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Equipe de Recherche Associée au C.N.R.S. N° 656

RELATIONAL SYSTEMS OF PREFERENCE WITH ONE OR SEVERAL PSEUDO-CRITERIA : NEW CONCEPTS AND NEW RESULTS

CAHIER N° 28 bis (*)
février 1982

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(*) Version modifiée du CAHIER N° 28.

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ABSTRACT

This paper proposes new concepts and new results which could lead to a more realistic preference modelling than in classical decision theory.

Sections I to III present four fundamental situations of preferences, their combinations and the concept of relational system of preferences. In section IV, a particular case of relational system of preference is studied. It is associated with the concept of pseudo-criterion derived from the classical concept of criterion by adjunction of two thresholds. Some results are given, generalizing the properties of such well-known structures as complete preorders and semiorders.

Sections V and VI emphasize the possibilities given by the preceding concepts to take into account the imprecisions, irresolutions and incomparabilities appearing in every concrete problem where several criteria must be considered.

SYSTEMES RELATIONNELS DE PREFERENCE AVEC UN OU PLUSIEURS PSEUDO-CRITERES : NOUVEAUX CONCEPTS ET NOUVEAUX RESULTATS

RESUME

Ce cahier propose de nouveaux concepts et de nouveaux résultats qui devraient conduire à une modélisation des préférences plus réaliste que celle de la théorie classique de la décision.

Les sections I à III présentent quatre situations fondamentales de préférence, leurs regroupements et le concept de système relationnel de préférence. Un cas particulier important de système relationnel de préférence est étudié section IV. Il s'agit du système associé au concept de pseudo-critère, lequel généralise celui, classique, de critère par l'adjonction de deux seuils. Quelques résultats sont donnés qui généralisent les propriétés des structures bien connues de préordre complet et de quasi-ordre.

Les sections V et VI mettent l'accent sur les possibilités apportées par les concepts précédents afin de prendre en compte les phénomènes d'imprécision, d'indétermination et d'incomparabilité présents dans tout problème concret où plusieurs critères doivent être considérés.

INTRODUCTION

Although it is seldom shown in practice, preference modelling plays a fundamental role in many different disciplines such as operational research, mathematical psychology, economic calculus and decision theory. The resulting theories and methods in these domains are only of practical interest if the preference modelling upon which they are inevitably based is correctly implemented, i.e. designed in a realistic way in relation to the information that can be obtained. It must be recognized that in general, this stage is completely neglected and the models which are usually employed are chosen because of their mathematical qualities. Simplifying hypotheses play a fundamental role in the modelling and moreover are often accepted without comment.

In this article, we would like to draw attention to the restrictive conditions assumed in classical decision theory and suggest an approach which leads to more realism in preference modelling. In practice, it is rare that the considered preferences are those of a single person. Moreover the different actors often have conflicting points of view. It is therefore important that models of collective choice and negotiation are developed to deal with these situations.

We will confine ourselves to modelling a unique actor's preferences. These preferences possibly take into account different points of view. Any individual, community or constituted body playing or not the role of the decision maker in the problem considered is called an actor.

The aim of this article is to present a number of fundamental concepts which, when used by the scientist, will allow him to integrate into his model as realistic a representation as possible of the particular actor's preferences.

It is not possible to develop in this paper the precise use of the presented concepts in decision-aid. For a better understanding of their full implications, the interested reader may refer to [6], [13], [14], [16], [23], [24].

Section I presents the four fundamental preference situations on which modelling is based. Incomparability and weak preference have been added to the usual situations of indifference and strict preference. Furthermore, we have excluded any hypotheses of transitivity of the associated relations. These points are discussed and justified in section II and the combined situations of preference are introduced. Based on the concepts of preceding section, section III introduces the so-called relational systems of preference. We will distinguish between different types of systems according to the preference situations used. The following section is devoted to the important problem of modelling preferences according to a <u>single</u> point of view (eventually aggregating several factors or attributes). We will present several relational systems of preference worthy of interest based on the idea of a pseudo-criterion. The fundamental structure used in this section is that of a pseudo-order. The structure used in classical decision theory is a particular case of a pseudo-order. Finally in section V, we give some examples of an actor's preference model according to <u>several heterogeneous</u> points of view. This model takes into account both the inevitable phenomena of imprecision and irresolution relative to each point of view, and the incomparability which necessarily results from the aggregation of different points of view.

We want to mention here that this paper deviates from the classical decision-making theory and so, from a perception of "rationality" which is accepted without any questions in U.S.A. but which seems to be more and more discussed in Europe.

