) \Rightarrow l(a) \geq r(b) + \epsilon$;
- iv) $T_0(a_r, b_l) \Rightarrow r(a) \geq l(b)$;

Proof.

- i) $T_0(a_l, b_l) \Rightarrow T_l(a, b) \Rightarrow (a, b) \in P \cup P.Q \cup P.Q^{-1} \cup P.I_l \cup P.I_r \cup Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l \subset P \cup Q \cup P.I_l \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l$. If aPb then $l(a) \geq r(b) + \epsilon \geq l(b) + \epsilon$.
If aQb then $l(a) \geq l(b) + \epsilon$.
If $aPcI_lb$ then $l(a) \geq r(c) + \epsilon \geq l(c) + \epsilon \geq l(b) + \epsilon$.
If aI_lcQb then $l(a) \geq l(c) \geq l(b) + \epsilon$.
If $aQcI_lb$ then $l(a) \geq l(c) + \epsilon \geq l(b) + \epsilon$.
If aI_lcQdI_lb then $l(a) \geq l(c) \geq l(d) + \epsilon \geq l(b) + \epsilon$.
- ii) Similar to i.
- iii) $T_0(a_l, b_r) \Leftrightarrow P(a, b) \Rightarrow l(a) \geq r(b) + \epsilon$.
- iv) $T_0(a_r, b_l) \Leftrightarrow \neg P(b, a) \Rightarrow r(a) \geq l(b)$.

■

The construction of the minimal ϵ -representation of a PQI interval order is a direct consequence of proposition 4.2 and theorems 4.5, 4.4.

Corollary 4.1 *Given a PQI interval order on a finite set A and a positive constant ϵ , let define*

- $l^*(a) = (i - j + 1)\epsilon$ where $a_l \in Z_i \subset A_j$;

- $r^*(a) = (i - j)\epsilon$ where $a_r \in Z_i \subset B_j$;

where A_j, B_j, Z_i defined as above.

Then (l^, r^*, ϵ) is the minimal ϵ -representation of the PQI interval order and the values of l^* and r^* are integral multiples of ϵ .*

Proof.

We consider the valued graph $G = ((A_l \cup A_r)/T_0^\sim, T_0, v)$ where v is defined as follows:

$$v(x, y) = \begin{cases} 0 & \text{if } x = [a_r], y = [b_l] \text{ for some } a, b \in A \\ \epsilon & \text{otherwise} \end{cases}$$

Since T_0 is a linear order $\Rightarrow$ there is no circuit $\Rightarrow$ there exists a potential function (theorem 2.3). We will prove that the maximal value of the paths

starting from a node a_l (a_r) (being also the smallest potential function) is :

$$g(a_l) = l^*(a)$$

$$g(a_r) = r^*(a)$$

The nodes of G can be presented as $Z_{l+r} T_0 Z_{l+r-1} T_0 \dots Z_1$. Let's remind that $Z_i T_0 Z_j$ iff $i \geq j$ and all the arcs of G are either 0 or $\epsilon > 0$. By proposition 4.2 and theorem 4.5, in two consecutive arcs, there is at least one arc with value ϵ .

For each Z_k , consider the path $\Phi = Z_k T_0 Z_{k-1} \dots T_0 Z_1$ and denote $V(\Phi)$ its value. Any other path Φ' starting from Z_k is obtained from Φ by applying (recursively) the following operation:

- drop out the last arc (x, y) , obviously $V(\Phi) \geq V(\Phi')$ ($v(x, y) \geq 0$).
- replacing a portion $(Z_i, Z_{i-1}, \dots, Z_j)$ by (Z_i, Z_j) . As $V(Z_i, Z_j) \leq \epsilon$ and $V(Z_i, Z_{i-1}, \dots, Z_j) \geq \epsilon$ then $V(\Phi) \geq V(\Phi')$.

Therefore, Φ is the path with maximal value starting from Z_k .

By theorem 4.5, along Φ , all the arcs are ϵ , except (a_r, b_l) which are transitive arcs connecting B_j to A_j . If $Z_i = a_l \in A_j$, then there are $(j-1)$ transitive arcs $\Rightarrow V(\Phi) = (i-j+1) * \epsilon$. If $Z_i = a_r \in B_j$, then there are j transitive arcs $\Rightarrow V(\Phi) = (i-j) * \epsilon$. ■

$M = l+r-m$ is called the magnitude of the PQI interval order as in the minimal ϵ -representation of a PQI interval order, the leftmost endpoint is ϵ and the rightmost endpoint is $M\epsilon$. It is easily to verify that when $l = r = m$, then $Q = \emptyset$, the preference structure in question is an interval order, and its magnitude is $M = m$.