I - FOUR FUNDAMENTAL PREFERENCE SITUATIONS

Consider an actor who must express his preference in relation to two potential actions (admissible solutions, possible decision). Suppose that the actor is aware of the consequences of these actions. The information he has on these consequences need not necessarily be exhaustive or precise. Classical decision theory ([2], [4], [10]) introduces only two fundamental preference situations which are usually called indifference ($\sim$) and strict preference ($\succ$). Moreover the two relations thus obtained are systematically supposed transitive. This is highlighted in studies containing utility functions, value functions or objective functions. In other words, this theory is based on the following axiom which we will call the complete transitive comparability axiom.

Axiom I.1 : The two incompatible situations of indifference and strict preference are appropriate to build a realistic model of any actor's preferences, irrespective of the potential actions under consideration, the points of view adopted to compare them and the information available. In this model, the actor's preferences can be described through assigning to each pair of potential actions one and only one of these two situations, the two binary relations $\sim$ and $\succ$ thus defined on the set of all potential actions being both transitive.

There are many reasons why the scientist, who models the preferences, ought to try to avoid the dilemma of indifference or strict preference when faced with the comparison of two actions a and b. He might in fact, at that level of modelling or of decision process :

a) Not be able to discriminate between the two : because of its subjectivity of quick elaboration, the information risks being unsuitable to a judgment of indifference or strict preference. To overestimate the significance of certain indices to consolidate a preference or infer from an absence of information an indifference involves taking arbitrary and non-coherent risks.

b) Not know how to discriminate : for certain pairs of actions the scientist may not be in a position to determine the decision maker's preferences. The decision maker may be an inaccessible and remote entity (chief-of-state, president of a large firm) or a fuzzy entity (public opinion) the preferences of which are badly defined and contradictory.

c) Not want to discriminate : to compare a and b means that we must consider on one hand the advantages of a over b, and on the other hand those of b over a without neglecting features common to the two. To be able to distinguish between two actions a and b, we must have at our disposition enough information about the decision maker's preferences or we must introduce volontarist hypotheses in order to arbitrate between antagonistic opinions. The scientist might not wish to take part in the decision process at this particular period in time and wait until a later stage or for complementary information.

For these reasons, we will assume that the comparison of any two potential actions a and b give use to four fundamental situations : indifference, strict preference, weak preference and incomparability which are defined in table 1.

The axiom of complete transitive comparability will be replaced by the following which we will call <u>the axiom of partial comparability</u>.

<u>Axiom I.2</u> : The four fundamental incompatible situations of indifference, strict preference, weak preference and incomparability defined in table 1 are appropriate to build a realistic model of any actor's preferences, irrespective of the potential actionsunder consideration, the points of view adopted to compare them and the available information. In this model, the actor's preferences can be described through assigning to each pair of potential actions either one or a combination of two or three of these four fundamental situations defined in table 2.

The expression "combination of two or three fundamental situations" will be explained and justified in the following section.

ABLE 1 : FUNDAMENTAL SITUATIONS TO WHICH MAY LEAD THE COMPARISON OF TWO POTENTIAL ACTIONS

SITUATION	DEFINITION	BINARY RELATION
Indifference	The two actions are indifferent in the sense that there exist clear and positive reasons to choose equivalence.	I : symmetric reflexive
Strict preference	There exist clear and positive reasons to justify that one (well specified) of the two actions is significatively preferred to the other.	P : asymmetric (irreflexive)
Weak preference	One (well specified) of the two actions is not strictly preferred to the other but it is impossible to say if the other is strictly preferred of indifferent to the first one because neither of the two former situations dominates.	Q : asymmetric (irreflexive)
Incomparability	The two actions are not comparable in the sense that neither of the three former situations dominates.	R : symmetric (irreflexive)

EXPLANATIONS ON SYMBOLISM AND TERMINOLOGY

Given a situation of indifference between two actions a and b, we will write :

$$a\ I\ b \quad \text{or} \quad b\ I\ a$$

ince the relation I is obviously symmetric. It is also reflexive : each action will be onsidered indifferent to itself. There are two possibilities for the situation of strict reference : a is strictly preferred to b (a P b) or b is strictly preferred to a b P a). The relation P is supposed asymmetric, i.e. :

$$a\ P\ b \Rightarrow \text{not } b\ P\ a\ ;$$

t follows therefore that P is irreflexive :

$$\forall\ a \in A, \text{ not } a\ P\ a.$$

The same symbolism and properties hold for the relation of weak preference, the leter Q replacing the letter P. Finally the relation of incomparability R will obiously be symmetric and irreflexive.

II - PROPERTIES AND COMBINATIONS OF FUNDAMENTAL RELATIONS

In what follows, we call "example of candidates" the problem of comparing candidates evaluated on 3 criteria as follows (preferences increasing with the values of the criteria, each one lying between 0 and 20).