Example 4.1 Let's consider the following PQI interval order (L stand for I_l)

	a	b	c	d	e	f	g	h
a		P	P	P	P	P	P	P
b			Q	P	P	P	P	P
c				Q	P	P	P	Q
d					P	P	P	L
e						Q	P	L
f							Q	L
g								L
h								

We have $m = 5, l = 7, r = 7, M = l + r - m = 9$.

$$A/\hat{T}_l^\sim = \{A_1 = \{h_l, g_l, f_l\}, A_2 = \{e_l\}, A_3 = \{d_l, c_l\}, A_4 = \{b_l\}, A_5 = \{a_l\}\}$$

$$A/(\hat{T})_r^\sim = \{B_1 = \{g_r\}, B_2 = \{f_r, e_r\}, B_3 = \{d_r, h_r\}, B_4 = \{c_r, b_r\}, B_5 = \{a_r\}\}$$

$A/T_l^\sim = \{X_1 = \{h_l, g_l\}, X_2 = \{f_l\}, X_3 = \{e_l\}, X_4 = \{d_l\}, X_5 = \{c_l\}, X_6 = \{b_l\}, X_7 = \{a_l\}\}$

$A/T_r^\sim = \{Y_1 = \{g_r\}, Y_2 = \{f_r\}, Y_3 = \{e_r\}, Y_4 = \{d_r, h_r\}, Y_5 = \{c_r\}, Y_6 = \{b_r\}, Y_7 = \{a_r\}\}$

After the reassignment of indices

$Z_1 = \{h_l, g_l\}, Z_2 = \{f_l\}, Z_3 = \{g_r\}, Z_4 = \{e_l\}, Z_5 = \{f_r\}, Z_6 = \{e_r\}, Z_7 = \{d_l\}, Z_8 = \{c_l\}, Z_9 = \{d_r, h_r\}, Z_{10} = \{b_l\}, Z_{11} = \{c_r\}, Z_{12} = \{b_r\}, Z_{13} = \{a_l\}, Z_{14} = \{a_r\}$

The 1-minimal representation of the PQI interval order is

	a	b	c	d	e	f	g	h
l^*	9	7	6	5	3	2	1	1
r^*	9	8	7	6	4	3	2	6

5 Algorithms

A direct application of the above results in order to determine a minimal ϵ -representation of a PQI interval order is rather complicated as it has to pass by the determination of $\hat{T}_l, \hat{T}_r, T_l, T_r, T_0, (A_l \cup A_r)/T_0^\sim \dots$. In this section, we will present some more results allowing us to determine a minimal ϵ -representation using two algorithms. The first algorithm (in $O(n^2)$) determines a numerical representation where all endpoints are distinct. The endpoints which should be identical will be unified in the second algorithm (in $O(n)$) to obtain a minimal ϵ -representation.

Proposition 5.1 *Let $\langle P, Q, I \rangle$ be a PQI interval order on a finite set A , (l, r) be a representation with all distinct endpoints, $B = \{l(x), r(x), x \in A\}$, the relation T defined on $(A_l \cup A_r)$ as:*

- $T(a_r, a_l)$;
- $T(a_l, b_l) \Leftrightarrow P(a, b) \text{ or } Q(a, b) \text{ or } I_l(a, b)$;
- $T(a_r, b_r) \Leftrightarrow P(a, b) \text{ or } Q(a, b) \text{ or } I_r(a, b)$;
- $T(a_l, b_r) \Leftrightarrow P(a, b)$;
- $T(a_r, b_l) \Leftrightarrow \neg P(b, a)$.

Then :

- i) $T_0 \subset T$, i.e. T is an extension of T_0 .
- ii) $(A_l \cup A_r, T)$ is a linear order and an isomorphism of the linear order $(B, >)$.

Proof.

- i) $(x, y) \in T_0$.

If $x = a_l, y = b_l$ then $(a, b) \in T_l \subset P \cup Q \cup I_r$ then $T(x, y)$. The same

argument for $x = a_r, y = b_r$.

By construction of T and T_0 , if $x = a_l, y = b_r$ or $x = a_r, y = b_l$ then $T(x, y)$.