Criteria \ Candidates	a_1	a_2	a_3	a_4	a_5	a_6
1	5	13	12	10	12	11
2	12	6	5	10	9	11
3	12	12	14	10	9	8

II.1 The relations of incomparability and weak preference

We are sometimes tempted to question the interest of incomparability since it is necessarily excluded by every decision. This contestation rests on a misunderstanding. The statement a R b simply implies a refusal to adopt a definite position at the level of preference modelling considered about the comparison of the values of a and b. This attitude cannot always be regarded as a proof of indifference between a and b. This is the case for example in a situation involving multiple criteria. The situation a Q b means that the actor might assert not b P a while wavering between a P b and a I b. The information available to him to discriminate between these last two possibilities might not appear to be conclusive. From a theoretical point of view, we could have limited the number of the fundamental situations to three and then introduced weak preference as a combination of the relations P and I. We have not adopted this form because in certain cases weak preference reflects an unyielding situation which merits to be placed at the same level as indifference and strict preference. In other words, it is not certain that additional information, no matter how important it might be, is of a nature to allow the actor to replace Q by P or I.

In the "example of candidates", a_1 and a_2 will be considered as incomparable. However, if the evaluation technique is such that a difference of 1 unity is not really significant, one may consider that a_3 is weakly preferred to a_2.

II.2 The transitivity of fundamental binary relations

Apart from the properties of symmetry and asymmetry mentioned in the preceding paragraph, the four relations I, P, Q and R do not have any distinguishing property. It is usual however to make I and P transitive, i.e. $a\ I\ b,\ b\ I\ c \Rightarrow a\ I\ c$ and $a\ P\ b,\ b\ P\ c \Rightarrow a\ P\ c$. As far as indifference is concerned, Luce's classical example of the cup of coffee ([11]) excellently illustrates how the intransitivity of indifference is presented in a natural way. We will not therefore impose transitivity on the relation I. Neither will we impose transitivity on Q as it is possible to have $a\ Q\ b,\ b\ Q\ c$ and $a\ P\ c$. We might imagine that the strict preference relation is transitive. Although this is usually accepted, it can however be contradicted by the following situation : $a\ P\ b,\ b\ P\ c$ and $a\ R\ c$. Such a case may arise when a and c have enough features in common with b to justify strict preference, but a and c have few common features which leads the actor to reserve his judgment. Therefore we will not suppose P to be necessarily transitive (even if usually it is).

In the "example of candidates", if the evaluation technique is such that a difference of 1 unity is not significant at all while a difference of 2 unities is significant, one may be led to $a_5\ P\ a_4$, $a_6\ P\ a_5$ and $a_4\ P\ a_6$. This situation does not result from "irrationality" but simply means the existence of a difficulty : it is not sure that such a situation implies that the preferences are not consistent.

Finally, it is clear that there is no reason to suppose that the incomparability relation is transitive.

II.3 The combination of fundamental situations

Is is sometimes useful to be able to introduce in the preference modelling, in addition to the relations I, P, Q and R, several binary relations combining two or three of the fundamental situations. The fact that at least two situations are considered for the pair of actions a, b means that the actor cannot, does not know how to or does not want to distinguish between them (at least at this stage of development of the decision process). The fact that two or three of the six possibilities $a \, I \, b$, $a \, P \, b$, $b \, P \, a$, $a \, Q \, b$, $b \, Q \, a$ and $a \, R \, b$ are considered does not mean they are simultaneously correct, but simply that the actor considers it is impossible, premature or useless to isolate the one which is correct. Admittedly it is necessary to determine which of those a priori possible combinations are worthy of interest. Table 2 defines five of these. It follows from the definitions of this table that the relation $\sim$ is symmetric and reflexive, the relation $\succ$ is asymmetric and for all a, b one and only one of the following statements is correct :

$$a \sim b, \; a \succ b, \; b \succ a.$$

The classical theory is based on the two relations $\sim$ and $\succ$ which are supposed transitive (cf. axiom I.1). Relation J must not be confused with the relation J^1 which can be defined as follows :

$$a \, J^1 \, b \quad \text{iff} \quad a \, Q \, b \text{ or } a \, I \, b.$$

The relation J models situations where discrimination between weak preference and indifference is (possibly momentarily) impossible. On the other hand, it follows from the properties of symmetry and asymmetry of I and Q respectively that knowing J^1 leads to a knowledge of I and Q. In fact, I and Q are simply the symmetric and asymmetric parts respectively of the relation J^1. In the case under consideration, the introduction of J^1 therefore would be totally redundant.