- ii) It is obvious that $(B, >)$ is a linear order as $l(x), r(x)$ have all distinct values. With the mapping $\phi : A_l \cup A_r \mapsto B$ defined as: $\phi(a_l) = l(a), \phi(a_r) = r(a), \forall a \in A$, it is easily to verify that ϕ is a bijection and $T(x, y) \Leftrightarrow \phi(x) > \phi(y)$.

■

We can consider now the valued graph $(A_l \cup A_r, T, v)$ where $v(x, y) = \epsilon, \forall x, y \in A$. It is obvious that $(l(a) = \epsilon \times g(a_l), r(a) = \epsilon \times g(a_r), \epsilon)$, where g is the rank function of G , is a minimal representation with all endpoints distinct.

From proposition 5.1, we have:

$\forall a_l \in A_l : T^+(a_l) = \{x_l, x_r : P(a, x), x \in A\} \cup \{x_l : Q(a, x), x \in A\} \cup \{x_l : I_l(a, x), x \in A\};$

$\forall a_r \in A_r : T^+(a_r) = \{a_l, x \in A\} \cup \{x_l, x_r : P(a, x), x \in A\} \cup \{x_l, x_r : Q(a, x), x \in A\} \cup \{x_l : Q^{-1}(a, x), x \in A\} \cup \{x_l : I_l(a, x), x \in A\} \cup \{x_l, x_r : I_r(a, x), x \in A\};$

This result leads us to the following formula of the rank function:

$\forall a \in A,$

$g(a_l) = |T^+(a_l)| + 1 = 2|P^+(a)| + |Q^+(a)| + |I_l^+(a)| + 1;$

$g(a_r) = |T^+(a_r)| + 1 = 2 + 2|P^+(a)| + 2|Q^+(a)| + |Q^{-1+}| + |I_l^+(a)| + 2|I_r^+(a)|.$

The function g can be implemented using the following algorithm whose complexity is $\frac{n(n-1)}{2}$, i.e. $O(n^2)$.

```
n=|A|
fl[1..n],fr[1..n] /* g(a_l),g(a_r) */
M[1..n,1..n]; /* matrix representing P,Q,I_l*/
procedure numerical_representation
  forall i
    fl[i]=0
    fr[i]=1
  endfor
  forall i, j, j > i
    switch (M[i,j])
      case P:
        fl[i]=fl[i]+2
        fr[i]=fr[i]+2
      case P-1:
        fl[j]=fl[j]+2
        fr[j]=fr[j]+2
```

```

case Q:
    fl[i]=fl[i]+1
    fr[i]=fr[i]+2
    fr[j]=fr[j]+1
case Q-1:
    fl[j]=fl[j]+1
    fr[j]=fr[j]+2
    fr[i]=fr[i]+1
case Il:
    fl[i]=fl[i]+1
    fr[i]=fr[i]+1
    fr[j]=fr[j]+2
case Ir:
    fl[j]=fl[j]+1
    fr[j]=fr[j]+1
    fr[i]=fr[i]+2
endswitch
endfor

```

Example 5.1 We keep on working with the same example.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	$g(x_l)$	$g(x_r)$
<i>a</i>		<i>P</i>	<i>P</i>	<i>P</i>	<i>P</i>	<i>P</i>	<i>P</i>	<i>P</i>	15	16
<i>b</i>			<i>Q</i>	<i>P</i>	<i>P</i>	<i>P</i>	<i>P</i>	<i>P</i>	12	14
<i>c</i>				<i>Q</i>	<i>P</i>	<i>P</i>	<i>P</i>	<i>Q</i>	9	13
<i>d</i>					<i>P</i>	<i>P</i>	<i>P</i>	<i>L</i>	8	10
<i>e</i>						<i>Q</i>	<i>P</i>	<i>L</i>	5	7
<i>f</i>							<i>Q</i>	<i>L</i>	3	6
<i>g</i>								<i>L</i>	2	4
<i>h</i>									1	11

By definition, $T_0 \subset T$, i.e., T is an extension of T_0 , furthermore, this extension adds only pairs of either type $T(a_l, b_l)$ and $T(a_r, b_r)$ to T_0 . We have seen in the previous section that the minimal ϵ -representation is based on T_0 . The unification of endpoints is indeed a reduction from T to T_0 : two consecutive left (right) endpoints (in T) which are not related by T_0 can be unified. Two consecutive endpoints $a_r T b_l$ can always be unified because $T_0(a_r, b_l)$ requires only $r(a) \geq l(b)$.