Note that :

$$a \, J \, b \text{ and } b \, J \, a \Rightarrow a \, I \, b,$$
$$a \, J \, b \text{ and not } b \, J \, a \Rightarrow a \, I \, b \text{ or } a \, Q \, b.$$

TABLE 2 : FIVE COMBINED SITUATIONS OF PREFERENCE IN THE COMPARISON OF TWO POTENTIAL ACTIONS a, b

SITUATION	DEFINITION	BINARY RELATION
Non-preference	There exist no clear and positive reasons to justify a strict or a weak preference for one of the two actions ; this situation combines indifference and incomparability without discriminating between them.	$\sim$: $a \sim b$ iff $a\ I\ b$ or $a\ R\ b$
Preference	There exist clear and positive reasons to justify a strict or a weak preference for one (well specified) of the two actions over the other ; this situation combines strict and weak preference without discriminating between them.	$\succ$: $a \succ b$ iff $a\ P\ b$ or $a\ Q\ b$
Presumed preference	There exist clear and positive reasons to justify a weak preference for one (well specified) action or an indifference between the two actions but there is no discrimination between the two situations.	J : $a\ J\ b$ only if $a\ Q\ b$ or $a\ I\ b$; $a\ Q\ b \Rightarrow a\ J\ b$, $a\ I\ b \Rightarrow a\ J\ b$ or $b\ J\ a$ (or not exclusive)
K-preference	There exist clear and positive reasons to justify a strict preference for one (well specified) action or an incomparability between the two actions but there is no discrimination between the two situations.	K : $a\ K\ b$ only if $a\ P\ b$ or $a\ R\ b$; $a\ P\ b \Rightarrow a\ K\ b$, $a\ R\ b \Rightarrow a\ K\ b$ or $b\ K\ a$ (or not exclusive)
Outranking	There exist clear and positive reasons to justify a preference or a presumed preference for one (well specified) action but there is no discrimination between strict preference, weak preference and indifference.	S : $a\ S\ b$ iff $a \succ b$ or $a\ J\ b$; thus, $a\ S\ b$ only if $a\ P\ b$ or $a\ Q\ b$ or $a\ I\ b$ $a\ I\ b \Rightarrow a\ S\ b$ or $b\ S\ a$ (or not exclusive)

Similarly a knowledge of K^1 defined by

$a\ K^1\ b$ iff $a\ P\ b$ or $a\ R\ b$

is sufficient to reconstitute P and R, which is not the case for K. In fact, K and J are complementary. It is essentially by this means that K can be introduced into the modelling. The relation S has a particularly simple interpretation : a outranks b means that a is judged at least as good as b. Also

$a\ S\ b$ and $b\ S\ a \Rightarrow a\ I\ b$,
$a\ S\ b$ and not $b\ S\ a \Rightarrow a\ P\ b$ or $a\ Q\ b$ or $a\ I\ b$.

This relation S is used in ELECTRE methods ([14], [16]) which have been applied in many concrete problems in France and in Europe. These applications are not generally published in English. However, the interested readel will find examples in [1], [3], [12], [15], [17], [18], [19].

III - RELATIONAL SYSTEMS OF PREFERENCE

III.1 Definitions

We will consider how the preceding binary relations can be used in conformity with the axiom of limited comparability to model an actor's preferences on a set A.

One way of doing this is to retain one and only one of the four fundamental situations for each pair of actions a and b. This is equivalent to regarding one and only one of the following six statements as being correct : a I b, a P b, b P a, a Q b, b Q a, a R b. In this case, the preference model obtained would be called a fundamental relational system of preference (f.r.s.p.).

Definition III.1 : Given four binary relations I, P, Q and R defined on a set of potential actions A, they constitute a f.r.s.p. of an actor if :

1°) In keeping with table 1 they can be taken as a model of this actor's preferences vis-à-vis the actions of A.

2°) They are exhaustive : for any pair of actions, at least one is satisfied.

3°) They are mutually exclusive : for any pair of actions, at most one is satisfied.

Another method consists in retaining for each pair of actions not necessarily one but possibly two or three of the fundamental situations. In this case, we will use the combined relations defined in the preceding section and will speak of a combined relational system of preference (c.r.s.p.).

Definition III.2 : Given nine binary relations I, R, $\sim$, P, Q, $\succ$, J, K, S defined on a set of potential actions A, they constitute a c.r.s.p. of an actor if :

1°) In keeping with tables 1 and 2 they can be taken as a model of this actor's preferences vis-à-vis the actions of A.

2°) They are exhaustive : for any pair of actions, at least one is satisfied.

3°) They are mutually exclusive : for any pair of actions, at most one is satisfied.

4°) At least one of the five relations $\sim$, $\succ$, J, K, S is not empty.

The expression relational system of preference (r.s.p.) will be used either for f.r.s.p., or for c.r.s.p. The r.s.p. of classical decision theory is of the form (I, P), or ($\sim$, P), or (I, $\succ$), or ($\sim$, $\succ$) : no difference exists between these forms in this theory. Moreover, the two relations are supposed to be transitive. Another type of important r.s.p. is of the form (S, R). ELECTRE methods are based on such r.s.p.