Proposition 5.2 Let $\langle P, Q, I \rangle$ be a PQI interval order on a finite set A , T_l, T_r, T as defined above then:

- i) if $a_l, b_l \in T$ are two consecutive endpoints and $T_0(a_l, b_l)$ then $Q(a, b)$;
- ii) if $a_r T b_r$ are two consecutive endpoints and $T_0(a_r, b_r)$ then $Q(a, b)$.

Proof.

- i) If $(a_l, b_l) \in T_0$ then $(a, b) \in T_l = P.\hat{I} \cup Q \cup I_l.Q \cup Q.I_l \cup I_l.Q.I_l$. With the exception of Q , there is always at least an endpoint x such that $a_l T x T b_l$, i.e., a_l, b_l are not consecutive. For example, $(a, b) \in I_l.Q$ then $\exists c \in A, a I_l c Q b$, and we have $a_l T c_l T b_l$. The other cases are similar.
- ii) Similar to i.

■

As a consequence, two consecutive endpoints xTy can be unified if, $\exists a, b \in A$ such that one of the following conditions is satisfied

- $x = a_r, y = b_l$;
- $x = a_l, y = b_l$ and $I_l(a, b)$;
- $x = a_r, y = b_r$ and $I_r(a, b)$.

We obtain the following algorithm in $O(n)$ to unify endpoints:

```

Rank[1..2n]; /* 1..2n rank of element  $x \in A_l \cup A_r$  */
Id[1..2n]; /* identification of element  $x \in A$  */
LR[1..2n]; /* left endpoint, right endpoint */
M[1..n, 1..n]; /* matrix representing  $P, Q, I_l$  */
X=0; /* number of unifications realised, to be subtracted
from the rank to obtain the minimal representation */
procedure minimal_numerical_representation
  for i=1..2n do
    Rank[i]=Rank[i]-X;
    if i=2n then stop endif;
    Rank[i]=Rank[i]-X;
    if [LR[i]=left and LR[i+1]=left and M[Id[i+1], Id[i]] =  $I_l$ ]
    or [LR[i]=right and LR[i+1]=right and M[Id[i+1], Id[i]] =  $I_r$ ]
    or [LR[i]=left and LR[i+1]=right] then
      X=X+1;
    endif;
  endfor;

```

Example 5.2 *We keep on working with the same example.*

<i>Id</i>	<i>Rank</i>	<i>X</i>	<i>Rank - X</i>	<i>observation</i>
h_l	1	0	1	$l, l, I_l(g, h)$
g_l	2	1	1	
f_l	3	—	2	l, r
g_r	4	2	2	
e_l	5	—	3	l, r
f_r	6	3	3	
e_r	7	—	4	
d_l	8	—	5	
c_l	9	—	6	l, r
d_r	10	4	6	$r, r, I_r(a, d)$
h_r	11	5	6	
b_l	12	—	7	l, r
c_r	13	6	7	
b_r	14	—	8	
a_l	15	—	9	l, r
a_r	16	7	9	

The result confirms those from the previous example 4.1.

6 More about the representation of PQI interval order

In this section we examine some other aspects of the representation of a PQI interval orders: the characteristic matrix and the synthetic graph (SG).

6.1 Characteristic matrix

Given a PQI interval order on a finite set A , by theorem 4.5, we have:

$$(A_l/T_l^\sim, T_l) = \{X_l T_l X_{l-1} T_l \dots X_1\};$$

$$(A_r/T_r^\sim, T_r) = \{Y_r T_r Y_{r-1} T_r \dots Y_1\}.$$

With such partitions, each one being a linear order, we define the characteristic matrix M of the PQI interval order as the 0 – 1 matrix defined on $\{(i, j) : i = 1..l, j = 1..r\}$ by:

$$M_{ij} = \begin{cases} 1 & \text{if } \exists x \in A, x \in X_i \cap Y_j \\ 0 & \text{otherwise} \end{cases}$$

Thus $M_{ij} = 1$ if some element $x \in A$ has the left endpoint in X_i and the right endpoint in Y_j .

We can observe that this matrix does not contain enough information to characterise a PQI -interval order as it does not contains information about the order of all (classes of) endpoints. It must be completed by a partition of the rows $(A_l/T_l^\sim)$ as well as of the columns $(A_r/T_r^\sim)$ using

$\hat{T}_l^\sim$ and $\hat{T}_r^\sim$. These two partitions enable the merge (see theorem 4.5 and proposition 4.2) of two linear orders, $(A_l/T_l^\sim, T_l)$ and $(A_r/T_r^\sim, T_r)$, into the linear order $((A_l \cup A_r)/T_0^\sim, T_0) = \{Z_1, Z_2, \dots, Z_{l+r}\}$. The characteristic matrix M completed by $((A_l \cup A_r)/T_0^\sim, T_0)$ totally characterises the associated *PQI* interval order.

The characteristic matrix associated with the example 4.1 is the following one. Here, next to each 1, the corresponding element in A is presented.

	$Z_3, 2$	$Z_5, 3$	$Z_6, 4$	$Z_9, 6$	$Z_{11}, 7$	$Z_{12}, 8$	$Z_{14}, 9$
$Z_1, 1$	1g			1h			
$Z_2, 2$		1f					
$Z_4, 3$			1e				
$Z_7, 5$				1d			
$Z_8, 6$					1c		
$Z_{10}, 7$						1b	
$Z_{13}, 9$							1a

It is easy to verify that if in M , an element 1 corresponding to Z_{l_i} (row) and Z_{r_j} (column) then $l_i < r_j$. Intuitively, as T_0 represents the order of all endpoints, for any interval, its left endpoint must precede its right endpoint. We can also verify that there is at least one 1 in every row and every column.

Conversely, given an $l \times r$ 0 – 1 matrix M , let $X = \{X_1, \dots, X_l\}$ be the set of rows and let $Y = \{Y_1, \dots, Y_r\}$ be the set of columns, two partitions $X = \{A_1, \dots, A_m\}$, $Y = \{B_1, \dots, B_m\}$, then we can enumerate the rows and columns, first by all the row in A_1 , and then all the columns in B_1 , then A_2 , B_2 ... The result of this rearrangement is $Z_1, \dots, Z_{l+r}$. If

- (i) there is at least one 1 in every row and every column of M ;
- (ii) for all element 1 whose row and column correspond respectively to Z_{l_i} and Z_{r_j} , we have $l_i < r_j$,

then M is a characteristic matrix of a *PQI* interval order.

We can also verify that two *PQI* interval orders without "ex æquo" are isomorphic if and only if they have the same characteristic matrix with partitions on rows and columns.

6.2 Synthetic graph (SG)

The idea of the synthetic graph is to find the graph as simple as possible to represent a preference structure. A *PQI* interval order, once identified (I_r determined), can be entirely described by the relations P, Q, I_r . We will procede by eliminating all redundant elements from the above relations in order to obtain simpler ones.

Let us consider the relation T^R on A defined as the restriction of T on A_r , i.e., $T^R = P \cup Q \cdot I_r$, T^R being a linear order (of all right end-points). If we arrange the elements of A by the linear order T^R such that $A = \{a_1 T^R a_2 T^R \dots a_n\}$, its representation matrix has the upper triangle containing only P, Q, I_r . On each row, all the elements on the right of each P are P too. We will define a relation P' containing only the leftmost P on each row, and then preserve P' instead of P . On the left of P' (until the diagonal), we have only Q and I_r . So, if we represent all the I_r elements, the other elements are obviously Q . In fact, we can choose to preserve Q instead of I_r , but here, we prefer I_r because each chain of I_r is a linear order, i.e., the elimination of redundant arcs and the reconstruction of I_r are quite straightforward. These observations will be formalised hereafter.

We define the following relations:

$P'(x, y)$ iff $P(x, y)$ and $\forall z \neq y, P(x, z) \Rightarrow R(y, z)$, P' preserves the leftmost P element on each row (if exists).

$Q' = Q \cap (T^R \setminus T^R \cdot T^R)$, Q' preserves only Q -arcs connecting consecutive elements (ordered by T^R). It is necessary to arrange the elements of A .

$I'_r = I_r \setminus I_r \cdot I_r$

Proposition 6.1 *let $T'^R = T^R \setminus T^R \cdot T^R$. Then $T'^R \subset P' \cup Q' \cup I'_r$.*

Proof. If $(x, y) \in (P \setminus P')$ then $\exists z \neq y, P(x, z)$ and $\neg T^R(y, z)$. Since T^R is a linear order, $\neg T^R(y, z) \Leftrightarrow T^R(z, y)$.