III.2 Remarks

1. A combined relational system of preference is not a juxtaposition of nine binary relations satisfying tables 1 and 2. Such a definition might lead to awkward and redundant models. Condition 3°) of definition III.2 avoids these problems by imposing the following two conditions :

$$a\ H_1\ b \Rightarrow \text{not}\ b\ H_2\ a$$
$$a\ H_1\ b \Rightarrow \text{not}\ a\ H_2\ b.$$

H_1 and H_2 are two distinct relations from the nine.

The reader will easily verify that the first condition is implied by tables 1 and 2, apart from certain exceptional cases which are not restrictive to exclude.

The second condition imposes no restriction. In fact :

$\forall \; H_1, H_2 \in \{I, R, \sim, P, Q, \succ, J, K, S\}$,
$\exists \; H_3 \in \{I, R, \sim, P, Q, \succ, J, K, S\}$
such that $a \, H_1 \, b$ and $a \, H_2 \, b \iff a \, H_3 \, b$.

The simultaneous realization of two distinct relations for a pair of potential actions a, b can always be expressed by a unique relation. Thus for example :

$a \, P \, b$ and $a \succ b \iff a \, P \, b$,
$a \succ b$ and $a \, J \, b \iff a \, Q \, b$;
$a \sim b$ and $a \, S \, b \iff a \, I \, b$.

2. In general, a combined relational system of preference will not bring into play the nine relations. Several of these may be empty.

3. The relations J, K, S may be satisfied by one or both couples of a pair ; the relations I, R, $\sim$ are symmetric ; the relations P, Q, $\succ$ are asymmetric.

IV - PSEUDO-CRITERION AND ASSOCIATED RELATIONAL SYSTEM OF PREFERENCE

In this section, we will introduce a preference model for an actor based on a single point of view (eventually aggregating several factors or attributes). This enables us to eliminate incomparabilities and to impose several additional properties on I, P and Q. In the following section, from this model, we will give an example of a fundamental relational system of outranking which simultaneously takes into account several points of view.

IV.1 Concept of pseudo-criterion

In practice, the imprecisions, irresolutions and uncertainties which lie in the definition of a given criterion g and the assessment of the numbers $g(a)$ are such that those numbers cannot be considered as an exact image of a hard reality. It is much too ambitious to accept, as classical theory does, that a positive difference $g(a) - g(b)$ is probative of strict preference. When this difference is smaller than a certain threshold (the value of which being dictated by the common sense), it must be viewed as non convincing of a strict preference, i.e. it can be interpreted as a weak preference or as an indifference.

The expression "discriminating power of a criterion" will be used to represent the ability recognized to a function g to discriminate between indifference, weak preference and strict preference on the basis of the difference $g(a) - g(b)$. In what follows, we are interested in the case where it is possible to choose 2 functions q and p in the following way :

- the difference $g(a) - g(b)$ only becomes really convincing of a strict preference if is is greater than $p(g(b))$;
- as long as this difference (supposed positive) does not exceed $q(g(b))$, it is not considered significant at all and b therefore is indifferent to a ;

- when the difference $g(a) - g(b)$ is between $q(g(b))$ and $p(g(b))$, it expresses a weak preference of a over b (see figure 1).

The thresholds $q(g)$ and $p(g)$ may be constant or may vary with $g(b)$. When they vary with $g(b)$, they must satisfy the following conditions :

$$\frac{q(g(b)) - q(g(a))}{(b) - g(a)} \geq -1 \quad \text{and} \quad \frac{p(g(b)) - p(g(a))}{g(b) - g(a)} \geq -1.$$

Definition IV.1 : A pseudo-criterion is a function g whose discriminating power is characterized in the following way by an indifference threshold $q(g)$ and a preference threshold $p(g)$:

$\forall\, a, b \in A$:

$a\ I\ b \iff -q(g(a)) \leq g(a) - g(b) \leq q(g(b))$,

$a\ Q\ b \iff q(g(b)) < g(a) - g(b) \leq p(g(b))$,

$a\ P\ b \iff p(g(b)) < g(a) - g(b)$,

$$\frac{q(g(b)) - q(g(a))}{g(b) - g(a)} \geq -1$$

$$\frac{p(g(b)) - p(g(a))}{g(b) - g(a)} \geq -1.$$

These last two conditions assume a minimum of coherence in the decision-maker's preferences. They exclude situations such as :

$g(a) < g(b) < g(b) + q(g(b)) < g(a) + q(g(a))$,

$g(a) < g(b) < g(b) + p(g(b)) \leq g(a) + p(g(b))$.

FIGURE 1 : PREFERENCE SITUATIONS MODELIZED BY A PSEUDO-CRITERION
(CASES WHERE THRESHOLDS ARE INDEPENDENT OF $g(a)$)

a strictly preferred to b a P b	a weakly preferred to b a Q b	a indifferent to b a I b and b I a	b weakly preferred to a b Q a	b strictly preferred to a b P a

$g(a)-p$ $\quad$ $g(a)-q$ $\quad$ $g(a)$ $\quad$ $g(a)+q$ $\quad$ $g(a)+p$ $\quad$ (b)

IV.2 Special cases

1. Complete preorder and relational system of preference

In the particular case where

$q(g) \equiv p(g) \equiv 0,$

we obtain the following preference model :

$$(1)\quad \begin{cases} \forall\ a, b \in A : \\ a\ I\ b \iff g(a) = g(b) \\ a\ P\ b \iff g(a) > g(b). \end{cases}$$

<u>Theorem IV.1</u> ([4], [21]) : Given a finite or countably infinite set A and two binary relations I, P defined on A, the necessary and sufficient conditions for the existence, on A, of a real valued function g satisfying conditions (1) is that (I, P) is a complete preorder on A, i.e. I is symmetric and transitive, P is asymmetric and transitive and each pair a, b satisfies one and only one of these two relations.

The pseudo-criterion which has thresholds of indifference and preference equal to 0 therefore corresponds to the model of classical decision theory. We call it "true-criterion".

A fundamental relational system of outranking of the form (I, S) induces a complete preorder structure iff the asymmetric part of S is transitive and the relation $I \cup \overset{\circ}{S}$ is transitive ($\overset{\circ}{S}$ being the symmetric part of S).

2. Semi-order and relational system of preference

In the particular case in which

$q(g) \equiv p(g),$

we obtain the following preference model :

$$(2) \begin{cases} \forall \ a, b \in A : \\ a \ I \ b \iff - q(g(a)) \leq g(a) - g(b) \leq q(g(b)) \\ a \ P \ b \iff \quad g(a) > g(b) + q(g(b)) \\ \dfrac{q(g(b)) - q(g(a))}{g(b) - g(a)} \geq - 1. \end{cases}$$

Theorem IV.2 ([4], [21]) : Given a finite or countably infinite set A and two binary relations I, P defined on A, the necessary and sufficient condition for the existence of two functions g and q satisfying conditions (2) is that (I, P) is a semi-order on A, i.e. :

- $\forall \ a, b \in A$: $a \ I \ b$ or $a \ P \ b$ or $b \ P \ a$ (or exclusive),
- I is symmetric, P is asymmetric,
- $\forall \ a, b, c, d \in A$: $a \ P \ b$, $b \ I \ c$, $c \ P \ d \Rightarrow a \ P \ d$,
- $\forall \ a, b, c, d \in A$: $a \ P \ b$, $b \ P \ c$, $a \ I \ d \Rightarrow$ not $c \ I \ d$.

The semi-order structure was introduced by Luce ([11]) to model situations where indifference is not necessarily transitive (see also [5], [7], [8], [13], [22]). When A is finite, the function $q(g(a))$ can be taken so as to be constant (the reader will find an algorithm in [21]). Interesting examples of semi-order situations may obtained with relational systems of preference of the form $(\dot{I}, P)$, $(\sim, P)$, $(\sim, \succ)$, $(I, \succ)$, (R, S), $(\sim, S)$.

3. Oriented semi-order and relational system of preference

When

$$q(g) \equiv 0,$$

we obtain the following preference model :

$$(3) \begin{cases} \forall \ a, b \in A : \\ a \ I \ b \iff g(a) = g(b), \\ a \ Q \ b \iff g(b) < g(a) \leq g(b) + p(g(b)), \\ a \ P \ b \iff p(g(b)) < g(a) - g(b), \\ \dfrac{p(g(b)) - p(g(a))}{g(b) - g(a)} \geq - 1. \end{cases}$$

Theorem IV.3 ([21]) : Given a finite or countably infinite set A and three binary relations I, Q, P defined on A, the necessary and sufficient condition for the existence of two functions g and p satisfying the conditions (3) is that :

a) (I, Q, P) is a r.s.p., i.e.
   - $\forall\ a, b \in A$: $a\ I\ b$ or $a\ Q\ b$ or $a\ P\ b$ or $b\ P\ a$ or $b\ Q\ a$ (or exclusive),
   - I is symmetric, Q and P are asymmetric.

b) $(I, Q \cup P)$ is a complete preorder on A.

c) $(\hat{P}, P)$ is a semi-ordre on A where $a\ \hat{P}\ b$ iff not $a\ P\ b$ and not $b\ P\ a$,

d) $\forall\ a, b, c \in A$: $\begin{cases} a\ P\ b,\ b\ Q\ c \Rightarrow a\ P\ c, \\ a\ Q\ b,\ b\ P\ c \Rightarrow a\ P\ c. \end{cases}$

e) $\forall\ a, b, c, d \in A$: $a\ I\ b,\ b\ P\ c,\ c\ I\ d \Rightarrow a\ P\ d$.

(I, Q, P) will then be called an oriented semi-order on A. When A is finite, the function $p(g(a))$ can be taken so as to be constant.