Therefore, $(x, y) \in P \cdot T^R \subset T^R \cdot T^R$, i.e., $(x, y) \neg \in T'^R$.

Obvious for the two other cases $((x, y) \in (Q \setminus Q'))$ or $(x, y) \in I_r \setminus I'_r$. ■

T'^R represent arcs connecting consecutive elements, necessary to arrange the elements of A (ordered by T^R). This results show that we don't have to preserve T'^R as it is included in $(P' \cup Q' \cup I'_r)$.

Proposition 6.2 *The relations P, Q, I_r can be reconstructed from P', Q', I'_r using:*

i) T^R, I_r being the transitive closures of, respectively, $(P' \cup Q' \cup I'_r)$ and I'_r .

ii) $P = P' \cup P' \cdot T^R$.

iii) $Q = (T^R \setminus (P \cup I_r))$

Proof i) and iii) are obvious.

ii) $(P \subset P' \cup P' \cdot R)$

If $P(x, y)$ and $\neg P'(x, y)$ then $\exists z, P(x, z)$ and $T^R(z, y)$.

If $P'(x, z)$ then we have $P' \cdot T^R(x, y)$, otherwise, we can find out another $z', P(x, z')$ and $T^R(z', z)$, or, by transitivity of R , $P(x, z')$ and $T^R(z', y)$. Since A is finite, the process must end up to a point such that we have $P' \cdot T^R(x, y)$

$$(P' \cup P'.T^R \subset P)$$

We have

$$P' \subset P$$

$$P'.T^R \subset P.T^R = P.(P \cup Q \cup I_r) = P.P \cup P.Q \cup P.I_r \subset P.$$

■

Example 6.1

After the rearrangement of elements of A by T^R , we have the following representation matrix (R stands for I_r).

	a	b	c	d	e	f	g	h	i	j	k	l	m	n
a		Q	Q	P	P	P	P	P	P	P	P	P	P	P
b			Q	P	P	P	P	P	P	P	P	P	P	P
c				Q	R	Q	Q	Q	P	P	P	P	P	P
d					R	R	Q	R	Q	P	P	P	P	P
e						Q	Q	P	P	P	P	P	P	P
f							Q	R	Q	P	P	P	P	P
g								R	Q	P	P	P	P	P
h									Q	P	P	P	P	P
i										P	P	P	P	P
j											Q	R	Q	P
k												R	R	R
l													P	P
m														P
n														

Using P', I'_r , the representation matrix become a synthetic representation matrix as follows (as T^{R_l} has already been considered in the order of rows and columns, there is no need to represent Q' in the matrix).

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>	<i>m</i>	<i>n</i>
<i>a</i>				P'										
<i>b</i>				P'										
<i>c</i>					R'				P'					
<i>d</i>					R'	R'				P'				
<i>e</i>								P'						
<i>f</i>								R'		P'				
<i>g</i>								R'		P'				
<i>h</i>										P'				
<i>i</i>										P'				
<i>j</i>												R'		P'
<i>k</i>												R'	R'	R'
<i>l</i>													P'	
<i>m</i>														P'
<i>n</i>														

The synthetic graph is the graph associated to the synthetic representation matrix.

7 Conclusion

In this paper we try to extend some well known results concerning the numerical representation of interval orders in the case of PQI interval orders. Such preference structures appear when, while comparing intervals, it might be interesting to distinguish a situation of hesitation between “sure” preference (empty intersection of the two intervals) and “sure” indifference (one interval included in the other).

The aim of this effort is to find under which foundations it is possible to construct a numerical representation of a PQI interval order as soon as it has been demonstrated that such a representation exists. Not surprisingly we are able to demonstrate that there exist two weak orders, one representing the order of the left extreme points and one representing the order of the right extreme points of the intervals and that on this basis it is possible to construct a numerical representation.

In the paper we demonstrate the theorems which enable to show what the numerical representation of a PQI interval order represents. We also show how it is possible to obtain a “minimal” representation and we provide alternative characterisations of PQI interval orders (in terms of characteristic matrix etc.). With such results we are able to define two algorithms, the first constructing a numerical representation for a given PQI interval order, the second minimising it. Both algorithms are shown to run in polynomial

time ($O(n^2)$ for the first and $O(n)$ for the second).

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