We do not want to include the mathematical proofs of this theorem and the following ones in this paper : they would take too much place. They can be found, in French, in a working paper ([21]) and we are preparing a more mathematical paper which will contain them.

IV.3 Structure of the r.s.p. in the general case

Theorem IV.4 ([21]) : Given a finite or countably infinite set A and three binary relations I, Q, P defined on A, the necessary and sufficient condition for the existence of three functions g, q, p such that

$\forall\ a, b \in A$:

$a\ I\ b \iff -q(g(a)) \leq g(a) - g(b) \leq q(g(a))$,

$a\ Q\ b \iff q(g(b)) < g(a) - g(b) \leq p(g(b))$

$a\ P\ b \iff p(g(b)) < g(a) - g(b)$,

$$\frac{q(g(b)) - q(g(a))}{g(b) - g(a)} \geq -1,$$

$$\frac{p(g(b)) - p(g(a))}{g(b) - g(a)} \geq -1$$

is that :

a) (I, Q, P) is a r.s.p., i.e. :

- $\forall$ a, b $\in$ A : a I b or a P b or a Q b or b Q a or b P a (or exclusive),
- I is symmetric, Q and P are asymmetric.

b) $(\hat{I}, Q \cup P)$ is a semi-order on A.

c) $(\hat{P}, P)$ is a semi-order on A where $a \hat{P} b$ iff not a P b and not b P a.

d) $\forall$ a, b, c, d $\in$ A : $\begin{cases} a\ P\ b,\ b\ I\ c,\ c\ Q\ d \Rightarrow a\ P\ d\ (P\ I\ Q \subset P), \\ a\ Q\ b,\ b\ I\ c,\ c\ P\ d \Rightarrow a\ P\ d\ (Q\ I\ P \subset P), \\ a\ P\ b,\ b\ Q\ c,\ c\ I\ d \Rightarrow a\ P\ d\ (P\ Q\ I \subset P), \\ a\ I\ b,\ b\ Q\ c,\ c\ P\ d \Rightarrow a\ P\ d\ (I\ Q\ P \subset P). \end{cases}$

(I, Q, P) will then be called a pseudo-order on A. When A is finite, one of the two thresholds functions can be taken so as to be constant.

As it is shown by the following example, it is not always possible to take both thresholds so as to be constant. We give below a necessary and sufficient condition to have two constant thresholds.

IV.4 Example of pseudo-order

Let A = {a, b, c, d, e} and the pseudo-order (I, Q, P) be defined as follows :

b Q a, c Q a, d Q a, e P a,
c I b, d Q b, e P b,
d I c, e I c,
e I d.

The verification of the pseudo-order axioms is left up to the reader. Tables 3 and 4 respectively define a pseudo-criterion with a constant indifference threshold and a pseudo-criterion with a constant preference threshold.

TABLE 3

	a	b	c	d	e
g	0	9	13	19	20
q	8	8	8	8	8
p	19.5	10.5	9	9	9

TABLE 4

	a	b	c	d	e
g	0	4	6	7	13
q	2	2.5	7.5	7	4
p	8	8	8	8	8

It is impossible to represent this pseudo-order by means of a pseudo-criterion with two constant thresholds q and p. In fact, $d\ I\ e$, $e\ P\ b$ and $b\ Q\ a$ imply respectively :

$$\begin{cases} |g(d) - g(e)| < q, \\ g(e) > g(b) + p, \\ g(b) > g(a) + q, \end{cases}$$

therefore

$$g(d) + q > g(e) > g(b) + p > g(a) + p + q$$

therefore

$$g(d) > g(a) + p$$

which contradicts $d\ Q\ a$.

Note also that a complete preorder M may be associated with any pseudo-order in the following way :

$$a\ M\ b \quad \text{iff} \quad \forall\ c \in A : \begin{cases} c\ Q\ a \Rightarrow c \succ b, \\ c\ P\ a \Rightarrow c\ P\ b, \\ b\ Q\ c \Rightarrow a \succ c, \\ b\ P\ c \Rightarrow a\ P\ c. \end{cases}$$

In our example we obtain

$$e\ M\ d\ M\ c\ M\ b\ M\ a.$$

In this example, the pseudo-order (I, Q, P) does not verify the conditions $Q\ P\ I \subset P$ and $I\ P\ Q \subset P$; these conditions are necessary to have two constant thresholds but are not sufficient.

IV.5 Condition for a pseudo-order to have two constant thresholds

Let (I, Q, P) be a triplet of relations in a finite set A such that $I \cup Q \cup P$ is a complete relation and I is symmetric.

Let us denote :

$$a\ Q^{-1}\ b \quad \text{iff} \quad b\ Q\ a.$$

We will call a circuit in (I, Q, P) a sequence of the form

$$a_1\ R_1\ a_2\ R_2\ a_3 \ldots R_1\ a_1$$

where

$$a_1, a_2, \ldots, a_1 \in A,$$
$$R_1, R_2, \ldots, R_1 \in \{I, Q, Q^{-1}, P\}.$$

By definition, the k-values of this circuit is the sum

$$v_k(R_1) + v_k(R_2) + \ldots + v_k(R_1)$$

where $v_k(R_i) = \begin{cases} -1 & \text{iff } R_i = I, \\ +1 & \text{iff } R_i = Q, \\ -k & \text{iff } R_i = Q^{-1}, \\ +k & \text{iff } R_i = P. \end{cases}$

Theorem IV.5 ([21]) : Let (I, Q, P) be a triplet of relations on a finite set A, such that $I \cup Q \cup P$ is complete and I is symmetric ; the necessary and sufficient condition for the existence of a function g and 2 constants q and p such that
$\forall\ a, b \in A$:
$a\ I\ b \iff |g(a) - g(b)| \leq q$,
$a\ Q\ b \iff g(b) + q < g(a) \leq g(b) + p$,
$a\ P\ b \iff g(b) + p < g(a)$
is that there exists $k > 1$ such that every circuit in (I, Q, P) has a strictly negative k-value.

V - EXAMPLES OF RELATIONAL SYSTEMS OF OUTRANKING ASSOCIATED WITH A FAMILY F OF PSEUDO-CRITERIA

Let $F = \{g_1, g_2, \ldots, g_n\}$ be a family of pseudo-criteria characterizing preferences attached to heterogeneous points of view. We respectively denote by P_j, Q_j and I_j $(j = 1, 2, \ldots, n)$ the associated strict preference, weak preference and indifference relations. Let us define the relation D_j by

$$b \; D_j \; a \quad \text{iff} \quad g_j(b) \geq g_j(a).$$

First, let us consider the case in which the relative importance of each criterion is not known or not definable. Classicaly, when thresholds are null, the only relation allowing global preference comparisons is the one of dominance D defined by

$$b \; D \; a \quad \text{iff} \quad b \; D_j \; a \quad \forall j.$$

When thresholds are not all null, a relational system of outranking (S, R) can be defined from F as follows (see [15] for an application) :

$$b \; S \; a \quad \text{iff} \quad \begin{cases} a \; P_j \; b \quad \text{is false} \quad \forall j \; ; \\ \|\{j : a \; Q_j \; b\}\| \leq \|\{j : b \; P_j \; a\}\| + \|\{j : b \; Q_j \; a\}\| \end{cases}$$

$b \; R \; a$ in other cases

where $\|X\|$ denotes the cardinality of the set X.

This relation S, which is not necessarily transitive, appears as a possible generalization of dominance. Other ones are considered in [15].

The more complex case in which some information in the relative importance of the pseudo-criteria must be taken into account would entail too long developments for this paper. Let us only mention that two cases can be distinguished :

- First, the information is too poor to assess substitution rates. The basic ideas of concordance and discordance ([13]) can be used to define a relational system of outranking (S, R) in the same spirit than in ELECTRE I or II.

- Second, the information is rich enough to eliminate every incomparability, through compensatory rules. However, the classical concept of substitution rate is defined without any threshold consideration and the reconciliation of the two concepts appears an interesting open field of research. The objective could be to aggregate the pseudo-criteria g_1, ..., g_n so as to build an r.s.p. of the form (I, Q, P) (see theorem IV.4) satisfying the pseudo-order conditions.

VI - CONCLUSION

The models commonly used in operations research and in decision theory often constitute strong idealizations of the reality. The out and out simplification of a model allows us to obtain nice mathematical properties, but the question is how realistic the model is then.

We think that the classical approaches of decision theory are only applicable to rather a restricted number of concrete problems. This aspect probably has a part in the actual crisis of decision science and in the mistrust of the practitionners towards it.

In order to right the situation, new concepts must be defined. The diversity of the models which can be deduced from the notions introduced here (and which contain the classical models as particular cases) perhaps constitutes a progress in this direction.

We terminate with two suggestions in this sense.

1. It is more realistic, in many concrete problems, to use a multicriteria method based on pseudo-criteria than on true-criteria. Although the choice of the values of the thresholds is not easy, it is a particularly arbitrary decision to consider them as null.

2. We do not think it is very good to systematically eliminate incomparability through hiding discrepancies in the value systems, accepting compensatory rules based on theoretical considerations not testable in practice, systematically representing imprecisions and irresolutions by probability distributions more often than not artificial, ... In fact, it is possible to develop multicriteria methods which take this incomparability into account. They are based on r.s.p. which are poorer, but more flexible and reliable than the classical one (I, P) which results in a systematic elimination of incomparabilities, both relations having

to be transitive. Obviously, such r.s.p. do not allow to determine an optimum and to impose the best decision : the prescription is more modest but the conclusions are more convincing. "Decision-making", in this case, is replaced by decision-aid.

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